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L^{φ}-Embedding inequalities for some operators on differential forms

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Abstract
In this paper, the local Poincaré inequality and embedding inequality are proved first. Then the global embedding inequality of composite operators for differential forms on L^{φ}-averaging domains with L^{φ}-norm is established. Some examples are also given to illustrate applications.

Keywords: homotopy operator; Dirac operator; Green's operator; L^{φ}-averaging domains

1 Introduction

The purpose of this paper is to establish the global Sobolev embedding inequality with L^{φ}-norm for the composite operators applied to differential forms on L^{φ}-averaging domains. As extensions of functions, differential forms have been deeply studied and widely applied in many fields, such as global analysis, nonlinear analysis, PDEs and differential geometry; see [1–10] for more information. The Sobolev embedding inequality, as a fundamental tool in the Sobolev space of functions, has been very well studied in [11]. The homotopy operator T, Dirac operator D, and Green's operator G are three key operators acting on differential forms and each of them has received much attention recently. However, the compositions of these operators are so complicated that the results concerned on them are quite few. Even though they are very complicated, the study for the composite operators has been increasing during the recent years; see [6–9, 12]. In many cases, we need to study or to use the compositions of operators. For example, we have to deal with the composition of the Dirac operator D and Green's operator G when we consider the Poisson equation $D^2(G(u)) = u - H(u)$, where $D^2 = \Delta$ and H is the projection operator. In 2013, Ding and Liu initiated the study of the composition $D \circ G$ and obtained some basic inequalities in [2]. During our recent investigation of the operator theory of differential forms, we realized that for applying the decomposition theorem to $D \circ G$, it is necessary to deal with the composition $T \circ D \circ G$. Hence, we are motivated to study the composition $T \circ D \circ G$. We obtain some basic estimates for $T \circ D \circ G$ and establish the global Sobolev embedding inequality with L^{φ}-norm

$$\left\| TDG(u) - \left(TDG(u)\right)_E \right\|_{W_E^{1,\varphi}} \le C\|u\|_{L_E^{1,\varphi}},$$

where E is an L^{φ}-averaging domain [13].

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For convenience, we keep using the traditional notations and terminologies. Except for special instructions, $E \subseteq \mathbb{R}^n$ is a bounded domain, and $|E|$ denotes the Lebesgue measure of E, $n \geq 2$. Suppose that B_x^r is a ball with a radius r, centered at x. For any $\sigma > 0$, $B \subseteq E$ and $\sigma B \subseteq E$ have the same center and satisfy $\operatorname{diam}(\sigma B) = \sigma \operatorname{diam}(B)$. Let $\Lambda^l(\mathbb{R}^n)$ be the space of all l-forms in $\mathbb{R}^n$, which is expanded by the exterior product of $e^{\mathfrak{B}} = e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_l}$, where $\mathfrak{B} = (i_1, \dots, i_l)$, $1 \leq i_1 < \cdots < i_l \leq n$, $l = 1, 2, \dots, n$. $C^\infty(\Lambda^l E)$ is the space of a smooth l-form on E. We use $D'(E, \Lambda^l)$ to denote the space of all differential l-forms on E, that is, $u(x)$ belongs to $D'(E, \Lambda^l)$ if and only if there exist some lth-differential functions $u_{\mathfrak{B}}$ in E such that $u(x) = \sum_{\mathfrak{B}} u_{\mathfrak{B}}(x)\,dx_{\mathfrak{B}} = \sum u_{i_1 i_2 \cdots i_l}(x)\,dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l}$. $L^p(E, \Lambda^l)$ is a Banach space with the norm equipped by $\|u\|_{p,E} = (\int_E |u(x)|^p\,dx)^{1/p}$, where $u(x) \in D'(E, \Lambda^l)$ and every coefficient function $u_{\mathfrak{B}} \in L^p(E)$, $0 < p < \infty$. In fact, $u(x)$ on E is the Schwarz distribution. If $w(x) > 0$ a.e. and $w(x) \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$, $w(x)$ is called a weight. Let $d\mu = w(x)\,dx$, then $L^p(E, \Lambda^l, w)$ is a weighted Banach space with the norm expressed by $\|u(x)\|_{p,E,w} = (\int_E |u(x)|^p w(x)\,dx)^{1/p}$. In this notation, the exterior derivative is denoted by d and the Hodge codifferential operator is expressed by $d^\star$. Refer to [1] for more details. Moreover, the Dirac operator, designated by D, was initially set forth by Paul Dirac, and we get its extensions, such as the Hodge-Dirac operator and the Euclidean Dirac operator. In this paper, the Dirac operator we choose is the Hodge-Dirac operator $D = d^* + d$. The Green's operator is a bound self-adjoint linear operator. It is commonly used to define the Poisson equation for differential forms; here it is represented by G. The homotopy operator T [10], advanced by Iwaniec, is defined on $C^\infty(D, \Lambda^l)$ and constructed as follows:

