Guo et al. *Journal of Inequalities and Applications* (2015) 2015:367
DOI 10.1186/s13660-015-0890-2

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH | Open Access

Strong converse inequality for left Bernstein-Durrmeyer quasi-interpolants

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Abstract

For the left Bernstein-Durrmeyer quasi-interpolants $M_n^{(2r-1)}f$, we prove that, for some l,

$$\omega_\varphi^{2r}\left(f,\frac{1}{\sqrt{n}}\right)_p \le C\left(\left\|M_n^{(2r-1)}f-f\right\|_p+\left\|M_{ln}^{(2r-1)}f-f\right\|_p\right) \quad (1<p\le\infty).$$

This is a strong converse inequality of type B.

MSC: 41A27; 41A40

Keywords: strong converse inequality; modulus of smoothness; K-functional; Bernstein-Durrmeyer left quasi-interpolants

1 Introduction

For $f\in L_p[0,1]$ the Bernstein-Durrmeyer operators are given by

$$M_n(f,x)=(n+1)\sum_{k=0}^{n}p_{n,k}(x)\int_0^1 p_{n,k}(t)f(t)\,dt,$$

where $p_{n,k}(x)=\binom{n}{k}x^k(1-x)^{n-k}$ (*cf.* [1, 2] and [3, 4] for more integral type operators). The rate of convergence and the inverse theorem for $M_n(f,x)$ and their combination have been investigated in [5]. Recently Sablonnière (*cf.* [6, 7]) introduced a family of operators, so-called quasi-interpolants. Many quasi-interpolants of different operators were studied (*e.g.* [8–12]).

In the following Π_j denotes the space of algebraic polynomials of degree at most j. Because M_n is an automorphism of Π_n, M_n and its inverse M_n^{-1} can be expressed as linear differential operators with polynomial coefficients in the forms $M_n=\sum_{j=0}^{n}\beta_j^n(x)D^j$ and $M_n^{-1}=\sum_{j=0}^{n}\alpha_j^n(x)D^j$, where $D^0=id$, $D=\frac{d}{dx}$. The polynomials $\alpha_j^n(x)\in\Pi_j$ are expressed explicitly in terms of shifted Jacobi polynomials (*cf.* [6, 9, 12]) as

$$\alpha_j^n(x)=\sum_{s=0}^{[j/2]}(-1)^sX^sJ_{j-2s}^{(s,s)}(x)/s!(n)_{j-s},$$

where $X=x(1-x)$, $(n)_j=n(n-1)\cdots(n+j-1)$, and

$$J_{j-2s}^{(s,s)}(x)=\sum_{i=0}^{j-2s}\binom{j-s}{i}\binom{j-s}{j-2s-i}(x-1)^{j-2s-i}x^i.$$

Now we give the definition of left Bernstein-Durrmeyer quasi-interpolants (*cf.* [6, 9, 12]):

$$M_n^{(r)}(f,x) = \sum_{j=0}^{r} \alpha_j^n(x) D^j M_n(f,x) =: \sum_{j=0}^{r} \alpha_j^n(x) M_{n,j}(f,x), \tag{1.1}$$

where $M_{n,j} = D^j M_n$. It is well known that $\alpha_0^n(x) = 1$, $M_n^{(r)}$ is exact on Π_r, *i.e.* $M_n^{(r)}p = p$ for all $p \in \Pi_r$, $0 \le r \le n$.

For $M_n^{(2r-1)}(f,x)$ the global approximation equivalent theorem has been obtained in [9] as follows.

Theorem [9] *Let* $f \in L_p[0,1]$, $1 < p \le \infty$, $\varphi(x) = \sqrt{x(1-x)}$, $n \ge 4r$, $r \in N$, $0 < \alpha < r$, *then*

$$\left\| M_n^{(2r-1)} f - f \right\|_p = O\left(n^{-\alpha}\right) \quad \Leftrightarrow \quad \omega_\varphi^{2r}(f,t)_p = O\left(t^{2\alpha}\right).$$

