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L_p-Dual geominimal surface area and general L_p-centroid bodies

Jinsheng Guo and Yibin Feng*

*Correspondence:
fengyibin001@163.com
School of Mathematics and Statistics, Hexi University, Zhangye, 734000, China

Abstract

In this article, we consider the Shephard type problems and obtain the affirmative and negative parts of the version of L_p-dual geominimal surface area for general L_p-centroid bodies. Combining with the L_p-dual geominimal surface area we also give a negative form of the Shephard type problems for L_p-centroid bodies.

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Keywords: L_p-dual geominimal surface area; general L_p-centroid bodies; L_p-centroid bodies; Shephard type problems

1 Introduction and main results

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in Euclidean space $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interiors and the set of origin-symmetric convex bodies in $\mathbb{R}^n$, we write $\mathcal{K}_o^n$ and $\mathcal{K}_c^n$, respectively. S_o^n and S_c^n, respectively, denote the set of star bodies (about the origin) and the set of origin-symmetric star bodies in $\mathbb{R}^n$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$, and let $V(K)$ denote the n-dimensional volume of a body K. For the standard unit ball B in $\mathbb{R}^n$, we use $\omega_n = V(B)$ to denote its volume.

The notion of geominimal surface area was discovered by Petty (see [1]). For $K \in \mathcal{K}^n$, the geominimal surface area, $G(K)$, of K is defined by

$$\omega_n^{\frac{1}{n}} G(K) = \inf\bigl\{nV_1(K,Q)V\bigl(Q^*\bigr)^{\frac{1}{n}} : Q \in \mathcal{K}^n\bigr\}.$$

Here Q^* denotes the polar of body Q and $V_1(M,N)$ denotes the mixed volume of $M,N \in \mathcal{K}^n$ (see [2]).

The geominimal surface area serves as a bridge connecting a number of areas of geometry: affine differential geometry, relative geometry, and Minkowskian geometry. Hence it receives a lot of attention (see, *e.g.*, [3, 4]). Lutwak in [5] showed that there were natural extensions of geominimal surface areas in the Brunn-Minkowski-Firey theory. It motivates extensions of some known inequalities for geominimal surface areas to L_p-geominimal surface areas. The inequalities for L_p-geominimal surface areas are stronger than their classical counterparts (see [6–10]).

Based on L_p-mixed volume, Lutwak [5] introduced the notion of L_p-geominimal surface area. For $K \in \mathcal{K}_o^n$, $p \geq 1$, the L_p-geominimal surface area, $G_p(K)$, of K is defined by

$$\omega_n^{\frac{p}{n}} G_p(K) = \inf\{nV_p(K,Q)V(Q^*)^{\frac{p}{n}} : Q \in \mathcal{K}_o^n\}.$$

Here $V_p(M,N)$ denotes the L_p-mixed volume of $M,N \in \mathcal{K}_o^n$ (see [5, 11]). Obviously, if $p = 1$, $G_p(K)$ is just the geominimal surface area $G(K)$.

Recently, Wang and Qi [12] introduced a concept of L_p-dual geominimal surface area, which is a dual concept for L_p-geominimal surface area and belongs to the dual L_p-Brunn-Minkowski theory for star bodies also developed by Lutwak (see [13, 14]). For $K \in S_o^n$, and $p \geq 1$, the L_p-dual geominimal surface area, $\widetilde{G}_{-p}(K)$, of K is defined by

$$\omega_n^{-\frac{p}{n}} \widetilde{G}_{-p}(K) = \inf\{n\widetilde{V}_{-p}(K,Q)V(Q^*)^{-\frac{p}{n}} : Q \in \mathcal{K}_c^n\}. \tag{1.1}$$

Here, $\widetilde{V}_{-p}(M,N)$ denotes the L_p-dual mixed volume of $M,N \in S_o^n$ (see [5]).

