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Sharp energy criteria of blow-up for the energy-critical Klein-Gordon equation

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Abstract
In this paper, we study the sharp energy criteria of blow-up and global existence for the nonlinear Klein-Gordon equation by the sharp Gagliardo-Nirebergy-Sobolev inequality.

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Keywords: Klein-Gordon equation; energy-critical; energy criteria; blow-up

1 Introduction

We study the Klein-Gordon equation involving the H^1-energy-critical nonlinearity

$$u_{tt} - \triangle u + mu - |u|^{p-1}u = 0, \quad t \geq 0, x \in \mathbb{R}^N, \tag{1.1}$$

where $\triangle = \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2}$ is the Laplacian in $\mathbb{R}^N$ and N is the space dimension. $u = u(t,x)$: $[0,T) \times \mathbb{R}^N \to \mathbb{R}$ and $0 < T \leq +\infty$. $m > 0$ is a positive parameter. $1 < p \leq 2^* - 1$ ($2^* = +\infty$ for $N = 1,2$ and $2^* = \frac{2N}{N-2}$ for $N \geq 3$), and 2^* is the critical Sobolev embedding exponent, which is also called the H^1-energy-critical exponent due to the embedding: $\dot{H}^1 \hookrightarrow L^{2^*}$ (see [1]). When $p = 2^* - 1$, equation (1.1) is called the energy-critical nonlinear Klein-Gordon equation.

We supplement equation (1.1) with the initial data.

$$u(0,x) = u_0, \qquad u_t(0,x) = u_1. \tag{1.2}$$

The local existence of the Cauchy problem (1.1)-(1.2) was developed in many papers (see for instance [2–7]. The existence of blow-up solutions and the blow-up properties are investigated by Ball [8], Payne and Sattinger [9], Keel and Tao [10], Jeanjean and Le Coz [11], Ohta and Todorova [12], *etc.* The solution with small initial data exists globally in all time is obtained in [6, 7]. Pecher [13], Ibrahim *et al.* [3] study the global existence and scattering properties of the solutions to equation (1.1).

Then the question how to distinguish the domains of blow-up and global existence is of particular interest and significance for both mathematicians and physicists. Zhang [14] investigates the sharp threshold of blow-up and global existence for equation (1.1) with H^1-sub-critical nonlinearity (*i.e.* $1 < p < 2^* - 1$) by the variational argument. We remark that the sharp threshold obtained in [14] is not the energy criteria due to the threshold is not fully determined by the $\dot{H}^1$-norm of the corresponding ground state solutions. The H^1-energy-critical case (*i.e.* $p = 2^* - 1$) has not been solved.

Motivated by these problems, we study the sharp energy criteria of blow-up and global existence for equation (1.1) in the H^1-energy-critical case: $p = 2^* - 1$. The main difficulty is the lack of scaling invariance. By injecting the best constant of the critical Sobolev embedding inequality and some new estimates into the energy, we find a convex property of the energy inequality. Then in terms of Kenig and Merle's arguments in [15], we obtain the sharp energy criteria of blow-up and global existence for equation (1.1) by constructing two invariant evolution flows. Define a functional $E((u, u_t))$ in $H^1 \times L^2$ by

$$E\big((u,u_t)\big) := \int \left[\frac{1}{2}\left|\frac{\partial}{\partial t}u(t)\right|^2 + \frac{1}{2}\big|\nabla u(t)\big|^2 + \frac{1}{2}\big|u(t)\big|^2 - \frac{1}{p+1}|u(t)|^{p+1}\right] dx.$$

Then the main theorem is the following.

Theorem 1.1 *Let $m = 1$, $N \geq 2$, and $p = 2^* - 1$. If $(u_0, u_1) \in H^1 \times L^2$ and*

$$E\big((u_0,u_1)\big) < \frac{1}{N}\|\nabla W\|_{L^2}^2, \tag{1.3}$$

then we have the following.