Here

$$\omega_\varphi^s(f,t)_p = \sup_{0<h\le t} \left\| \triangle_{h\varphi}^s f(x) \right\|_p,$$

$$\triangle_{h\varphi}^s f(x) = \sum_{k=0}^{s} (-1)^k \binom{s}{k} f(x + sh/2 - kh).$$

This is Ditzian-Totik modulus of smoothness, it is equivalent to K-functional

$$K_\varphi^s\left(f,t^s\right)_p = \inf_{g\in W^s(\varphi)} \left\{ \|f-g\|_p + t^s \left\| \varphi^s g^{(s)} \right\|_p \right\}, \tag{1.2}$$

where $W^s(\varphi) =: \{g \in L_p[0,1], g^{(s-1)} \in A.C._{[0,1]}, \|\varphi^s g^{(s)}\|_p < \infty\}$. It was proved that $\omega_\varphi^s(f,t)_p \sim K_\varphi^s(f,t^s)_p$, *i.e.* there exists $A > 0$ such that (*cf.* [13])

$$A^{-1}\omega_\varphi^s(f,t)_p \le K_\varphi^s\left(f,t^s\right)_p \le A\omega_\varphi^s(f,t)_p. \tag{1.3}$$

The strong converse inequality is an important problem of operator approximation theory. The strong converse inequalities for various operators have been investigated in subsequent papers (*e.g.* [14, 15]). In most of these results the second order moduli of smoothness $\omega_\varphi^2(f,t)_p$ were used. The intention of this paper is to prove a strong converse inequality of type B for the quasi-interpolants $M_n^{(2r-1)}f$ by using high order modulus. To this end we have to prove several key lemmas presented in Section 2. Application of these lemmas enables us to prove our main result in Section 3.

Throughout this paper C denotes a positive constant independent of n and x not necessarily the same at each occurrence.

2 Lemmas

In this section we give some lemmas.

Lemma 2.1 (*cf.* [9, 10]) *For* $j \ge 1$, $r \in N$, *we have*

$$\left|\alpha_j^n(x)\right| \le Cn^{-\frac{j}{2}} \delta_n^j(x), \qquad \left|D^r \alpha_j^n(x)\right| \le Cn^{\frac{-j+r}{2}} \delta_n^{j-r}(x), \tag{2.1}$$

where $\delta_n(x) = \varphi(x) + \frac{1}{\sqrt{n}} \sim \max\{\varphi(x), \frac{1}{\sqrt{n}}\}$.

Lemma 2.2 *Let* $E_n = [\frac{1}{n}, 1-\frac{1}{n}]$, $\varphi(x) = \sqrt{x(1-x)}$, $f \in W^{2r+1}(\varphi)$ *and* $R_{2r+1}(f,t,x) = \frac{1}{(2r)!}\int_x^t (t-u)^{2r} f^{(2r+1)}(u)\,du$, *then we have, for* $1 < p \le \infty$,

$$\bigl\|M_n^{(2r-1)}\bigl(R_{2r+1}(f,\cdot,x),x\bigr)\bigr\|_p^{E_n} \le Cn^{-r-\frac{1}{2}}\bigl\|\varphi^{2r+1}f^{(2r+1)}\bigr\|_p. \tag{2.2}$$

Proof Let $\psi(u) = \varphi^{2r+1}(u)f^{(2r+1)}(u)$, $G(x) = M(\psi,x) = \sup_t |\frac{1}{t-x}\int_x^t |\psi(u)|\,du|$, *i.e.* $G(x)$ is the maximal function of ψ. Noting that (*cf.* [7])

$$\bigl|D^j p_{n,k}(x)\bigr| \le C\sum_{i=0}^{j}\left(\frac{\sqrt{n}}{\varphi(x)}\right)^{j+i}\left|\frac{k}{n}-x\right|^i p_{n,k}(x), \quad x \in E_n,$$

we have, for $x \in E_n$,

$$\bigl|D^j M_n(f,x)\bigr| = \bigl|M_{n,j}(f,x)\bigr| \le C\sum_{i=0}^{j}\left(\frac{\sqrt{n}}{\varphi(x)}\right)^{j+i}\sum_{k=0}^{n}p_{n,k}(x)\left|\frac{k}{n}-x\right|^i\bigl|a_k(n)\bigr|,$$

