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Continuity of Riesz potential operator in the supercritical case on unbounded domain

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Abstract

The aim of this paper is to prove continuity of the Riesz potential operator $R^s : E \mapsto \mathcal{C}H$ in optimal couple $E, \mathcal{C}H$, for the supercritical case on unbounded domain, where E is a rearrangement invariant function space and $\mathcal{C}H$ is the generalized Hölder-Zygmund space generated by a function space H. We also construct optimal domain and target quasi-norms for R^s on unbounded domain.

MSC: 46E30; 46E35

Keywords: Riesz potential operator; rearrangement invariant function spaces; Hölder-Zygmund space; real interpolation

1 Introduction

Let L_{loc} be the space of all locally integrable functions f on $\mathbf{R}^n$ with lebesgue measure. The Riesz potential operator R^s, $0 < s < n$, $n \geq 1$ is defined by

$$R^s f(x) = \int_{\mathbf{R}^n} f(y)|x-y|^{s-n}\,dy,$$

where $f \in L_{\mathrm{loc}}$.

It is well known that in the supercritical case $s > n/p$,

$$R^s : L^p \mapsto \mathcal{C}^{s-n/p}, \quad s > n/p, \tag{1.1}$$

where $\mathcal{C}^\gamma$; $\gamma > 0$ is Hölder-Zygmund space [1], but in the critical case $s = n/p$ the function $R^s f$ may not be even continuous. We prove the optimal one is obtained if in above L^p is replaced by Marcinkiewicz space $L^{p,\infty}$. In this paper we prove similar optimal results, when $L^{p,\infty}$ is replaced by more general rearrangement invariant spaces E. More precisely, we consider quasi-norm rearrangment invariant space E, consisting of functions $f \in L^1 + L^\infty$, such that the quasi-norm $\|f\|_E = \rho(f^*) < \infty$, where ρ_E a monotone quasi-norm, defined on M^+ with values in $[0,\infty]$. Here M^+ is the cone of all locally integrable functions $g \geq 0$ on $(0,\infty)$ with Lebesgue measure.

Monotonicity means that $g_1 \leq g_2$ implies $\rho_E(g_1) \leq \rho_E(g_2)$. We suppose that $L^1 \cap L^\infty \hookrightarrow E \hookrightarrow L^1 + L^\infty$, which means continuous embeddings. Here f^* is the decreasing rearrangement of f, given by

$$f^*(t) = \inf\{\lambda > 0 : \mu_f(\lambda) \leq t\}, \quad t > 0,$$

and μ_f is the distribution function of f, defined by

$$\mu_f(\lambda) = \left|\left\{x \in \mathbf{R}^n : \left|f(x)\right| > \lambda\right\}\right|_n,$$

$|\cdot|_n$ denoting the Lebesgue n-measure.

Finally,

$$f^{**}(t) := \frac{1}{t}\int_0^t f^*(s)\,ds.$$

Let α_E, β_E be the Boyd indices of E (see [2–4]). For example, if $E = L^p$, then $\alpha_E = \beta_E = 1/p$ and the condition $1 > s/n \geq 1/p$ means $p > 1$, $\beta_E < 1$. For these reasons we suppose that for the general E, $0 < \alpha_E = \beta_E \leq 1$, and the case $s/n > \alpha_E$ is called supercritical, while the case $s/n = \alpha_E$ is called critical. In the supercritical case the function $R^s f; f \in E$ is always continuous [5], while the spaces in the critical case $\alpha_E = s/n$, can be divided into two subclasses: in the first subclass the functions $R^s f$ may not be continuous; then the target space is rearrangement invariant, while these functions in the second subclass are continuous and the target space is the generalized Hölder-Zygmund space $\mathcal{C}H$ [6, 7]. The separating space for these two subclasses is given by the Lorentz space $L^{n/s,1}$. The continuity of fractional maximal operator and Bessel potential operator is discussed in [8] and [9]. Gogatishvili and Ovchinnikov in [10] discussed the optimal Sobolev's embeddings. The problem of the optimal target rearrangement invariant spaces for potential type operators is considered in [11] by using L_p-capacities. The problem of mapping properties of the Riesz potential in optimal couples of rearrangement invariant spaces is treated in [12–15]. The characterization of the continuous embedding of the generalized Bessel potential spaces into Hölder-Zygmund spaces $\mathcal{CH}$, when H is a weighted Lebesgue space, is given in [7]. For further literature and reviews, we refer the reader to [16–20].

The main goal of this paper is to prove continuity of the Riesz potential operator $R^S : E \mapsto \mathcal{C}H$ in an optimal couple E, $\mathcal{C}H$, for the supercritical case on unbounded domain. The same problem was considered in [5] for bounded domain. The critical and subcritical case for the continuity of Riesz potential operator was considered in [12] and [14].

The plane of this paper is as follows. In Section 2 we provide some basic definitions and known results. In Section 3 we characterize the continuity of the Riesz potential operator $R^S : E \mapsto \mathcal{C}H$. The optimal quasi-norms are constructed in Section 4.

2 Preliminaries

We use the notations $a_1 \lesssim a_2$ or $a_2 \gtrsim a_1$ for nonnegative functions or functionals to mean that the quotient a_1/a_2 is bounded; also, $a_1 \approx a_2$ means that $a_1 \lesssim a_2$ and $a_1 \gtrsim a_2$. We say that a_1 is equivalent to a_2 if $a_1 \approx a_2$.