Centroid bodies are a classical notion from geometry which have attracted increased attention in recent years (see [13, 15–22]). In particular, Lutwak and Zhang [18] introduced the notion of L_p-centroid bodies. For each compact star-shaped (about the origin) K in $\mathbb{R}^n$ and real number $p \geq 1$, the L_p-centroid body, $\Gamma_p K$, of K is an origin-symmetric convex body whose support function is defined by

$$\begin{aligned} h^p_{\Gamma_p K}(u) &= \frac{1}{c_{n,p}V(K)} \int_K |u \cdot x|^p \, dx \\ &= \frac{1}{c_{n,p}(n+p)V(K)} \int_{S^{n-1}} |u \cdot v|^p \rho_K^{n+p}(v) \, dS(v) \end{aligned} \tag{1.2}$$

for all $u \in S^{n-1}$, where

$$c_{n,p} = \omega_{n+p}/\omega_2\omega_n\omega_{p-1}, \quad \text{and} \quad \omega_n = \pi^{\frac{n}{2}}/\Gamma\left(1+\frac{n}{2}\right). \tag{1.3}$$

More recently, Feng *et al.* [23] defined a new notion of general L_p-centriod bodies, which generalized the concept of L_p-centroid bodies. For $K \in S_o^n$, $p \geq 1$, and $\tau \in [-1,1]$, the general L_p-centroid body, $\Gamma_p^\tau K$, of K is a convex body whose support function is defined by

$$\begin{aligned} h^p_{\Gamma_p^\tau K}(u) &= \frac{1}{c_{n,p}(\tau)V(K)} \int_K \varphi_\tau(u \cdot x)^p \, dx \\ &= \frac{1}{c_{n,p}(\tau)(n+p)V(K)} \int_{S^{n-1}} \varphi_\tau(u \cdot v)^p \rho_K^{n+p}(v) \, dv, \end{aligned} \tag{1.4}$$

where

$$c_{n,p}(\tau) = \frac{1}{2}c_{n,p}\left[(1+\tau)^p + (1-\tau)^p\right],$$

and $\varphi_\tau : \mathbb{R} \to [0,\infty)$ is a function defined by $\varphi_\tau(t) = |t| + \tau t$. We note that general L_p-centroid bodies are an essential part of the rapidly evolving asymmetric L_p-Brunn-Minkowski theory (see [20, 24–32]).

The normalization is chosen such that $\Gamma_p^\tau B = B$ for every $\tau \in [-1,1]$, and $\Gamma_p^0 K = \Gamma_p K$. Let $\varphi_+(u\cdot x) = \max\{u\cdot x, 0\}$ ($\tau = 1$) in (1.4), then a special case of the definition of $\Gamma_p^\tau K$ is $\Gamma_p^+ K$, *i.e.*,

$$h^p_{\Gamma_p^+ K}(u) = \frac{1}{c_{n,p}V(K)}\int_K \varphi_+(u\cdot x)^p\,dx$$
$$= \frac{1}{c_{n,p}(n+p)V(K)}\int_{S^{n-1}} \varphi_+(u\cdot v)^p \rho_K^{n+p}(v)\,dv. \tag{1.5}$$

Besides, we also define

$$\Gamma_p^- K = \Gamma_p^+(-K). \tag{1.6}$$

From the definition of $\Gamma_p^\pm K$ and (1.4), we see that if $K \in S_o^n$, $p \geq 1$, and $\tau \in [-1,1]$, then

$$\Gamma_p^\tau K = f_1(\tau)\cdot \Gamma_p^+ K +_p f_2(\tau)\cdot \Gamma_p^- K, \tag{1.7}$$

where '$+_p$' denotes the Firey L_p-combination of convex bodies, and

$$f_1(\tau) = \frac{(1+\tau)^p}{(1+\tau)^p + (1-\tau)^p}, \qquad f_2(\tau) = \frac{(1-\tau)^p}{(1+\tau)^p + (1-\tau)^p}. \tag{1.8}$$