where $a_k(n) = (n+1)\int_0^1 p_{n,k}(t)f(t)\,dt$. So, for $x \in E_n$,

$$\begin{aligned}
&\bigl|M_n^{(2r-1)}\bigl(R_{2r+1}(f,\cdot,x),x\bigr)\bigr| \\
&\quad\le \sum_{j=0}^{2r-1}\bigl|\alpha_j^n(x)\bigr|\bigl|M_{n,j}\bigl(R_{2r+1}(f,\cdot,x),x\bigr)\bigr| \\
&\quad\le C\sum_{j=0}^{2r-1}\bigl|\alpha_j^n(x)\bigr|\sum_{i=0}^{j}\left(\frac{\sqrt{n}}{\varphi(x)}\right)^{j+i}\sum_{k=0}^{n}p_{n,k}(x)\left|\frac{k}{n}-x\right|^i\bigl|\overline{a}_k(n)\bigr| \\
&\quad=: C\sum_{j=0}^{2r-1}I_j,
\end{aligned}$$

where $\overline{a}_k(n) = \frac{n+1}{(2r)!}\int_0^1 p_{n,k}(t)\int_x^t (t-u)^{2r}f^{(2r+1)}(u)\,du\,dt$. Using (9.6.1) in [13], we have

$$\begin{aligned}
\bigl|\overline{a}_k(n)\bigr| &= \frac{n+1}{(2r)!}\int_0^1 p_{n,k}(t)\left|\int_x^t \frac{(t-u)^{2r}}{\varphi^{2r+1}(u)}\varphi^{2r+1}(u)f^{(2r+1)}(u)\,du\right|dt \\
&\le \frac{n+1}{(2r)!}\varphi^{-(2r+1)}(x)G(x)\int_0^1 p_{n,k}(t)|t-x|^{2r+1}\,dt.
\end{aligned}$$

Hence by Hölder's inequality, we have, for $x \in E_n$,

$$\begin{aligned}
\|I_j\|_p^{E_n} &\le \bigl\|G(x)\bigr\|_p\left\|\alpha_j^n(x)\varphi^{-(2r+1)}(x)\sum_{i=0}^{j}\left(\frac{\sqrt{n}}{\varphi(x)}\right)^{j+i}\sum_{k=0}^{n}p_{n,k}(x)\left|\frac{k}{n}-x\right|^i\right. \\
&\qquad\left.\times (n+1)\int_0^1 p_{n,k}(t)|t-x|^{2r+1}\,dt\right\|_p^{E_n} \\
&\le \bigl\|G(x)\bigr\|_p\left\|\alpha_j^n(x)\varphi^{-(2r+1)}(x)\sum_{i=0}^{j}\left(\frac{\sqrt{n}}{\varphi(x)}\right)^{j+i}\left(\sum_{k=0}^{n}p_{n,k}(x)\left(\frac{k}{n}-x\right)^{2i}\right)^{1/2}\right. \\
&\qquad\left.\times\left(\sum_{k=0}^{n}p_{n,k}(x)(n+1)\int_0^1 p_{n,k}(t)(t-x)^{4r+2}\,dt\right)^{1/2}\right\|_p^{E_n}.
\end{aligned}$$

From (9.4.14) in [13] and (6.4) in [5], we have, for $x \in E_n$,

$$\left(\sum_{k=0}^{n} p_{n,k}(x)\left(\frac{k}{n}-x\right)^{2i}\right)^{1/2} \leq C\frac{\varphi^{i}(x)}{n^{\frac{i}{2}}},$$

$$\left(\sum_{k=0}^{n} p_{n,k}(x)(n+1)\int_0^1 p_{n,k}(t)(t-x)^{4r+2}\,dt\right)^{1/2} \leq C\frac{\varphi^{2r+1}(x)}{n^{r+\frac{1}{2}}}.$$