(i) *If*

$$\|\nabla u_0\|_{L^2}^2 + \|u_0\|_{L^2}^2 < \|\nabla W\|_{L^2}^2, \tag{1.4}$$

then the solution $u(t,x)$ of the Cauchy problem (1.1)-(1.2) *is bounded in H^1. Moreover, $u(t,x)$ satisfies*

$$\big\|\nabla u(t)\big\|_{L^2}^2 + \big\|u(t)\big\|_{L^2}^2 < \|\nabla W\|_{L^2}^2. \tag{1.5}$$

(ii) *If*

$$\|\nabla u_0\|_{L^2}^2 + \|u_0\|_{L^2}^2 > \|\nabla W\|_{L^2}^2, \tag{1.6}$$

then the solution $u(t,x)$ of the Cauchy problem (1.1)-(1.2) *blows up in a finite time $0 < T < +\infty$, i.e.* $\lim_{t\to T}\|u(t)\|_{H^1} = +\infty$.

Here W is the solution of the following elliptic equation:

$$\triangle W + |W|^{\frac{4}{N-2}} W = 0, \qquad W \in \dot{H}^1. \tag{1.7}$$

Finally, we extend this method to equation (1.1) in the H^1-sub-critical case: $1 < p < 2^* - 1$. The main difficulty comes from that there is no best constant of the Sobolev inequality. We use the best constant of the Gagliardo-Nirenberg inequality and some new estimates to obtain the energy inequality containing the convex property as in the H^1-energy-critical case. Then we can obtain the sharp energy criteria of blow-up and global existence for equation (1.1) in the H^1-sub-critical case. We should point out that we just consider the case $m = 1$ for simplicity, and the case $m \neq 1$ can be handled by the same argument. The method in this paper may have potential applications for nonlinear wave equations with damping term, forcing term, *etc.*

We conclude this section with several notations. We abbreviate $L^q(\mathbb{R}^N)$, $\|\cdot\|_{L^q(\mathbb{R}^N)}$, $H^1(\mathbb{R}^N)$, $\dot{H}^1(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N}\cdot\,dx$ by L^q, $\|\cdot\|_q$, H^1, $\dot{H}^1$, and $\int\cdot\,dx$. The various positive constants will be simply denoted by C.

2 Preliminaries

In this paper, the space we work in $H^1 := \{v\in L^2 \mid \nabla v\in L^2\}$, is a Hilbert space. The norm of H^1 is denoted by $\|\cdot\|_{H^1}$. Ginibre *et al.* [2], Nakanishi [4] established the local well-posedness of the Cauchy problem (1.1)-(1.2) in energy space.

Proposition 2.1 *Let $1<p\le 2^*-1$ and $(u_0,u_1)\in H^1\times L^2$. There exists an unique solution $u(t,x)$ of the Cauchy problem* (1.1)-(1.2) *on the maximal time $[0,T)$ such that $u(t,x)\in C([0,T);H^1\times L^2)$. Moreover, $u(t,x)\in L_t^q(I;L_x^r)$ for any (q,r) admissible and every compact time interval $I\subset(0,T)$ and the following properties hold: if $0<T<+\infty$ then $\lim_{t\to T}\|u(t,x)\|_{H^1}=+\infty$ or $\sup\|u(t,x)\|_{L_t^q([0,T);L_x^r)}=+\infty$ (blow-up). Furthermore, for all $t\in[0,T)$, $u(t,x)$ satisfies the following conservation law:*

$$E\bigl((u,u_t)\bigr)=E\bigl((u_0,u_1)\bigr). \tag{2.1}$$

Here, (q,r) being admissible means $\frac{1}{q}=N(\frac{1}{2}-\frac{1}{r})-1$ and

$$\|u(t,x)\|_{L_t^q([0,T);L_x^r)}=\left(\int_0^T\left(\int|u(t,x)|^r\,dx\right)^{\frac{q}{r}}dt\right)^{\frac{1}{q}}.$$

Remark 2.2 If $1<p<2^*-1$, then according to the local well-posedness, for the solution $u(t,x)\in C([0,T);H^1\times L^2)$ of the Cauchy problem (1.1)-(1.2), we have the following alternative: either $T=+\infty$ (global existence), or $0<T<+\infty$ and $\lim_{t\to T}\|u(t,x)\|_{H^1}=+\infty$ (blow-up).