If $\tau = \pm 1$ in (1.7) and using (1.8), then

$$\Gamma_p^{+1} K = \Gamma_p^+ K, \qquad \Gamma_p^{-1} K = \Gamma_p^- K.$$

In [16] Grinberg and Zhang discussed an investigation of Shephard type problems for L_p-centriod bodies. Namely, let K and L be two origin-symmetric star bodies such that

$$\Gamma_p K \subset \Gamma_p L.$$

They proved that if the space $(\mathbb{R}^n, \|\cdot\|_L)$ embeds in L_p, then we necessarily have

$$V(K) \leq V(L).$$

On the other hand, if $(\mathbb{R}^n, \|\cdot\|_K)$ does not embed in L_p, then there is a body L so that $\Gamma_p K \subset \Gamma_p L$, but $V(K) \leq V(L)$.

In this article, we first investigate the Shephard type problems for general L_p-centroid bodies and give the affirmative and negative parts of the version of L_p-dual geominimal surface area.

Theorem 1.1 *For $K \in \mathcal{K}_o^n$, $L \in \mathcal{K}_c^n$, and $p \geq 1$, if $\Gamma_p^+ K = \Gamma_p^+ L$ and $\Gamma_p^- K = \Gamma_p^- L$, then*

$$\widetilde{G}_{-p}(K) \leq \widetilde{G}_{-p}(L), \tag{1.9}$$

with equality if and only if $K = L$.

Theorem 1.2 *For $L \in S_o^n$, $p \geq 1$ and $\tau \in (-1,1)$, if L is not origin-symmetric, then there exists $K \in S_o^n$, such that*

$$\Gamma_p^+ K \subset \Gamma_p^\tau L, \qquad \Gamma_p^- K \subset \Gamma_p^{-\tau} L.$$

But

$$\widetilde{G}_{-p}(K) > \widetilde{G}_{-p}(L).$$

Further, taking together the L_p-dual geominimal surface area with L_p-centroid bodies we establish the following Shephard type problem.

Theorem 1.3 *For $L \in S_o^n$ and $1 \leq p < n$, if L is not origin-symmetric star body, then there exists $K \in S_o^n$, such that*

$$\Gamma_p K \subset \Gamma_p L.$$

But

$$\widetilde{G}_{-p}(K) > \widetilde{G}_{-p}(L).$$

The proofs of Theorems 1.1-1.3 will be given in Section 3.

2 Preliminaries

2.1 Support functions, radial functions, and polars of convex bodies

The support function, $h_K = h(K,\cdot) : \mathbb{R}^n \to (-\infty,\infty)$, of $K \in \mathcal{K}^n$ is defined by (see [33, 34])

$$h(K,x) = \max\{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n, \tag{2.1}$$

where $x \cdot y$ denotes the standard inner product of x and y.

If K is a compact star-shaped (about the origin) set in $\mathbb{R}^n$, then its radial function, $\rho_K = \rho(K,\cdot) : \mathbb{R}^n \setminus \{0\} \to [0,\infty)$, is defined by (see [33, 34])

$$\rho(K,u) = \max\{\lambda \geq 0 : \lambda \cdot u \in K\}, \quad u \in S^{n-1}. \tag{2.2}$$

If ρ_K is continuous and positive, then K will be called a star body. Two star bodies K, L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

If $K \in \mathcal{K}_o^n$, the polar body, K^*, of K is defined by (see [33, 34])

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, y \in K\}. \tag{2.3}$$

For $K, L \in \mathcal{K}_o^n$, $p \geq 1$, and $\lambda, \mu \geq 0$ (not both zero), the Firey L_p-combination, $\lambda \cdot K +_p \mu \cdot L$, of K and L is defined by (see [35])

$$h(\lambda \cdot K +_p \mu \cdot L, \cdot)^p = \lambda h(K,\cdot)^p + \mu h(L,\cdot)^p, \tag{2.4}$$

where '$\cdot$' in $\lambda \cdot K$ denotes the Firey scalar multiplication. Obviously, the L_p-Firey and the usual scalar multiplications are related by $\lambda \cdot K = \lambda^{\frac{1}{p}} K$.