Together with (2.1) and the fact that

$$\|G(x)\|_p \leq C_p \|\varphi^{2r+1}f^{(2r+1)}\|_p,$$

we obtain (2.2). □

Lemma 2.3 *For $n \geq 2r$, we have*

$$\begin{aligned} &M_n^{(2r-1)}\left((t-x)^{2r},x\right) \\ &\quad = (-1)^{r+1}n^{-r}\varphi^{2r}(x)\frac{(2r)!}{2^r(r!)} + \varphi^{2r}(x)o\left(\frac{1}{n^r}\right) \\ &\qquad + (2r)!\left(b_{2r}^n\frac{1}{n^{2r}} + b_{2r-1}^n\frac{\varphi^2(x)}{n^{2r-1}} + \cdots + b_{r+1}^n\frac{\varphi^{2r-2}(x)}{n^{r+1}}\right)\left(1+O\left(\frac{1}{n}\right)\right), \end{aligned} \tag{2.3}$$

where b_j^n are uniformly bounded in n and independent of x.

Proof First we note $M_n^{(2r)}p = p$ for all $p \in \Pi_{2r}$, so we have

$$M_n^{(2r)}\left((t-x)^{2r},x\right) = 0,$$

then

$$M_n^{(2r)}\left((t-x)^{2r},x\right) - M_n^{(2r-1)}\left((t-x)^{2r},x\right) = \alpha_{2r}^n(x)M_{n,2r}\left((t-x)^{2r},x\right).$$

Therefore we have

$$M_n^{(2r-1)}\left((t-x)^{2r},x\right) = -\alpha_{2r}^n(x)M_{n,2r}\left((t-x)^{2r},x\right). \tag{2.4}$$

Using (*cf.* [5])

$$M_{n,2r}(f,x) = \frac{(n+1)!n!}{(n-2r)!(n+2r)!}\sum_{k=0}^{n-2r} p_{n-2r,k}(x)\int_0^1 p_{n+2r,k+2r}(t)f^{(2r)}(t)\,dt,$$

we have

$$M_{n,2r}\left((t-x)^{2r},x\right) = \frac{(n+1)!n!(2r)!}{(n-2r)!(n+2r)!} = (2r)!\left(1+O\left(\frac{1}{n}\right)\right).$$

Therefore

$$M_n^{(2r-1)}\left((t-x)^{2r},x\right) = -(2r)!\alpha_{2r}^n(x)\left(1+O\left(\frac{1}{n}\right)\right). \tag{2.5}$$

Also we have (*cf.* (3.11) in [10])

$$\alpha_{2r}^n(x) = b_{2r}^n \frac{1}{n^{2r}} + b_{2r-1}^n \frac{\varphi^2(x)}{n^{2r-1}} + \cdots + b_r^n \frac{\varphi^{2r}(x)}{n^r}, \tag{2.6}$$

where b_j^n are uniformly bounded in n and independent of x.

By Theorem 4.2 and Table 2 in [7] we know that $\lim_n n^r \alpha_{2r}^n(x)$ exists and

$$\lim_n n^r \alpha_{2r}^n(x) = \frac{(-1)^r \varphi^{2r}(x)}{2^r(r!)}. \tag{2.7}$$

With this relation and (2.6), we get the representation of the coefficient b_r^n in (2.6), *i.e.*

$$\lim_n b_r^n = \frac{(-1)^r}{2^r(r!)}. \tag{2.8}$$

From (2.5)-(2.8) we get (2.3). □

Lemma 2.4 *For $f \in W^{2r}(\varphi)$, $1 < p \le \infty$, we have*

$$\left\| \varphi^{2r+1} D^{2r+1}\left(M_n^{(2r-1)} f\right) \right\|_p \le C\sqrt{n} \left\| \varphi^{2r} f^{(2r)} \right\|_p. \tag{2.9}$$

Proof By (2.6) in [5] one has, for $x \in [0,1]$, $1 \le p \le \infty$, $r, s \in N_0 = N \cup \{0\}$,

$$\left\| \delta_n^s(x)\varphi^{2r}(x) D^{2r+s} M_n(f,x) \right\|_p \le C n^{\frac{s}{2}} \left\| \varphi^{2r} f^{(2r)} \right\|_p.$$