There is an equivalent quasi-norm $\rho_p \approx \rho_E$ that satisfies the triangle inequality $\rho_p^p(g_1 + g_2) \leq \rho_p^p(g_1) + \rho_p^p(g_2)$ for some $p \in (0,1]$ that depends only on the space E (see [21]). We say that the quasi-norm ρ_E satisfies Minkowski's inequality if for the equivalent quasi-norm ρ_p,

$$\rho_p^p\Big(\sum g_j\Big) \lesssim \sum \rho_p^p(g_j), \quad g_j \in M^+.$$

Usually we apply this inequality to functions $g \in M^+$ with some kind of monotonicity.

Recall the definition of the lower and upper Boyd indices α_E and β_E. Let $g_u(t) = g(t/u)$ where $g \in M^+$, and let

$$h_E(u) = \sup\left\{\frac{\rho_E(g_u^*)}{\rho_E(g^*)} : g \in M^+\right\}, \quad u > 0,$$

be the dilation function generated by ρ_E. Suppose that it is finite. Then

$$\alpha_E := \sup_{0<t<1} \frac{\log h_E(t)}{\log t} \quad \text{and} \quad \beta_E := \inf_{1<t<\infty} \frac{\log h_E(t)}{\log t}.$$

The function h_E is sub-multiplicative, increasing, $h_E(1) = 1$, $h_E(u)h_E(1/u) \geq 1$ hence $0 \leq \alpha_E \leq \beta_E$. We suppose that $0 < \alpha_E = \beta_E \leq 1$ and $g^{**}(\infty) = 0$.

If $\beta_E < 1$ we have by using Minkowski's inequality that $\rho_E(f^*) \approx \rho_E(f^{**})$.

Recall that $w \in M^+$ is slowly varying function, if for every $\epsilon > 0$, the function $t^\epsilon w(t)$ is equivalent to increasing function and $t^{-\epsilon} w(t)$ is equivalent to a decreasing function.

In order to introduce the Hölder-Zygmund class of spaces, we denote the modulus of continuity of order k by

$$\omega^k(t,f) = \sup_{|h|\leq t} \sup_{x \in \mathbf{R}^n} \left|\Delta_h^k f(x)\right|,$$

where $\Delta_h^k f$ are the usual iterated differences of f. When $k = 1$ we simply write $\omega(t,f)$. Let H be a quasi-normed space of locally integrable functions on the interval $(0,1)$ with the Lebesgue measure, continuously embedded in $L^\infty(0,1)$ and $\|g\|_H = \rho_H(|g|)$, where ρ_H is a monotone quasi-norm on M^+ which satisfies Minkowski's inequality. The dilation function h_H, generated by ρ_H, is defined as follows:

$$h_H(u) = \sup\left\{\frac{\rho_H(\chi_{(0,1)}\tilde{g}_u)}{\rho_H(\chi_{(0,1)}g)} : g \in M_a\right\},$$

where $(\tilde{g}_u)(t) = g(ut)$ if $ut < 1$, $(\tilde{g}_u)(t) = g(1)$ if $ut \geq 1$, and

$$M_a := \left\{g \in M^+ : t^{-a/n} g(t) \text{ is decreasing } g \text{ is increasing and } g(+0) = 0\right\}.$$

The choice of the space M_a is motivated by the fact that $\omega^n(t^{1/n},f)$, is equivalent to a function $g \in M_a$.

The function $h_H(u)$ is sub-multiplicative and $u^{-1}h_H(u)$ is decreasing and

$$h_H(1) = 1, \qquad h_H(u)h_H(1/u) \geq 1.$$

Suppose that h_H is finite. Then the Boyd indices of H are well defined,

$$\alpha_H = \sup_{0<t<1} \frac{\log h_H(t)}{\log t} \quad \text{and} \quad \beta_H = \inf_{1<t<\infty} \frac{\log h_H(t)}{\log t},$$

and they satisfy $\alpha_H \leq \beta_H \leq 1$. In the following, we suppose that $0 \leq \alpha_H = \beta_H < 1$.

For example, let $H = L_*^q(b(t)t^{-\gamma/n})$. Here $0 \leq \gamma < a/n$ and b is a slowly varying function, and $L_*^q(w)$, or simply L_*^q if $w = 1$, is the weighted Lebesgue space with a quasi-norm $\|g\|_{L_*^q(w)} = \rho_{w,q}(|g|)$. It turns out that $\alpha_H = \beta_H = \gamma/n$.

Definition 2.1 Let $j = 0, 1, \ldots$ and let C^j stand for the space of all functions f, defined on $\mathbf{R}^n$, that have bounded and uniformly continuous derivatives up to the order j, normed by $\|f\|_{C^j} = \sup \sum_{l=0}^{j} |P^l f(x)|$, where $P^l f(x) = \sum_{|\nu|=l} D^\nu f(x)$.

- If $j/n < \alpha_H < (j+1)/n$ for $j \geq 1$ or $0 \leq \alpha_H < 1/n$ for $j = 0$, then $\mathcal{C}H$ is formed by all functions f in C^j having a finite quasi-norm

$$\|f\|_{\mathcal{C}H} = \|f\|_{C^j} + \rho_H\bigl(t^{j/n}\omega\bigl(t^{1/n}, P^j f\bigr)\bigr).$$

- If $\alpha_H = (j+1)/n$, then $\mathcal{C}H$ consists of all functions f in C^j having a finite quasi-norm

$$\|f\|_{\mathcal{C}H} = \|f\|_{C^j} + \rho_H\bigl(t^{j/n}\omega^2\bigl(t^{1/n}, P^j f\bigr)\bigr).$$

In particular, if $H = L^\infty(t^{-\gamma/n})$, $\gamma > 0$, then $\mathcal{C}H$ coincides with the usual Hölder-Zygmund space $\mathcal{C}^\gamma$ (see [1]). Also, if $H = L^\infty$, then $\mathcal{C}H = C^0$. We need the following result about the equivalent quasi-norm in the generalized Hölder-Zygmund spaces.