At the end of this section, we introduce two important inequalities (see [6, 16–19]).

Lemma 2.3 *Let $N\ge 2$ and*

$$W(x)=\left(\frac{N(N-2)}{N(N-2)+|x|^2}\right)^{\frac{N-2}{2}} \tag{2.2}$$

solve the nonlinear elliptic equation (1.7). *Then the best constant $C_N>0$ of the Sobolev embedding inequality*

$$\|v\|_{2^*}\le C_N\|\nabla v\|_2,\quad v\in\dot{H}^1, \tag{2.3}$$

is given by $C_N=\|\nabla W\|_2^{-\frac{2}{N}}$.

Lemma 2.4 *Let $p=N=3$. If $v\in H^1$, then*

$$\|v\|_4^4\le\frac{4}{\sqrt{3}\|\nabla Q\|_2^2}\|v\|_2\|\nabla v\|_2^3, \tag{2.4}$$

where Q is the ground state solution of

$$-\triangle Q + Q - |Q|^2 Q = 0, \quad Q \in H^1. \tag{2.5}$$

3 Main results

In this paper, the main strategy is that we will use the best constant of the Sobolev embedding inequality and the best constant of the Gagliardo-Nirenberg inequality to explore the convex properties of the energy inequality. Then, by constructing the invariant sets generated by the evolutional system, we can obtain the sharp energy criteria of blow-up and global existence for equation (1.1). Here, the energy criteria mean that the thresholds are fully expressed by the H^1-norm or the $\dot{H}^1$-norm of the corresponding ground state solutions.

At first, we prove Theorem 1.1, which gives the sharp energy criteria of blow-up and global existence for equation (1.1) in the H^1-energy-critical case.

Proof Inject the best constant of the Sobolev inequality (2.3) into the energy functional $E((u,u_t))$. We get

$$\begin{aligned} E\big((u,u_t)\big) &= \frac{1}{2}\int |u_t|^2\,dx + \frac{1}{2}\int \big(|\nabla u|^2 + |u|^2\big)\,dx - \frac{1}{2^*}\int |u|^{2^*}\,dx \\ &\geq \frac{1}{2}\big(\|\nabla u\|_2^2 + \|u\|_2^2\big) - \frac{C_N^{2^*}}{2^*}\big(\|\nabla u\|_2^2 + \|u\|_2^2\big)^{\frac{2^*}{2}}. \end{aligned} \tag{3.1}$$

Define a function $f(y)$ on $[0,+\infty)$ by $f(y) = \frac{1}{2}y^2 - \frac{C_N^{2^*}}{2^*}y^{2^*}$, then

$$f'(y) = y - C_N^{2^*} y^{2^*-1} = y\big(1 - C_N^{2^*} y^{2^*-2}\big). \tag{3.2}$$

Obviously, there are two roots for the equation $f'(y) = 0$: $y_1 = 0$, $y_2 = \|\nabla W\|_2$. Hence, y_1 and y_2 are two minimizers of $f(y)$, and $f(y)$ is increasing on the interval $[y_1, y_2)$ and decreasing on the interval $[y_2, +\infty)$.

Note that $f(y_1) = 0$ and $f(y_2) = \frac{\|\nabla W\|_2^2}{N}$. From (2.1) and (1.3), we get

$$f\Big(\sqrt{\|\nabla u\|_2^2 + \|u\|_2^2}\Big) \leq E\big((u,u_t)\big) = E\big((u_0,u_1)\big) < \frac{\|\nabla W\|_2^2}{N} = f_{\max} = f(y_2). \tag{3.3}$$