So we have

$$\left\| \varphi^{2r+m}(x) D^{2r+m} M_n(f,x) \right\|_p^{E_n} \le C n^{\frac{m}{2}} \left\| \varphi^{2r} f^{(2r)} \right\|_p, \quad m \ge 0. \tag{2.10}$$

Using (2.1) and (2.10), we have

$$\begin{aligned} &\left\| \varphi^{2r+1}(x) D^{2r+1} \sum_{j=0}^{2r-1} \alpha_j^n(x) M_{n,j}(f,x) \right\|_p^{E_n} \\ &\quad \le \sum_{j=0}^{2r-1} \sum_{i=0}^{j} \left\| \varphi^{2r+1}(x) \binom{2r+1}{i} \left(D^i \alpha_j^n(x)\right) M_{n,2r+1+j-i}(f,x) \right\|_p^{E_n} \\ &\quad \le C \sum_{j=0}^{2r-1} \sum_{i=0}^{j} \left\| \varphi^{2r+1}(x) n^{\frac{-j+i}{2}} \varphi^{j-i}(x) M_{n,2r+1+j-i}(f,x) \right\|_p^{E_n} \\ &\quad \le C\sqrt{n} \left\| \varphi^{2r} f^{(2r)} \right\|_p. \end{aligned} \tag{2.11}$$

Since $(\varphi^{2r+1}(x) D^{2r+1} M_n^{(2r-1)}(f,x))^2$ are polynomials, we can use a result of the weight polynomial approximation [13], Theorem 8.4.8, translating the interval $[-1,1]$ to $[0,1]$ to obtain the estimate

$$\left\| \left(\varphi^{2r+1} D^{2r+1}\left(M_n^{(2r-1)} f\right)\right)^2 \right\|_p^{[0,1]} \le M \left\| \left(\varphi^{2r+1} D^{2r+1}\left(M_n^{(2r-1)} f\right)\right)^2 \right\|_p^{E_n}, \tag{2.12}$$

where M does not depend on n. From (2.11) and (2.12) we obtain (2.9). □

Lemma 2.5 ((4.2) in [9]) *For $f \in L_p[0,1]$, $1 \le p \le \infty$, we have*

$$\left\|\varphi^{2r}D^{2r}\left(M_n^{(2r-1)}f\right)\right\|_p \le Cn^r\|f\|_p. \tag{2.13}$$

Lemma 2.6 *For $f \in W^{2r+1}(\varphi)$, we have*

$$\begin{aligned} &M_n^{(2r-1)}(f,x) - f(x) - \frac{(-1)^{r+1}\varphi^{2r}(x)}{2^r n^r (r!)} f^{(2r)}(x) \\ &\quad = o\left(\frac{1}{n^r}\right)\varphi^{2r}(x) f^{(2r)}(x) + \left(b_{2r}^n \frac{1}{n^{2r}} + b_{2r-1}^n \frac{\varphi^2(x)}{n^{2r-1}} + \cdots + b_{r+1}^n \frac{\varphi^{2r-2}(x)}{n^{r+1}}\right) \\ &\qquad \times f^{(2r)}(x)\left(1 + O\left(\frac{1}{n}\right)\right) + M_n^{(2r-1)}\left(R_{2r+1}(f,\cdot,x),x\right), \end{aligned} \tag{2.14}$$

where $\{b_{2r-1}^n, \ldots, b_{r+1}^n\}$ are uniformly bounded in n and independent of x.

Proof By Taylor's formula we expand f as follows:

$$f(t) = f(x) + (t-x)f'(x) + \cdots + \frac{(t-x)^{2r}}{(2r)!} f^{(2r)}(x) + R_{2r+1}(f,t,x),$$

where $R_{2r+1}(f,t,x) = \frac{1}{(2r)!}\int_x^t (t-u)^{2r} f^{(2r+1)}(u)\,du$.

Noting $M_n^{(2r-1)}p = p$ for all $p \in \Pi_{2r-1}$ (*cf.* [7]), we obtain

$$M_n^{(2r-1)}(f,x) - f(x) = M_n^{(2r-1)}\left(\frac{(t-x)^{2r}}{(2r)!}, x\right) f^{(2r)}(x) + M_n^{(2r-1)}\left(R_{2r+1}(f,\cdot,x),x\right).$$

Using Lemma 2.3 we obtain (2.14). □

3 Main result

Using the lemmas in Section 2 we are able to prove the following main result, which is the strong converse inequality for left Bernstein-Durrmeyer quasi-interpolants of type B.