Theorem 2.2 (equivalence) ([6]) *Let ρ_H be a monotone quasi-norm, satisfying Minkowski's inequality and let $0 \leq \alpha_H = \beta_H < m/n$. If $\rho_H(t^\alpha) < \infty$ for $\alpha > \alpha_H$, then, for all such m,*

$$\|f\|_{\mathcal{C}H} \approx \|f\|_{C^0} + \rho_H\bigl(\omega^m\bigl(t^{1/n}, f\bigr)\bigr). \tag{2.1}$$

Let N be the class of all admissible couples, it will be convenient to use the following definitions.

Definition 2.3 (admissible couple) We say that the couple $(\rho_E, \rho_H) \in N$ is admissible for the Riesz potential if

$$\bigl\|R^s f\bigr\|_{\mathcal{C}H} \lesssim \rho_E\bigl(f^*\bigr), \quad f \in E. \tag{2.2}$$

Then the couple E, H is called admissible. Moreover, ρ_E (E) is called domain quasi-norm (domain space), and ρ_H (H) is called the target quasi-norm (target space).

To prove our result we introduce the classes of the domain and target quasi-norms, where the optimality is investigated.

Let N_d consist of all domain quasi-norms ρ_E that are monotone, satisfy Minkowski's inequality, $0 < \alpha_E = \beta_E < 1$, and the condition

$$\int_0^\infty t^{s/n-1} g(t)\, dt \lesssim \rho_E(g), \quad g \downarrow, \tag{2.3}$$

$\int_0^\infty g^*(u)\, du \lesssim \rho_E(g^*)$ and $\rho_E(\chi_{(0,1)} t^{-\alpha}) < \infty$ if $\alpha < \alpha_E$.

Let N_t consist of all target quasi-norms ρ_H that are monotone, satisfy Minkowski's inequality, $0 \leq \alpha_H = \beta_H < 1$, $\rho_H(t^\alpha) < \infty$ if $\alpha > \alpha_H$ and $\sup \chi_{(0,1)} g(t) \lesssim \rho_G(\chi_{(0,1)} g)$, $g \in M_n$.

Finally

$$N := \bigl\{(\rho_E, \rho_H) \in N_d \times N_t : \rho_H\bigl(\chi_{(0,1)} t^{s/n} g(t)\bigr) \lesssim \rho_E(g), g \downarrow\bigr\}.$$

Definition 2.4 (optimal target quasi-norm) Given the domain quasi-norm ρ_E, the optimal target quasi-norm, denoted by $\rho_{H(E)}$, is the strongest target quasi-norm, such that $(\rho_E, \rho_{H(E)}) \in N$ and

$$\rho_H(\chi_{(0,1)}g) \lesssim \rho_{H(E)}(\chi_{(0,1)}g), \quad g \in M_n, \tag{2.4}$$

for any target quasi-norm ρ_H such that the couple $(\rho_E, \rho_H) \in N$ is admissible. Since $\mathcal{C}H(E) \hookrightarrow \mathcal{C}H$, we call $\mathcal{C}H(E)$ the optimal Hölder-Zygmund space. For shortness, the space $H(E)$ is also called an optimal target space.

Definition 2.5 (optimal domain quasi-norm) Given the target quasi-norm $\rho_H \in N_t$, the optimal domain quasi-norm, denoted by $\rho_{E(H)}$, is the weakest domain quasi-norm, such that $(\rho_{E(H)}, \rho_H) \in N$ and

$$\rho_{E(H)}(f^*) \lesssim \rho_E(f^*), \quad f \in E,$$

for any domain quasi-norm $\rho_E \in N_d$ such that the couple $(\rho_E, \rho_H) \in N$ is admissible. The space $E(H)$ is called an optimal domain space.

Definition 2.6 (optimal couple) The admissible couple $(\rho_E, \rho_H) \in N$ is said to be optimal if both ρ_E and ρ_H are optimal. Then the couple E, H is called optimal.

3 Admissible couples

Here we give a characterization of all admissible couples $(\rho_E, \rho_H) \in N$. By using the following Hardy-Littlewood inequality [2], p.44, we get the well-known mapping property:

$$R^s : \Lambda^1(t^{s/n}) \mapsto L^\infty.$$

Now from (2.3) it follows that

$$R^s : E \to L^\infty. \tag{3.1}$$

We have the following basic estimate.

Theorem 3.1 *If $f \in E$, then*

$$\chi_{(0,1)}\omega^m(t^{1/n}, R^s f) \lesssim S(f^*)(t), \quad s < m, \tag{3.2}$$

where

$$Sg(t) := \int_0^t u^{s/n-1} g(u)\, du, \quad g \in M^+. \tag{3.3}$$

Proof The proof of this result follows from Theorem 3.1 in [5]. □

Now we discuss the mapping property $R^s : E \mapsto C^0$.