Therefore, using the convexity and monotony of $f(y)$ and the conservation of energy, we can construct two invariant evolution flows generated by the evolutional system (1.1)-(1.2), as follows. Let u be the solution of equation (1.1). We have

$$K_1 := \bigg\{ u \in H^1 \setminus \{0\} \,\bigg|\, \|\nabla u\|_2^2 + \|u\|_2^2 < \|\nabla W\|_2^2, 0 < E\big((u,u_t)\big) < \frac{\|\nabla W\|_2^2}{N} \bigg\},$$

$$K_2 := \bigg\{ u \in H^1 \setminus \{0\} \,\bigg|\, \|\nabla u\|_2^2 + \|u\|_2^2 > \|\nabla W\|_2^2, 0 < E\big((u,u_t)\big) < \frac{\|\nabla W\|_2^2}{N} \bigg\}.$$

Indeed, if $u_0 \in K_1$, *i.e.* $\|\nabla u_0\|_2^2 + \|u_0\|_2^2 < \|\nabla W\|_2^2$, then $\sqrt{\|\nabla u_0\|_2^2 + \|u_0\|_2^2} < y_2$. Then, by the bootstrap and continuity argument, we can claim that the corresponding solution $u(t,x)$

is such that, for all $t \in I$,

$$\left\|\nabla u(t)\right\|_2^2 + \left\|u(t)\right\|_2^2 < \|\nabla W\|_2^2. \tag{3.4}$$

This implies that K_1 is invariant. Indeed, if (3.4) is not true for all $t \in I$, then there exists $t_1 \in I$ such that $\|\nabla u(t_1)\|_2^2 + \|u(t_1)\|_2^2 \geq \|\nabla W\|_2^2 = y_2^2$. But from the fact that the corresponding solution $u(t,x)$ of the Cauchy problem (1.1)-(1.2) is continuous with respect to t, there exists $0 < t_0 \leq t_1$ such that $\|\nabla u(t_0)\|_2^2 + \|u(t_0)\|_2^2 = \|\nabla W\|_2^2 = y_2^2$. Inject this fact into (3.3) and take $t = t_0$. We see that

$$f(y_2) = f\left(\sqrt{\left\|\nabla u(t_0)\right\|_2^2 + \left\|u(t_0)\right\|_2^2}\right) \leq E\big((u_0,u_1)\big) < \frac{\|\nabla W\|_2^2}{N} = f(y_2).$$

This is a contradiction because $f(y)$ is increasing on the interval $[0, y_2)$.

If $u_0 \in K_2$, *i.e.* $\|\nabla u_0\|_2^2 + \|u_0\|_2^2 > \|\nabla W\|_2^2$, then $\sqrt{\|\nabla u_0\|_2^2 + \|u_0\|_2^2} > y_2$. Since $f(y)$ is continuous on $[0,+\infty)$ and decreasing on $[y_2,+\infty)$, we deduce that for all $t \in I$ (maximal existence interval)

$$\sqrt{\left\|\nabla u(t,x)\right\|_2^2 + \left\|u(t,x)\right\|_2^2} > y_2, \tag{3.5}$$

which implies that K_2 is invariant. Here, we use the same argument of the proof of (3.4) and we omit the detailed proof in this paper.

Now, we return to the proof of Theorem 1.1. Equations (1.3) and (1.4) imply $u_0 \in K_1$. Applying the invariant of K_1, we can obtain (1.5), as well as (3.4). Then the solution $u(t,x)$ of the Cauchy problem (1.1)-(1.2) is bounded in H^1. This completes part (i) of the proof of Theorem 1.1. For (ii), we see that (1.3) and (1.6) imply $u_0 \in K_2$. Applying the invariant of K_2, (3.5) is true. From (2.1) and (1.3), we get

$$\|u\|_{2^*}^{2^*} > -\frac{2^*}{N}\|\nabla W\|_2^2 + \frac{2^*}{2}\|u_t\|_2^2 + \frac{2^*}{2}\|\nabla u\|_2^2 + \frac{2^*}{2}\|u\|_2^2. \tag{3.6}$$