Theorem 3.1 *Let $f \in L_p[0,1]$, $1 < p \le \infty$, $\varphi(x) = \sqrt{x(1-x)}$, $n \ge 4r$, $r \in N$, then there exists a constant k such that, for $l \ge kn$,*

$$\omega_\varphi^{2r}\left(f, \frac{1}{\sqrt{n}}\right)_p \le C\left(\frac{l}{n}\right)^r \left(\left\|M_n^{(2r-1)}f - f\right\|_p + \left\|M_l^{(2r-1)}f - f\right\|_p\right).$$

Proof To prove our result at first we estimate K-functional $K_\varphi^{2r}(f,n^{-r})_p$. We choose the function

$$g = K_n^{(2r-1)}\left(K_n^{(2r-1)}f\right) =: K_n^{2(2r-1)}f.$$

By the definition of the K-functional and the boundedness of $K_n^{(2r-1)}$ (*cf.* [7], p.243, (3.2) in [9]), we have

$$\begin{aligned} K_\varphi^{2r}\left(f,n^{-r}\right)_p &\le \|f-g\|_p + n^{-r}\left\|\varphi^{2r}g^{(2r)}\right\|_p \\ &= \left\|f - M_n^{2(2r-1)}f\right\|_p + n^{-r}\left\|\varphi^{2r}D^{2r}\left(M_n^{2(2r-1)}f\right)\right\|_p \end{aligned}$$

$$\begin{aligned}
&\leq \|f - M_n^{(2r-1)}f\|_p + \|M_n^{(2r-1)}f - M_n^{2(2r-1)}f\|_p \\
&\quad + n^{-r}\|\varphi^{2r}D^{2r}(M_n^{2(2r-1)}f)\|_p \\
&\leq C\|f - M_n^{(2r-1)}f\|_p + n^{-r}\|\varphi^{2r}D^{2r}(M_n^{2(2r-1)}f)\|_p.
\end{aligned}$$

Therefore we only need to estimate $\varphi^{2r}g^{(2r)} = \varphi^{2r}D^{2r}(M_n^{2(2r-1)}f)$. We recall Lemma 2.6 with $g = M_n^{2(2r-1)}f$ in place of f and l in place of n to obtain

$$\begin{aligned}
&M_l^{(2r-1)}(g,x) - g(x) - \frac{(-1)^{r+1}\varphi^{2r}(x)}{2^r l^r (r!)}g^{(2r)}(x) \\
&\quad = o\left(\frac{1}{l^r}\right)\varphi^{2r}(x)g^{(2r)}(x) + \left(b_{2r}^l\frac{1}{l^{2r}} + b_{2r-1}^l\frac{\varphi^2(x)}{l^{2r-1}} + \cdots + b_{r+1}^l\frac{\varphi^{2r-2}(x)}{l^{r+1}}\right) \\
&\qquad \times g^{(2r)}(x)\left(1 + O\left(\frac{1}{l}\right)\right) + M_l^{(2r-1)}\bigl(R_{2r+1}(g,\cdot,x),x\bigr).
\end{aligned} \tag{3.1}$$