$$Tu = \int_D \psi(y) K_y u\,dy,$$

where the linear operator K_y is denoted by $(K_y u)(x; \xi_1, \dots, \xi_{l-1}) = \int_0^1 t^{l-1} u(tx + y - ty; x - y, \xi_1, \dots, \xi_{l-1})\,dt$, and ψ from $C_0^\infty(D)$ is normalized so that $\int \psi(y)\,dy = 1$.

From [10], we obtain some good results. For any differential form u, the decomposition below holds:

$$u = d(Tu) + T(du). \tag{1.1}$$

Meanwhile, for $u_\Omega \in D'(\Omega, \Lambda^l)$, we define

$$u_\Omega = \begin{cases} |\Omega|^{-1} \int_\Omega u(y)\,dy, & l = 0, \\ dT(u), & l = 1, 2, \dots, n. \end{cases} \tag{1.2}$$

Then we know that, for any differential form $u \in L^s_{\mathrm{loc}}(B, \Lambda^l)$, $l = 1, \dots, n$, $1 < s < \infty$,

$$\big\|\nabla(Tu)\big\|_{s,B} \leq C|B| \|u\|_{s,B} \quad \text{and} \quad \|Tu\|_{s,B} \leq C|B| \operatorname{diam}(B) \|u\|_{s,B} \tag{1.3}$$

holds for any ball B in Ω.

Concerning the homotopy operator T, according to [1], we can obtain a nice property. That is, $\Omega \subset \mathbb{R}^n$ is the union of a collection of cube Ω_k, whose sides are parallel to the axes, whose interiors are mutually disjoint, and whose diameters are approximately proportional to their distances from F, where F is the complement of Ω in $\mathbb{R}^n$. In detail, $\Omega \subset \mathbb{R}^n$ can be expressed as follows:

(1) $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$;
(2) $\Omega_i^0 \cap \Omega_j^0 = \emptyset$, if $j \neq i$;
(3) there exist two constants $C_1 > 0$ and $C_2 > 0$, such that

$$C_1 \operatorname{diam}(\Omega_k) \leq \operatorname{distance}(\Omega_k, F) \leq C_2 \operatorname{diam}(\Omega_k).$$

Based on this fact, the homotopy operator T can be extended to any domain $\Omega \subset \mathbb{R}^n$. For any $x \in \Omega$, $x \in \Omega_k$ for some k, let T_{Ω_k} be the homotopy operator defined on Ω_k, which is the convex and bounded set. So, the following definition for T on any domain Ω holds:

$$T_\Omega = \sum_{k=1}^{+\infty} T_{\Omega_k} \chi_{\Omega_k},$$

where χ_{Ω_k} is the characteristic function defined on Ω.

Further, it is essential to recall some well-known results as regards the A-harmonic equation for differential forms, which appeared in [1]. To be more precise, we here consider the non-homogeneous A-harmonic equation as follows:

$$d^\star A(x, du) = B(x, du), \tag{1.4}$$

where $A : E \times \wedge^l(\mathbb{R}^n) \to \wedge^l(\mathbb{R}^n)$ and $B : E \times \wedge^l(\mathbb{R}^n) \to \wedge^{l-1}(\mathbb{R}^n)$ satisfy the conditions

$$\left|A(x,\xi)\right| \leq a|\xi|^{p-1}, \qquad A(x,\xi)\cdot\xi \geq |\xi|^p \quad \text{and} \quad \left|B(x,\xi)\right| \leq b|\xi|^{p-1}$$

for almost every $x \in E$ and all $\xi \in \wedge^l(\mathbb{R}^n)$. Meanwhile, the parameters $a, b > 0$ and $1 < p < \infty$ associated with (1.4) are constants and a fixed exponential, respectively. If $B = 0$, the equation $d^\star A(x, du) = 0$ is called a homogeneous A-harmonic equation.