For $K, L \in S_o^n$, $p \geq 1$, and $\lambda, \mu \geq 0$ (not both zero), the L_p-harmonic radial combination, $\lambda \star K +_{-p} \mu \star L \in S_o^n$, of K and L is defined by (see [5])

$$\rho(\lambda \star K +_{-p} \mu \star L, \cdot)^{-p} = \lambda \rho(K, \cdot)^{-p} + \mu \rho(L, \cdot)^{-p}, \tag{2.5}$$

where $\lambda \star K$ denotes the L_p-harmonic radial scalar multiplication. Here, we have $\lambda \star K = \lambda^{-\frac{1}{p}} K$.

2.2 L_p-Dual mixed volume

Using L_p-harmonic radial combination, Lutwak [5] introduced the notion of L_p-dual mixed volume. For $K, L \in S_o^n$, $p \geq 1$, and $\varepsilon > 0$, the L_p-dual mixed volume, $\widetilde{V}_{-p}(K, L)$, of K and L is defined by

$$\frac{n}{-p} \widetilde{V}_{-p}(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K +_{-p} \varepsilon \star L) - V(K)}{\varepsilon}.$$

The definition above and de l'Hospital's rule yield the following integral representation of L_p-dual mixed volume (see [5]):

$$\widetilde{V}_{-p}(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho_K^{n+p}(u) \rho_L^{-p}(u)\, du, \tag{2.6}$$

where the integration is with respect to spherical Lebesgue measure on S^{n-1}.

From (2.6), it follows immediately that, for each $K \in S_o^n$ and $p \geq 1$,

$$\widetilde{V}_{-p}(K, K) = V(K) = \frac{1}{n} \int_{S^{n-1}} \rho_K^n(u)\, du. \tag{2.7}$$

Minkowski's inequality for a L_p-dual mixed volume can be stated as follows (see [5]).

Theorem 2.A *If $K, L \in S_o^n$, $p \geq 1$, then*

$$\widetilde{V}_{-p}(K, L) \geq V(K)^{\frac{n+p}{n}} V(L)^{-\frac{p}{n}}, \tag{2.8}$$

with equality if and only if K and L are dilates.

2.3 General L_p-harmonic Blaschke bodies

For $K \in S_o^n$, $p \geq 1$, and $\tau \in [-1, 1]$, the general L_p-harmonic Blaschke body, $\widehat{\nabla}_p^\tau K$, of K is defined by (see [36])

$$\frac{\rho(\widehat{\nabla}_p^\tau K, \cdot)^{n+p}}{V(\widehat{\nabla}_p^\tau K)} = f_1(\tau) \frac{\rho(K, \cdot)^{n+p}}{V(K)} + f_2(\tau) \frac{\rho(-K, \cdot)^{n+p}}{V(-K)}. \tag{2.9}$$

Operators of this type and related maps compatible with linear transformations appear essentially in the theory of valuations in connection with isoperimetric and analytic inequalities (see [37–43]).

Theorem 2.B [36] *If $K \in S_o^n$, $p \geq 1$, and $\tau \in (-1,1)$, then*

$$\widetilde{G}_{-p}\big(\widehat{\nabla}_p^\tau K\big) \geq \widetilde{G}_{-p}(K), \tag{2.10}$$

with equality if and only if K is origin-symmetric.

3 Proofs of main results

In this section, we complete the proofs of Theorems 1.1-1.3. The proof of Theorem 1.1 requires the following lemma.