Letting $J(t) := \int |u(t,x)|^2\,dx$, by some basic computations, we see that

$$J'(t) = 2\int uu_t\,dx \tag{3.7}$$

and

$$J''(t) = 2\int \left(|u_t|^2 + |u|^{2^*} - |\nabla u|^2 - |u|^2\right) dx. \tag{3.8}$$

It follows from (3.6)-(3.8) that

$$\begin{aligned} J''(t) &\geq \frac{4(N-1)}{N-2}\|u_t\|_2^2 - \frac{4}{N-2}\|\nabla W\|_2^2 + \frac{4}{N-2}\|\nabla u\|_2^2 + \frac{4}{N-2}\|u\|_2^2 \\ &> \frac{4(N-1)}{N-2}\|u_t\|_2^2. \end{aligned} \tag{3.9}$$

Applying the Hölder inequality for (3.7), we get $J'(t)^2 \leq 4\|u\|_2^2\|u_t\|_2^2$. Moreover, multiplying (3.9) with $J(t)$, we get

$$J(t)J''(t) > \frac{4(N-1)}{N-2}\|u_t\|_2^2\|u\|_2^2 > \frac{N-1}{N-2}J'(t)^2. \tag{3.10}$$

It follows from (3.9) that $J''(t)$ is positive and has a lower bound. Hence, there exists a $t_0 > 0$ such that $J'(t) > 0$ for $t > t_0$. From (3.10), we get

$$\frac{J''(t)}{J'(t)} > \frac{N-1}{N-2}\frac{J'(t)}{J(t)} \quad \text{for } t > t_0,$$

which implies that

$$J'(t) > KJ(t)^{\frac{N-1}{N-2}}, \tag{3.11}$$

where $K > 0$. Since $\frac{N-1}{N-2} > 1$, for $t > t_0$, by integrating (3.11) from t_0 to t, we deduce that

$$J(t) > \left(\frac{1}{J(t_0)^{\frac{-1}{N-2}} - \frac{K(t-t_0)}{N-2}}\right)^{N-2}. \tag{3.12}$$

Then there exists a finite time $0 < T < +\infty$ such that $\lim_{t\to T} J(t) = +\infty$, and $\lim_{t\to T}\|u(t)\|_{H^1} = +\infty$. That is, the solution $u(t,x)$ of the Cauchy problem (1.1)-(1.2) blows up in the finite time $0 < T < +\infty$. □

Next, we want to extend the argument of Theorem 1.1 to equation (1.1) with the H^1-sub-critical nonlinearity: $1 < p < 2^* - 1$. Here, we consider the special case $p = N = 3$. There are two main difficulties: One is that if we directly use the methods of Kenig and Merle for the H^1-energy-critical wave and Schrödinger equations [1, 15], the best constant of the Sobolev inequality $\|u\|_4 \leq C(\|\nabla u\|_2 + \|u\|_2)$ is not determined; the other is that if we directly use Holmer and Roudenko's arguments [20] for the H^1-sub-critical nonlinear Schrödinger equation, the L^2-norm of the solutions is not conserved. Our main strategy is to add some new estimates to the sharp Gagliardo-Nirenberg inequality. Then we can balance the L^2-norm of the solutions, and control the energy by the $\|\nabla u(t)\|_2$. Finally, applying the convexity of the energy inequality, we establish two types of invariant evolution flows, and we obtain the sharp energy criterion of blow-up and global existence of the solutions to the Cauchy problem (1.1)-(1.2), as follows.

Theorem 3.1 *Let $m = 1$, $p = N = 3$, and Q be the ground state solution of* (2.5). *Assume that $(u_0, u_1) \in H^1 \times L^2$ and*

$$E\big((u_0,u_1)\big) < \frac{1}{4}\|\nabla Q\|_2^2. \tag{3.13}$$

Then the following hold.

(i) *If*

$$\|\nabla u_0\|_2 < \|\nabla Q\|_2, \tag{3.14}$$

then the solution $u(t,x)$ *of the Cauchy problem* (1.1)-(1.2) *exists globally. Moreover,* $u(t,x)$ *satisfies, for all* $t \in I$,

$$\|\nabla u(t)\|_2 < \|\nabla Q\|_2. \tag{3.15}$$

(ii) *If*

$$\|\nabla u_0\|_2 > \|\nabla Q\|_2, \tag{3.16}$$

then the solution $u(t,x)$ *of the Cauchy problem* (1.1)-(1.2) *blows up in a finite time* $0 < T < +\infty$.