For $x \in E_n$, $n\varphi^2(x) \geq 1$. So we have

$$\begin{aligned}
&\left\|\frac{1}{l^{2r}}g^{(2r)}(x)\right\|_p^{E^n} = \left\|\frac{n^r\varphi^{2r}(x)}{l^{2r}n^r\varphi^{2r}(x)}g^{(2r)}(x)\right\|_p^{E^n} \leq \frac{1}{l^r}\left(\frac{n}{l}\right)^r\|\varphi^{2r}g^{(2r)}\|_p, \\
&\left\|\frac{\varphi^2(x)}{l^{2r-1}}g^{(2r)}(x)\right\|_p^{E^n} = \left\|\frac{n^{r-1}\varphi^{2r}(x)}{l^{2r-1}n^{r-1}\varphi^{2r-2}(x)}g^{(2r)}(x)\right\|_p^{E^n} \leq \frac{1}{l^r}\left(\frac{n}{l}\right)^{r-1}\|\varphi^{2r}g^{(2r)}\|_p, \\
&\ldots, \\
&\left\|\frac{\varphi^{2r-2}(x)}{l^{r+1}}g^{(2r)}(x)\right\|_p^{E^n} = \left\|\frac{n\varphi^{2r}(x)}{l^{r+1}n\varphi^2(x)}g^{(2r)}(x)\right\|_p^{E^n} \leq \frac{1}{l^r}\left(\frac{n}{l}\right)\|\varphi^{2r}g^{(2r)}\|_p.
\end{aligned} \tag{3.2}$$

By Lemma 2.2 we have

$$\|M_l^{(2r-1)}\bigl(R_{2r+1}(g,\cdot,x),x\bigr)\|_p^{E_n} \leq Cl^{-r-\frac{1}{2}}\|\varphi^{2r+1}g^{(2r+1)}\|_p. \tag{3.3}$$

Combining (3.1)-(3.3), we obtain

$$\begin{aligned}
&\frac{1}{2^r l^r (r!)}\|\varphi^{2r}g^{(2r)}\|_p^{E_n} \\
&\quad \leq \|M_l^{(2r-1)}g - g\|_p + o\left(\frac{1}{l^r}\right)\|\varphi^{2r}g^{(2r)}\|_p \\
&\qquad + C\frac{1}{l^r}\left[\left(\frac{n}{l}\right)^r + \left(\frac{n}{l}\right)^{r-1} + \cdots + \frac{n}{l}\right]\left(1 + O\left(\frac{1}{l}\right)\right)\|\varphi^{2r}g^{(2r)}\|_p \\
&\qquad + Cl^{-r-\frac{1}{2}}\|\varphi^{2r+1}g^{(2r+1)}\|_p.
\end{aligned} \tag{3.4}$$

Next we estimate the first term and the last term of the right side in (3.4). By the boundedness of $M_n^{(2r-1)}f$ we have

$$\begin{aligned}
\|M_l^{(2r-1)}g - g\|_p &= \|M_l^{(2r-1)}(M_n^{2(2r-1)}f) - M_n^{2(2r-1)}f\|_p \\
&\leq \|M_l^{(2r-1)}(M_n^{2(2r-1)}f - M_n^{(2r-1)}f)\|_p + \|M_l^{(2r-1)}(M_n^{(2r-1)}f - f)\|_p
\end{aligned}$$

$$\begin{aligned}&+ \|M_l^{(2r-1)}f - f\|_p + \|f - M_n^{(2r-1)}f\|_p + \|M_n^{(2r-1)}(f - M_n^{(2r-1)}f)\|_p \\ &\le C(\|M_n^{(2r-1)}f - f\|_p + \|M_l^{(2r-1)}f - f\|_p). \end{aligned} \tag{3.5}$$

Using (2.9) and (2.13), we obtain

$$\begin{aligned}\|\varphi^{2r+1}g^{(2r+1)}\|_p &= \|\varphi^{2r+1}D^{2r+1}(M_n^{2(2r-1)}f)\|_p \\ &\le C\sqrt{n}\|\varphi^{2r}D^{2r}(M_n^{(2r-1)}f)\|_p \\ &\le C\sqrt{n}(\|\varphi^{2r}D^{2r}(M_n^{2(2r-1)}f)\|_p + \|\varphi^{2r}D^{2r}(M_n^{(2r-1)}(M_n^{(2r-1)}f - f))\|_p) \\ &\le C\sqrt{n}(\|\varphi^{2r}g^{(2r)}\|_p + n^r\|M_n^{(2r-1)}f - f\|_p). \end{aligned} \tag{3.6}$$