Theorem 3.2 *A necessary and sufficient condition for the mapping*

$$R^s : E \mapsto C^0$$

is the following:

$$\int_0^\infty t^{s/n-1} g(t)\,dt \lesssim \rho_E(g), \quad g \downarrow. \tag{3.4}$$

Proof We already know that

$$R^s : E \to L^\infty. \tag{3.5}$$

To prove that $R^s(E) \subset C^0$, it remains to show that $R^s f$ is a uniformly continuous function. It is enough to show that

$$\lim_{t\to 0} \omega\bigl(t^{\frac{1}{n}}, R^s f\bigr) = 0 \quad \text{if } f \in E.$$

By using Marchaud's inequality,

$$\omega\bigl(t^{\frac{1}{n}}, R^s f\bigr) \lesssim t^{\frac{1}{n}} \int_t^\infty u^{\frac{-1}{n}} \omega^m\bigl(u^{\frac{1}{n}}, R^s f\bigr)\frac{du}{u},$$

L'Hôpital's rule, and (3.2), we get

$$\begin{aligned} \lim_{t\to 0} \omega\bigl(t^{\frac{1}{n}}, R^s f\bigr) &\lesssim \lim_{t\to 0} \frac{t^{\frac{-1}{n}} \omega^m(t^{\frac{1}{n}}, R^s f)}{t^{\frac{-1}{n}}} \\ &= \lim_{t\to 0} \omega^m\bigl(t^{\frac{1}{n}}, R^s f\bigr) \\ &\lesssim \lim_{t\to 0} Sf^*(t) = 0. \end{aligned}$$

Hence

$$R^s f \in C^0.$$

It remains to prove that if $R^s : E \to C^0$, then (3.4) is true for $\alpha_E \le s/n$. To this end we choose a test function h as follows. Let $g \in D_{n-s}$, $\rho_E(g) < \infty$ and

$$h(x) = \int_0^\infty g(u)\varphi\bigl(xu^{-1/n}\bigr)\frac{du}{u}, \tag{3.6}$$

where $\varphi \ge 0$ is a smooth function with compact support in $(-c^{-1/n}, c^{-1/n})$ such that if $\psi = R^s\varphi$, then $\psi(0) > 0$. To see that this is possible, we calculate $\psi(0)$. Since

$$\psi(x) = \int_{R^n} \varphi(y)|x-y|^{s-n}\,dy,$$

we have for appropriate $d > 0$,

$$\psi(0) \ge \int_{|y|\le d} \varphi(y)|y|^{s-n}\,dy \gtrsim \int_{|y|\le d} \varphi(y)\,dy > 0.$$

Note also that, for large $c > 0$,

$$\psi(x) \lesssim |x|^{s-n}, \quad u > c. \tag{3.7}$$

Indeed

$$\psi(x) = \int_{|y|\leq d} \varphi(y)|x-y|^{s-n}\,dy \lesssim |x|^{s-n} \int_{|y|\leq d} \varphi(y)\,dy$$

since

$$|x-y| \geq |x| - |y| \geq |x| - d \geq |x|/2, \quad \text{if } c > 2d.$$

We also have

$$R^s\bigl(\varphi\bigl(xu^{-1/n}\bigr)\bigr) = u^{s/n}\psi\bigl(xu^{-1/n}\bigr).$$

Hence

$$f(x) := R^s h(x) = \int_0^\infty u^{s/n} g(u)\psi\bigl(xu^{-1/n}\bigr)\frac{du}{u}.$$

We may take

$$h(x) \lesssim \int_{c|x|^n}^\infty g(u)\,du/u,$$

hence, for appropriate $c > 0$,

$$h^*(t) \lesssim \int_t^\infty g(u)\,du/u.$$

Applying Minkowski's inequality and using $\alpha_E > 0$, we have

$$\rho_E\bigl(h^*\bigr) \lesssim \rho_E(g). \tag{3.8}$$

Given that

$$\sup\bigl|R^s h(x)\bigr| \lesssim \|h\|_E,$$

we have in particular

$$\bigl|R^s h(0)\bigr| \lesssim \|h\|_E,$$

whence

$$R^s h(0) = \psi(0)\int_0^\infty u^{s/n-1} g(u)\,du \lesssim \|h\|_E \lesssim \rho_E(g).$$

Thus (3.4) is proved. □

In the following theorem, we characterize the admission couple. Note that this result cannot obtained directly from Theorem 3.4 [5], because here we consider an unbounded domain.

Theorem 3.3 *The couple* $(\rho_E, \rho_H) \in N$ *is admissible if and only if*

$$\rho_H(\chi_{(0,1)} Sg) \lesssim \rho_E(g), \quad g \downarrow. \tag{3.9}$$

Proof Let (3.9) be true. By using (3.2) and (3.9), we get

$$\rho_H\bigl(\chi_{(0,1)}\omega^m\bigl(t^{\frac{1}{n}}, R^s f\bigr)\bigr) \lesssim \rho_H\bigl(\chi_{(0,1)} S\bigl(f^*\bigr)\bigr) \lesssim \rho_E\bigl(f^*\bigr), \quad m > s.$$

Therefore

$$\begin{aligned}\bigl\|R^s f\bigr\|_{CH} &\approx \bigl\|R^s f\bigr\|_{C^0} + \rho_H\bigl(\omega^m\bigl(t^{\frac{1}{n}}, R^s f\bigr)\bigr)\\ &\lesssim \rho_E\bigl(f^*\bigr) + \bigl\|R^s f\bigr\|_{C^0}\\ &\lesssim \rho_E\bigl(f^*\bigr) + \rho_E\bigl(f^*\bigr)\\ &\lesssim \rho_E\bigl(f^*\bigr).\end{aligned}$$

Thus ρ_E, ρ_H is an admissible couple.