Proof From the sharp Gagliardo-Nirenberg inequality (2.4) and the Young inequality, we deduce that

$$\begin{aligned} E\bigl((u,u_t)\bigr) &= \frac{1}{2}\int |u_t|^2\,dx + \frac{1}{2}\int \bigl(|\nabla u|^2 + |u|^2\bigr)\,dx - \frac{1}{4}\int |u|^4\,dx \\ &\geq \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|u\|_2^2 - \frac{1}{\sqrt{3}\|\nabla Q\|_2^2}\|u\|_2\|\nabla u\|_2^3 \\ &\geq \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|u\|_2^2 - \frac{1}{2}\|u\|_2^2 - \frac{1}{6\|\nabla Q\|_2^4}\|\nabla u\|_2^6 \\ &\geq \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{6\|\nabla Q\|_2^4}\|\nabla u\|_2^6. \end{aligned} \tag{3.17}$$

Define a function $f(y)$ on $[0,+\infty)$ by $f(y) = \frac{1}{2}y^2 - \frac{1}{6\|\nabla Q\|_2^4}y^6$. We see that

$$f'(y) = y - \frac{1}{\|\nabla Q\|_2^4}y^5 = y\left(1 - \frac{1}{\|\nabla Q\|_2^4}y^4\right). \tag{3.18}$$

It is obvious that there are two roots for the equation $f'(y) = 0$: $y_1 = 0$, $y_2 = \|\nabla Q\|_2$. Hence, y_1 and y_2 are two minimizers of $f(y)$, and $f(y)$ is increasing on the interval $[y_1, y_2)$ and decreasing on the interval $[y_2, +\infty)$. Note that $f(y_1) = 0$ and $f(y_2) = \frac{1}{3}\|\nabla Q\|_2^2$. By the conservation of energy and (1.3), we get

$$f\bigl(\|\nabla u\|_2\bigr) \leq E(u) = E\bigl((u_0,u_1)\bigr) < \frac{1}{3}\|\nabla Q\|_2^2 = f(y_2). \tag{3.19}$$

Therefore, using the convexity and monotony of $f(y)$ and the conservation of energy, one obtains two invariant evolution flows generated by the Cauchy problem (1.1)-(1.2), as follows. Let u be the solution of equation (1.1).

$$K_1 := \left\{ u \in H^1 \setminus \{0\} \,\middle|\, \|\nabla u\|_2 < \|\nabla Q\|_2, 0 < E\bigl((u,u_t)\bigr) < \frac{\|\nabla Q\|_2^2}{4} \right\},$$

$$K_2 := \left\{ u \in H^1 \setminus \{0\} \,\middle|\, \|\nabla u\|_2 > \|\nabla Q\|_2, 0 < E\bigl((u,u_t)\bigr) < \frac{\|\nabla Q\|_2^2}{4} \right\}.$$

Indeed, if the initial data is such that $E((u_0,u_1)) < \frac{\|\nabla Q\|_2^2}{4}$, then from (2.1), the corresponding solution satisfies $E((u,u_t)) < \frac{\|\nabla Q\|_2^2}{4}$. If $u_0 \in K_1$, *i.e.* $\|\nabla u_0\|_2^2 < \|\nabla Q\|_2^2$, then $\|\nabla u_0\|_2 < y_2$.