Therefore with (3.4)-(3.6) we get

$$\begin{aligned}&\frac{1}{2^r l^r (r!)}\|\varphi^{2r}g^{(2r)}\|_p^{E_n} \\ &\quad\le C(\|M_n^{(2r-1)}f - f\|_p + \|M_l^{(2r-1)}f - f\|_p) \\ &\qquad + C\left(\frac{n}{l}\right)^{r+\frac{1}{2}}\|M_n^{(2r-1)}f - f\|_p + Cl^{-r}\left(\frac{n}{l}\right)^{\frac{1}{2}}\|\varphi^{2r}g^{(2r)}\|_p \\ &\qquad + C\frac{1}{l^r}\left[\left(\frac{n}{l}\right)^r + \left(\frac{n}{l}\right)^{r-1} + \cdots + \frac{n}{l} + o(1)\right]\|\varphi^{2r}g^{(2r)}\|_p. \end{aligned} \tag{3.7}$$

Since $\varphi^{2r}g^{(2r)} = \varphi^{2r}D^{2r}(M_n^{2(2r-1)}f)$ are polynomials, for the same reason as (2.12) we have

$$\|\varphi^{2r}g^{(2r)}\|_p^{[0,1]} \le M\|\varphi^{2r}g^{(2r)}\|_p^{E_n}, \tag{3.8}$$

where M does not depend on n. Hence by (3.7) and (3.8) we obtain

$$\begin{aligned}&\frac{1}{2^r l^r (r!)}\|\varphi^{2r}g^{(2r)}\|_p \\ &\quad\le \frac{M}{2^r l^r (r!)}\|\varphi^{2r}g^{(2r)}\|_p^{E_n} \\ &\quad\le CM(\|M_n^{(2r-1)}f - f\|_p + \|M_l^{(2r-1)}f - f\|_p) + CM\left(\frac{n}{l}\right)^{r+\frac{1}{2}}\|M_n^{(2r-1)}f - f\|_p \\ &\qquad + CM\frac{1}{l^r}\left[\left(\frac{n}{l}\right)^r + \left(\frac{n}{l}\right)^{r-1} + \cdots + \frac{n}{l} + \left(\frac{n}{l}\right)^{\frac{1}{2}} + o(1)\right]\|\varphi^{2r}g^{(2r)}\|_p. \end{aligned} \tag{3.9}$$

We now choose $l \ge kn$ with k large enough such that

$$CM\frac{1}{l^r}\left[\left(\frac{n}{l}\right)^r + \left(\frac{n}{l}\right)^{r-1} + \cdots + \frac{n}{l} + \left(\frac{n}{l}\right)^{\frac{1}{2}} + o(1)\right] \le \frac{1}{2\cdot 2^r l^r (r!)}. \tag{3.10}$$

By (3.9) and (3.10) we get

$$\frac{1}{2\cdot 2^r l^r (r!)}\|\varphi^{2r}g^{(2r)}\|_p \le C\{\|M_n^{(2r-1)}f - f\|_p + \|M_l^{(2r-1)}f - f\|_p\}.$$

Therefore

$$\begin{aligned} K_\varphi^{2r}(f, n^{-r})_p &\leq C\|f - M_n^{(2r-1)}f\|_p + n^{-r}\|\varphi^{2r}g^{(2r)}\|_p \\ &\leq C\|f - M_n^{(2r-1)}f\|_p + C\left(\frac{l}{n}\right)^r \left(\|M_n^{(2r-1)}f - f\|_p + \|M_l^{(2r-1)}f - f\|_p\right) \\ &\leq C\left(\frac{l}{n}\right)^r \left(\|M_n^{(2r-1)}f - f\|_p + \|M_l^{(2r-1)}f - f\|_p\right). \end{aligned}$$

With (1.3) we obtain

$$\omega_\varphi^{2r}\left(f, \frac{1}{\sqrt{n}}\right)_p \leq C\left(\frac{l}{n}\right)^r \left(\|M_n^{(2r-1)}f - f\|_p + \|M_l^{(2r-1)}f - f\|_p\right).$$

The proof is complete. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors conceived of the study, participated its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

Acknowledgements
This work was supported by the National Natural Science Foundation of China (Grant Nos. 11371119 and 11401162) and by the Natural Science Foundation of Education Department of Hebei Province (Grant No. Z2014031).

Received: 13 August 2015 Accepted: 9 November 2015 Published online: 21 November 2015

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