For the converse, we have to prove that (2.2) implies (3.9). To this end we choose a test function in the form $f(x) = R^s h(x)$, where h is given by (3.6). We have

$$f(x) = R^s h(x) = \int_0^\infty u^{s/n} g(u)\psi\bigl(xu^{-\frac{1}{n}}\bigr)\frac{du}{u}.$$

To estimate the modulus of continuity of f from below, we split f as follows:

$$f = f_{1t} + f_{2t},$$

where

$$f_{1t}(x) = \int_0^t u^{\frac{s}{n}} g(u)\psi\bigl(xu^{-\frac{1}{n}}\bigr)\frac{du}{u}, \qquad f_{2t}(x) = \int_t^\infty u^{\frac{s}{n}} g(u)\psi\bigl(xu^{-\frac{1}{n}}\bigr)\frac{du}{u}.$$

First we prove that, for some large $C > 0$,

$$\omega^m\bigl(Ct^{\frac{1}{n}}, f_{1t}\bigr) \ge \frac{\psi(0)}{2} Sg(t).$$

To this aim consider

$$\Delta_h^m f_{1t}(x) = \int_0^t u^{\frac{s}{n}} g(u)\Delta_h^m \psi\bigl(xu^{-\frac{1}{n}}\bigr)\frac{du}{u}.$$

Also consider

$$\begin{aligned}\Delta_h^m \psi\bigl(xu^{-\frac{1}{n}}\bigr) &= \sum_{k=0}^m (-1)^{m-k}\binom{m}{k}\psi\bigl((x+hk)u^{-\frac{1}{n}}\bigr)\\ &= (-1)^m\psi(0) + \sum_{k=1}^m (-1)^{m-k}\binom{m}{k}\psi\bigl(hku^{-\frac{1}{n}}\bigr) \quad \text{at } x = 0.\end{aligned}$$

If $|h| = Ct^{\frac{1}{n}}$, then for $u < t$, $k \geq 1$, $|h|ku^{-\frac{1}{n}} \geq Ck \geq C$, hence by (3.7) and for large $C > 0$,

$$\psi\left(hku^{-\frac{1}{n}}\right) \lesssim C^{s-n}, \quad u < t, k \geq 1.$$

Therefore,

$$\Delta_h^m f_{1t}(0) = \int_0^t u^{\frac{s}{n}} g(u)\left[(-1)^m \psi(0) + \sum_{k=1}^m (-1)^{m-k} \psi\left(hku^{-\frac{1}{n}}\right)\right]\frac{du}{u}$$

and, for large $C > 0$,

$$\begin{aligned}\left|\Delta_h^m f_{1t}(0)\right| &= \left|(-1)^m \psi(0)\int_0^t u^{\frac{s}{n}} g(u)\frac{du}{u} + \sum_{k=1}^m (-1)^{m-k}\int_0^t u^{\frac{s}{n}} g(u)\psi\left(hku^{-\frac{1}{n}}\right)\frac{du}{u}\right| \\ &\geq \psi(0)\int_0^t u^{\frac{s}{n}} g(u)\frac{du}{u} - c_m \int_0^t u^{\frac{s}{n}} g(u)\psi\left(hku^{-\frac{1}{n}}\right)\frac{du}{u} \\ &\geq \psi(0)\int_0^t u^{\frac{s}{n}} g(u)\frac{du}{u} - C^{s-n} c_m \int_0^t u^{\frac{s}{n}} g(u)\frac{du}{u} \\ &= \frac{\psi(0)}{2}\int_0^t u^{\frac{s}{n}} g(u)\frac{du}{u}.\end{aligned}$$

Hence

$$\omega^m\left(Ct^{\frac{1}{n}}, f_{1t}\right) \geq \frac{\psi(0)}{2} Sg(t)$$

or

$$\omega^m\left(t^{\frac{1}{n}}, f_{1t}\right) \approx \omega^m\left(Ct^{\frac{1}{n}}, f_{1t}\right) \geq \frac{\psi(0)}{2} Sg(t). \tag{3.10}$$

Further,

$$\omega^m\left(t^{\frac{1}{n}}, f\right) \geq \omega^m\left(t^{\frac{1}{n}}, f_{1t}\right) - \omega^m\left(t^{\frac{1}{n}}, f_{2t}\right).$$

Now we estimate the modulus of continuity of the second function from above. To this aim, by using the formula [2], p.336, we get

$$\begin{aligned}\left|\Delta_h^m f_{2t}(x)\right| &= \left|\int_{-\infty}^{\infty} M_m(u)\sum_{|\nu|=m}\frac{m!}{\nu!} D^\nu f_{2t}(x+uh)h^\nu\,du\right| \\ &\lesssim \int_{-\infty}^{\infty} M_m(u)\sum_{|\nu|=m}\frac{m!}{\nu!}\left|D^\nu f_{2t}(x+uh)\right||h|^{|\nu|}\,du.\end{aligned}$$

Hence

$$\sup_x\left|\Delta_h^m f_{2t}(x)\right| \lesssim |h|^m \int_{-\infty}^{\infty} M_m(u)\sup\left|P^m f_{2t}(x+uh)\right|du \lesssim |h|^m \left\|P^m f_{2t}\right\|_{L_\infty}.$$