Then, by the bootstrap and continuity argument, we claim that the solution $u(t,x)$ is such that, for all $t \in I$,

$$\|\nabla u(t)\|_2^2 < \|\nabla Q\|_2^2. \tag{3.20}$$

This implies that K_1 is invariant. Indeed, if (3.20) is not true for all $t \in I$, then there exists $t_1 \in I$ such that $\|\nabla u(t_1)\|_2^2 \geq \|\nabla Q\|_2^2 = y_2^2$. But from the fact that the corresponding solution $u(t,x)$ of the Cauchy problem (1.1)-(1.2) is continuous with respect to t, there exists $0 < t_0 \leq t_1$ such that $\|\nabla u(t_0)\|_2^2 = \|\nabla Q\|_2^2 = y_2^2$. Inject this fact into (3.19) and take $t = t_0$. We see that

$$f(y_2) = f\left(\|\nabla u(t_0)\|_2\right) \leq E\left((u_0, u_1)\right) < \frac{\|\nabla Q\|_2^2}{4} < f(y_2).$$

This is a contradiction because $f(y)$ is increasing on the interval $[0, y_2)$. Moreover, by the same argument, we can give the proof of the invariant of K_2 (see also the proof of Theorem 1.1. Here, we omit the detailed proof).

Now, we return to the proof the Theorem 3.1. From (3.13) and (3.14), we get $u_0 \in K_1$. Applying the invariant of K_1 and the local well-posedness, we deduce that (3.15) is true and the solution $u(t,x)$ of the Cauchy problem (1.1)-(1.2) exists globally by the local well-posedness theory (see Remark 2.2). This completes part (i) of the proof. Next, we give the proof of (ii). By (3.13) and (3.16), we see that $u_0 \in K_2$. Applying the invariant of K_2, we have

$$\|\nabla u(t)\|_2^2 > \|\nabla Q\|_2^2 \quad \text{for all } t \in I. \tag{3.21}$$

From (2.1) and (3.13), we get

$$2\|u\|_4^4 > -2\|\nabla Q\|_2^2 + 4\|u_t\|_2^2 + 4\|\nabla u\|_2^2 + 4\|u\|_2^2. \tag{3.22}$$

Let $J(t) := \int |u(t,x)|^2\,dx$. By some basic computations, we deduce that $J'(t) = 2\int uu_t\,dx$ and

$$J''(t) = 2\int \left(|u_t|^2 + |u|^4 - |\nabla u|^2 - |u|^2\right) dx. \tag{3.23}$$

It follows from (3.21)-(3.23) that

$$J''(t) \geq 6\|u_t\|_2^2 - 2\|\nabla Q\|_2^2 + 2\|\nabla u\|_2^2 + 2\|u\|_2^2 > 6\|u_t\|_2^2. \tag{3.24}$$

Multiplying (3.24) with $J(t)$ and injecting $J'(t)^2 \leq 4\|u\|_2^2\|u_t\|_2^2$, we get

$$J(t)J''(t) > 6\|u_t\|_2^2\|u\|_2^2 > \frac{3}{2}J'(t)^2. \tag{3.25}$$

From (3.24) and (3.25), there exists a $t_0 > 0$ such that for $t > t_0$

$$J'(t) > KJ(t)^{\frac{3}{2}}, \tag{3.26}$$

where $K > 0$. Since $\frac{3}{2} > 1$, by the same argument as the proof of Theorem 1.1, we can deduce that the solution $u(t,x)$ of the Cauchy problem (1.1)-(1.2) blows up in the finite time $0 < T < +\infty$. □

Remark 3.2 Theorem 3.1 can be extended to the general H^1-sub-critical case by the same argument. For the H^1-sub-critical case, we extend Holmer and Roudenko's arguments [20] for the nonlinear Schrödinger equations to the nonlinear Klein-Gordon equations, which is one of the novelties in this paper. This argument has potential applications in the nonlinear wave equations with damping term, forcing term, *etc.*

It is well known that the scaling invariance brings about a lot of algebraic or geometric structures and simplifications, which are significant to analyze the nonlinear waves (see [1, 15, 17]). The nonlinear Klein-Gordon equation is lack of scaling invariance. However, one of the interesting features resulting from the breakdown of the scaling is that the sharp energy criterion of the blow-up solutions is not given by the ground state of the original nonlinear Klein-Gordon equation but that of a modified equation.

Competing interests

The author declares to have no competing interests.

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