Lemma 3.1 *If $K, L \in S_o^n$ and $p \geq 1$, if $\Gamma_p^+ K = \Gamma_p^+ L$ and $\Gamma_p^- K = \Gamma_p^- L$, then for any $Q \in S_c^n$*

$$\frac{\widetilde{V}_{-p}(K,Q)}{V(K)} = \frac{\widetilde{V}_{-p}(L,Q)}{V(K)}. \tag{3.1}$$

Proof Since $\Gamma_p^+ K = \Gamma_p^+ L$ and $\Gamma_p^- K = \Gamma_p^- L$, it easily follows that for any $u \in S^{n-1}$

$$h^p_{\Gamma_p^+ K}(u) + h^p_{\Gamma_p^- K}(u) = h^p_{\Gamma_p^+ L}(u) + h^p_{\Gamma_p^- L}(u).$$

Together (1.5) with (1.6), we get

$$\int_{S^{n-1}} \varphi_+(u\cdot v)^p \left[\frac{\rho_K^{n+p}(v)}{V(K)} + \frac{\rho_{-K}^{n+p}(v)}{V(-K)} - \frac{\rho_L^{n+p}(v)}{V(L)} - \frac{\rho_{-L}^{n+p}(v)}{V(-L)}\right] dv = 0.$$

Let

$$\mu(v) = \frac{\rho_K^{n+p}(v)}{V(K)} + \frac{\rho_{-K}^{n+p}(v)}{V(-K)} - \frac{\rho_L^{n+p}(v)}{V(L)} - \frac{\rho_{-L}^{n+p}(v)}{V(-L)},$$

then have

$$\int_{S^{n-1}} \varphi_+(u\cdot v)^p \mu(v)\, dv = 0. \tag{3.2}$$

Notice that $\rho_{-K}(v) = \rho_K(-v)$ for all $v \in S^{n-1}$, thus we know that $\mu(v)$ is a finite even Borel measure. Together with (3.2), then $\mu(v) = 0$, *i.e.*,

$$\frac{\rho_K^{n+p}(v)}{V(K)} + \frac{\rho_K^{n+p}(-v)}{V(-K)} = \frac{\rho_L^{n+p}(v)}{V(L)} + \frac{\rho_L^{n+p}(-v)}{V(L)}.$$

For any $Q \in S_c^n$, then use $\rho_Q(v) = \rho_{-Q}(v) = \rho_Q(-v)$ to get

$$\frac{\rho_K^{n+p}(v)\rho_Q^{-p}(v)}{V(K)} + \frac{\rho_K^{n+p}(-v)\rho_Q^{-p}(-v)}{V(K)} = \frac{\rho_L^{n+p}(v)\rho_Q^{-p}(v)}{V(L)} + \frac{\rho_L^{n+p}(-v)\rho_Q^{-p}(-v)}{V(L)}.$$

From (2.6), this yields for any $Q \in S_c^n$

$$\frac{\widetilde{V}_{-p}(K,Q)}{V(K)} = \frac{\widetilde{V}_{-p}(L,Q)}{V(L)}.$$

□

Proof of Theorem 1.1 Together with definition (1.1), we know

$$\frac{\omega_n^{-\frac{p}{n}}\widetilde{G}_{-p}(K)}{V(K)} = \inf\left\{n\frac{\widetilde{V}_{-p}(K,Q)}{V(K)}V\left(Q^*\right)^{-\frac{p}{n}} : Q \in \mathcal{K}_c^n\right\}. \tag{3.3}$$

Since $\Gamma_p^+K = \Gamma_p^+L$ and $\Gamma_p^-K = \Gamma_p^-L$, from (3.1), we get, for any $Q \in \mathcal{K}_c^n$,

$$\frac{\widetilde{V}_{-p}(K,Q)}{V(K)} = \frac{\widetilde{V}_{-p}(L,Q)}{V(L)}. \tag{3.4}$$