Therefore

$$\sup_x\left|\Delta_h^m f_{2t}(x)\right| \lesssim |h|^m \left\|P^m f_{2t}\right\|_{L_\infty}. \tag{3.11}$$

To simplify (3.11), consider

$$\begin{aligned} &\left|P^m f_{2t}\right| = \left|\int_t^\infty u^{\frac{s}{n}} g(u) P^m\left(\psi\left(xu^{-\frac{1}{n}}\right)\right)\frac{du}{u}\right|, \\ &\sup_x\left|P^m f_{2t}\right| \lesssim \int_t^\infty u^{\frac{s}{n}} g(u) u^{-\frac{m}{n}} \left\|P^m\psi\right\|_{L_\infty}\frac{du}{u}, \\ &\left\|P^m f_{2t}\right\|_{L_\infty} \lesssim \int_t^\infty u^{\frac{s-m}{n}} g(u)\frac{du}{u}. \end{aligned} \tag{3.12}$$

So (3.11) becomes

$$\omega^m\left(t^{\frac{1}{n}}, f_{2t}\right) \lesssim t^{\frac{m}{n}} \int_t^\infty u^{\frac{s-m}{n}} g(u)\frac{du}{u} \tag{3.13}$$

whence for $m > s$, we have

$$\omega^m\left(t^{\frac{1}{n}}, f_{2t}\right) \lesssim \int_t^\infty t^{\frac{s}{n}} g(u)\frac{du}{u}.$$

Hence

$$\chi_{(0,1)} Sg(t) \lesssim \chi_{(0,1)}\omega^m\left(t^{\frac{1}{n}}, f\right) + \chi_{(0,1)}\int_t^\infty t^{\frac{s}{n}} g(u)\frac{du}{u}, \tag{3.14}$$

$$\rho_H(\chi_{(0,1)} Sg) \lesssim \rho_H\left(\chi_{(0,1)}\omega^m\left(t^{1/n}, f\right)\right) + \rho_H\left(\chi_{(0,1)}\int_t^\infty t^{\frac{s}{n}} g(u)\frac{du}{u}\right).$$

Now since $(\rho_E, \rho_H) \in N$, we get

$$\rho_H(\chi_{(0,1)} Sg) \lesssim \rho_E(g). \qquad \square$$

4 Optimal quasi-norms

Here we give a characterization of the optimal domain and optimal target quasi-norms.

4.1 Optimal domain quasi-norms

We can construct an optimal domain quasi-norm $\rho_{E(H)}$ by Theorem 3.3 as follows.

Definition 4.1 (construction of an optimal domain quasi-norm) For a given target quasi-norm $\rho_H \in N_t$ we set

$$\rho_{E(H)}(g) := \rho_H(\chi_{(0,1)} Sg), \quad g \in M^+. \tag{4.1}$$

Note that

$$\alpha_{E(H)} = \beta_{E(H)} = s/n - \alpha_H.$$

Theorem 4.2 *The couple $\rho_{E(H)}$, ρ_H is admissible and the domain quasi-norm $\rho_{E(H)}$ is optimal. Moreover, the target quasi-norm ρ_H is also optimal and*

$$\rho_{E(H)}(g) \approx \rho_H\left(\chi_{(0,1)} t^{s/n} g\right), \quad g\downarrow \ \text{if}\ \alpha_H > 0. \tag{4.2}$$

Proof The couple $\rho_{E(H)}$, ρ_H is admissible since

$$\rho_H(\chi_{(0,1)}Sg) = \rho_{E(H)}(g).$$

Moreover, $\rho_{E(H)}$ is optimal, since for any admissible couple $(\rho_{E_1}, \rho_H) \in N$ we have

$$\rho_H(\chi_{(0,1)}Sg) \lesssim \rho_{E_1}(g).$$

Therefore,

$$\rho_{E(H)}(f^*) = \rho_H\big(\chi_{(0,1)}S(f^*)\big) \lesssim \rho_{E_1}(f^*), \quad f \in E.$$

To prove that ρ_H is also optimal, let $(\rho_{E(H)}, \rho_{H_1}) \in N$ be an arbitrary admissible couple. Then

$$\rho_{H_1}(\chi_{(0,1)}Sg) \lesssim \rho_{E(H)}(g).$$

We have to show that

$$\rho_{H_1}(\chi_{(0,1)}g) \lesssim \rho_H(\chi_{(0,1)}g), \quad g \in M_n. \tag{4.3}$$

Since $g \in M_n$ is a quasi-concave, it is equivalent to a concave one, hence

$$g(t) \approx \int_0^t h_1(u)\,du, \quad h_1 \downarrow.$$

Let

$$h(t) = t^{1-s/n}h_1(t).$$

Therefore

$$\rho_{H_1}(\chi_{(0,1)}g) \lesssim \rho_{H_1}(\chi_{(0,1)}Sh) \lesssim \rho_{E(H)}(h) \lesssim \rho_H(\chi_{(0,1)}Sh) \lesssim \rho_H(\chi_{(0,1)}g).$$

Thus (4.3) is proved.

To prove the equivalence (4.2), first we prove that

$$\rho_{E(H)}(g) \lesssim \rho_H\big(\chi_{(0,1)}t^{\frac{s}{n}}g\big), \quad g \downarrow \text{ if } \alpha_H > 0.$$

To this aim we consider

$$\begin{aligned}\rho_H(\chi_{(0,1)}Sg) &= \rho_H\left(\chi_{(0,1)}\int_0^t u^{\frac{s}{n}}g(u)\frac{du}{u}\right)\\ &= \rho_H\left(\chi_{(0,1)}\int_0^1 (tv)^{\frac{s}{n}}g(tv)\frac{dv}{v}\right).\end{aligned}$$

Applying Minkowski's inequality and using $\alpha_H > 0$, we have

$$\rho_{E(H)}(g) = \rho_H(\chi_{(0,1)}Sg) \lesssim \rho_H\big(\chi_{(0,1)}t^{\frac{s}{n}}g(t)\big).$$

For the reverse we use

$$t^{\frac{s}{n}} g(t) \lesssim Sg(t), \quad g \downarrow,$$

whence

$$\rho_H\big(\chi_{(0,1)} t^{\frac{s}{n}} g(t)\big) \lesssim \rho_H\big(\chi_{(0,1)} Sg(t)\big) = \rho_{E(H)}(g). \qquad \square$$

Example 4.3 Consider the space $H = L^1_*(v)$, where v is slowly varying and $v > 1$. Then $\rho_H \in N_t$ and by Theorem 4.2, we can construct an optimal domain $E(H)$, where

$$\begin{aligned}\rho_{E(H)}(g) = \rho_H(Sg) &= \int_0^1 v(t) Sg(t)\, dt/t \\ &= \int_0^1 v(t) \int_0^t u^{\frac{s}{n}} g(u) \frac{du}{u} \frac{dt}{t} = \int_0^1 w(u) g(u) \frac{du}{u},\end{aligned}$$

and $w(u) = \int_u^1 v(t) \frac{dt}{t}$. Hence $E(H) = \Lambda^1(t^{s/n} w)$ and this couple is optimal. Also $\alpha_E = \beta_E = s/n$.

Example 4.4 Let $H = L^\infty(v)$, where v is slowly varying and $v > 1$. Then $\rho_H \in N_t$. Let

$$\rho_E(g) = \sup v(t) \int_0^t u^{s/n} g^*(u)\, du/u.$$

Then by Theorem 4.2 this is an optimal domain quasi-norm and the couple ρ_E, ρ_H is optimal. In particular, the couple $\Lambda^1(t^{s/n})$, C^0 is optimal.

4.2 Optimal target quasi-norms

Definition 4.5 (construction of an optimal target quasi-norm) For a given domain quasi-norm $\rho_E \in N_d$, we set

$$\rho_{H(E)}(\chi_{(0,1)} g) := \inf\big\{\rho_E(h) : \chi_{(0,1)} g \le Sh, h \downarrow\big\}, \quad g \in M^+. \tag{4.4}$$

Note that

$$\alpha_{H(E)} = \beta_{H(E)} = s/n - \alpha_E.$$

Theorem 4.6 *The target quasi-norm $\rho_{H(E)} \in N_t$, the couple ρ_E, $\rho_{H(E)}$ is admissible, and the target quasi-norm is optimal.*

Proof The couple ρ_E, $\rho_{H(E)}$ is admissible since

$$\rho_{H(E)}(\chi_{(0,1)} Sh) \le \rho_E(h), \quad h \downarrow.$$

Now to prove that $\rho_{H(E)}$ is optimal, we take any admissible couple $\rho_E, \rho_{H_1} \in N_t$. Then

$$\rho_{H_1}(\chi_{(0,1)} Sh) \lesssim \rho_E(h), \quad h \downarrow.$$

Therefore, if $g \le Sh$, $h \downarrow$, then

$$\rho_{H_1}(\chi_{(0,1)}g) \le \rho_{H_1}(\chi_{(0,1)}Sh) \lesssim \rho_E(h),$$

whence, taking the infimum, we get

$$\rho_{H_1}(\chi_{(0,1)}g) \lesssim \rho_{H(E)}(\chi_{(0,1)}g).$$

Hence $\rho_{H(E)}$ is optimal. □

Theorem 4.7 *If $\alpha_E < s/n$, then*

$$\rho_{H(E)}(\chi_{(0,1)}g) \approx \rho_E\bigl(t^{-s/n}g(t)\bigr), \quad g \in M_n.$$

Moreover, the couple ρ_E, $\rho_{H(E)}$ is optimal.

Proof Consider

$$\begin{aligned}\rho_E\bigl(t^{-s/n}Sh(t)\bigr) &= \rho_E\left(t^{-s/n}\int_0^t u^{s/n}h(u)\frac{du}{u}\right)\\ &= \rho_E\left(\int_0^1 v^{s/n}h(tv)\frac{dv}{v}\right), \quad h\downarrow.\end{aligned}$$

Applying Minkowski's inequality and using $\beta_E < s/n$, we have

$$\rho_E\bigl(t^{-s/n}Sh(t)\bigr) \lesssim \rho_E(h), \quad h\downarrow.$$

If $\chi_{(0,1)}g \le Sh$, $g \in M_n$, then

$$\rho_E\bigl(t^{-s/n}g(t)\bigr) \lesssim \rho_E\bigl(t^{s/n}Sh(t)\bigr) \lesssim \rho_E(h)$$

and, taking the infimum, we get

$$\rho_E\bigl(t^{-s/n}g(t)\bigr) \lesssim \rho_{H(E)}(\chi_{(0,1)}g).$$

On the other hand, for $g \in M_n$, let $h(t) = t^{-s/n}g(t)\chi_{(0,1)}(t)$. Then $h\downarrow$ and

$$\begin{aligned}Sh(t) &= \int_0^t u^{s/n}h(u)\frac{du}{u}\\ &= \int_0^t u^{s/n}u^{-s/n}g(u)\frac{du}{u}\\ &\ge g(t).\end{aligned}$$