Hence, from (3.3) and (3.4), we can get

$$\frac{\widetilde{G}_{-p}(K)}{V(K)} = \frac{\widetilde{G}_{-p}(L)}{V(L)},$$

i.e.,

$$\frac{\widetilde{G}_{-p}(K)}{\widetilde{G}_{-p}(L)} = \frac{V(K)}{V(L)}. \tag{3.5}$$

Taking $Q = L$ in (3.4) and associating this with (2.8), since $L \in \mathcal{K}_c^n$, we obtain

$$V(K) = \widetilde{V}_{-p}(K,L) \geq V(K)^{\frac{n+p}{n}}V(L)^{-\frac{p}{n}},$$

i.e.,

$$V(K) \leq V(L). \tag{3.6}$$

Combining (3.5) with (3.6), we get (1.9).

According to the equality condition of (3.6), we see that equality holds in (1.9) if and only if $K = L$. □

Lemma 3.2 [44] *If* $K \in S_o^n$, $p \geq 1$, $\tau \in (-1,1)$, *then*

$$\Gamma_p^+\left(\widehat{\nabla}_p^\tau K\right) = \Gamma_p^\tau K \tag{3.7}$$

and

$$\Gamma_p^-\left(\widehat{\nabla}_p^\tau K\right) = \Gamma_p^{-\tau} K. \tag{3.8}$$

Proof of Theorem 1.2 Since L is not origin-symmetric and $\tau \in (-1,1)$, it follows from Theorem 2.B that $\widetilde{G}_{-p}(\widehat{\nabla}_p^\tau L) > \widetilde{G}_{-p}(L)$. Choose $\varepsilon > 0$, such that $K = (1-\varepsilon)\widehat{\nabla}_p^\tau L$ satisfies

$$\widetilde{G}_{-p}(K) = \widetilde{G}_{-p}\left((1-\varepsilon)\widehat{\nabla}_p^\tau L\right) > \widetilde{G}_{-p}(L).$$

By (3.7) and (3.8), we, respectively, have

$$\Gamma_p^+K = \Gamma_p^+\left[(1-\varepsilon)\widehat{\nabla}_p^\tau L\right] = (1-\varepsilon)\Gamma_p^+\left(\widehat{\nabla}_p^\tau L\right) = (1-\varepsilon)\Gamma_p^\tau L \subset \Gamma_p^\tau L$$

and

$$\Gamma_p^- K = \Gamma_p^-\big[(1-\varepsilon)\widehat{\nabla}_p^\tau L\big] = (1-\varepsilon)\Gamma_p^-\big(\widehat{\nabla}_p^\tau L\big) = (1-\varepsilon)\Gamma_p^{-\tau} L \subset \Gamma_p^{-\tau} L. \qquad \square$$

Lemma 3.3 [44] *If $K \in S_o^n$, $p \geq 1$, and $\tau \in [-1,1]$, then*

$$\Gamma_p\big(\widehat{\nabla}_p^\tau K\big) = \Gamma_p K. \tag{3.9}$$

Proof of Theorem 1.3 Since L is not origin-symmetric, Theorem 2.B has $\widetilde{G}_{-p}(\widehat{\nabla}_p^\tau L) > \widetilde{G}_{-p}(L)$ for $\tau \in (-1,1)$. Take $\varepsilon > 0$, and let $K = (1-\varepsilon)\widehat{\nabla}_p^\tau L$ such that

$$\widetilde{G}_{-p}(K) = \widetilde{G}_{-p}\big((1-\varepsilon)\widehat{\nabla}_p^\tau L\big) > \widetilde{G}_{-p}(L).$$

It follows from (3.9) that

$$\Gamma_p K = \Gamma_p\big[(1-\varepsilon)\widehat{\nabla}_p^\tau L\big] = (1-\varepsilon)\Gamma_p\big(\widehat{\nabla}_p^\tau L\big) = (1-\varepsilon)\Gamma_p L \subset \Gamma_p L. \qquad \square$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors completed the writing of this paper and read and approved the final manuscript.

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