Therefore

$$\rho_{H(E)}(\chi_{(0,1)}g) \lesssim \rho_E(h) = \rho_E\bigl(t^{-s/n}g(t)\bigr).$$

Now we show that the domain quasi-norm ρ_E is also optimal. We have

$$\begin{aligned}\rho_{E(H(E))}(f^*) &= \rho_{H(E)}(\chi_{(0,1)}Sf^*)\\ &\approx \rho_E(t^{-s/n}Sf^*(t))\\ &= \rho_E\left(t^{-s/n}\int_0^t u^{s/n}f^*(u)\frac{du}{u}\right)\\ &\gtrsim \rho_E(f^*), \quad f \in E.\end{aligned}$$

Therefore

$$\rho_{E(H(E))}(f^*) \gtrsim \rho_E(f^*); \quad f \in E. \qquad \square$$

Example 4.8 Consider the space $E = \Lambda^q(t^\alpha w(t))$, $0 < q \leq \infty$, where w is slowly varying and $s/n > \alpha > 0$. Then $\beta_E = \alpha_E = \alpha$ and $\rho_E \in N_d$. Hence by Theorem 4.7,

$$\rho_{H(E)}(g) \approx \rho_E(t^{-s/n}g(t)) = \left(\int_0^1 (t^{-s/n}w(t)g^*(t))^q\frac{dt}{t}\right)^{1/q},$$

which implies that $H(E) = L_*^q(t^{-s/n}w)$.

Moreover, the couple ρ_E, $\rho_H(E)$ is optimal. In particular, the couple

$$L^{p,\infty}, \mathcal{C}^{s-n/p}, \quad s > n/p, 1 < p < \infty,$$

is optimal.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References
1. Triebel, H: Theory of Function Spaces. Birkhäuser, Basel (1983)
2. Bennett, C, Sharpley, R: Interpolation of Operators. Academic Press, New York (1988)
3. Krein, SG, Petunin, UI, Semenov, EM: Interpolation of Linear Operators. Nauka, Moscow (1978)
4. Nazeer, W, Mehmood, Q, Nizami, AR, Kang, SM: Boyd indices for quasi-normed function spaces with some bounds. J. Inequal. Appl. **2015**, 235 (2015). doi:10.1186/s13660-015-0754-9
5. Karadzhov, GE, Mehmood, Q: Optimal regularity properties of the Riesz potential operator. Commun. Math. Anal. **14**(1), 118-132 (2013)
6. Bashir, Z, Cobos, F, Karadzhov, GE: Optimal embeddings of Calderón spaces into generalized Hölder-Zygmund spaces. Math. Scand. **114**, 120-148 (2014)
7. Gogatishvili, A, Neves, JS, Opic, B: Optimal embeddings of Bessel-potential-type spaces into generalized Hölder spaces involving *k*-modulus of smoothness. Potential Anal. **32**, 201-228 (2010)
8. Ahmed, I, Karadzhov, GE, Nazeer, W: Optimal couples of rearrangement invariant spaces for the fractional maximal operator. C. R. Acad. Bulgare Sci. **64**(9), 1233-1240 (2011)

9. Karadzhov, GE, Nazeer, N: Optimal couples of rearrangment invariant spaces for the Bessel potential. C. R. Acad. Bulgare Sci. **64**(6), 767-774 (2011)
10. Gogatishvili, A, Ovchinnikov, VI: Interpolation orbits and optimal Sobolev's embeddings. J. Funct. Anal. **253**, 1-17 (2007)
11. Hansson, K: Imbedding theorems of Sobolev type in potential theory. Math. Scand. **45**, 77-102 (1979)
12. Capone, C, Fiorenza, A, Karadhov, GE, Nazeer, W: Optimal couples of rearrangement invariant spaces for the Riesz potential operator. Z. Anal. Anwend. **30**, 219-236 (2011)
13. Cianchi, A: Symmetrization and second order Sobolev inequalities. Ann. Mat. Pura Appl. (4) **183**, 45-77 (2004)
14. Kang, SM, Rafiq, A, Nazeer, W, Ahmad, I, Ali, F, Kwun, YC: Optimal couples of rearrangement invariant spaces for the Riesz potential on the bounded domain. J. Inequal. Appl. **2014**, 60 (2014). doi:10.1186/1029-242X-2014-60
15. Vybiral, J: Optimality of Function Spaces for Boundedness of Integral Operators and Sobolev Embeddings. Master Thesis, Faculty of Mathematics and Physics, Charles University, Prague (2002)
16. Berg, J, Löfström, J: Interpolation Spaces. An Introduction. Springer, New York (1976)
17. Martin, J, Milman, M, Pustylnik, E: Sobolev inequalities: symmetrization and self-improvement via truncation. J. Funct. Anal. **252**, 677-695 (2007)
18. Milman, M, Pustylnik, E: On sharp higher order Sobolev embeddings. Commun. Contemp. Math. **6**, 495-511 (2004)
19. Muckenhoupt, B: Hardy's inequality with weights. Stud. Math. **XLIV**, 31-38 (1972)
20. Stein, EM: Singular Integrals and Differentiability Properties of Functions. Princeton University Press, Princeton (1970)
21. Köthe, G: Topol. Lineare Räume. Springer, Berlin (1966)