In addition, for proving the global embedding inequality on the L^φ-averaging domain, we also need the following definitions and notations.

A function $\varphi(x)$ is called an Orlicz function, if $\varphi(x)$ satisfies: (1) $\varphi(x)$ is continuously increasing; (2) $\varphi(0) = 0$. Furthermore, if the Orlicz function $\varphi(x)$ is a convex function, $\varphi(x)$ is named a Young function. Therefore, based on this type of functions, the Orlicz norm for differential forms can be denoted as follows.

Take $\varphi(x)$ as a Orlicz function, and $E \subset \mathbb{R}^n$ as a bounded domain, for any $u(x) \in L^p_{\mathrm{loc}}(E, \Lambda^l)$, $l = 0, 1, 2, \ldots, n$, the Orlicz norm for the differential form is equipped with

$$\|u\|_{L^\varphi_E} = \inf\left\{\lambda > 0 : \int_E \varphi\left(\frac{|u|}{\lambda}\right) d\mu \leq 1\right\},$$

where the measure μ is expressed by $d\mu = w(x)\,dx$, $w(x)$ is a weight.

It is easy to prove that L^φ_E is a Banach space. Actually, provided that $\varphi(x)$ is taken as $\varphi(x) = x^s$ ($s > 0$), then $\varphi(x)$ is an Orlicz function, and it trivially corresponds to a $L^s(\mu)$ space. As a result of that, we can say that L^φ_E is the generalization of L^s_E.

Similarly, we intend to give L^φ-averaging domains. Based on the Orlicz norm above, the Orlicz-Sobolev space of l-form is denoted by $W^{1,\varphi}(E, \Lambda^l)$, with the norm

$$\|u\|_{W^{1,\varphi}_E} = \|u\|_{W^{1,\varphi}(E,\Lambda^l)} = \operatorname{diam}(E)^{-1}\|u\|_{L^\varphi_E} + \|\nabla u\|_{L^\varphi_E}. \tag{1.5}$$

From the definition of $W_E^{1,\varphi}$, we know that $W_E^{1,\varphi}$ is equal to $L_E^{\varphi} \cap L_{1,E}^{\varphi}$. In detail, if $\varphi(x)$ is expressed by $\varphi(x) = t^s$ ($s > 0$), we can conclude the norm of $W_E^{1,s}$ for the differential form:

$$\|u\|_{W_E^{1,s}} = \|u\|_{W^{1,s}(E,\Lambda^l)} = \operatorname{diam}(E)^{-1}\|u\|_{L_E^s} + \|\nabla u\|_{L_E^s}.$$

Besides that, we will show another definition, which was initially introduced by Ding.

Definition 1.1 [13] Let $\varphi(x)$ be a Young function, the proper domain $E \subseteq \mathbb{R}^n$ is called the L^{φ}-averaging domain, if $\mu(E) < \infty$ and there exists a constant $C > 0$ such that for any $B_0 \subseteq E$ and $\varphi(|u|) \in L^1_{\mathrm{loc}}(E,\mu)$, u satisfies

$$\frac{1}{\mu(E)}\int_E \varphi\bigl(\tau|u - u_{B_0}|\bigr)\,d\mu \le C \sup_{4B\subset E}\frac{1}{\mu(B)}\int_B \varphi\bigl(\sigma|u - u_B|\bigr)\,d\mu,$$

where the measure μ is denoted by $d\mu = w(x)\,dx$, $w(x)$ is a weight, σ and τ are constants with $0 < \tau, \sigma < 1$, and the supremum is over all balls $B \subset E$ with $4B \subset E$.

Using the same analysis method as of the L_E^{φ}-norm, we can conclude that L^{φ}-averaging domains are the generalization of L^s-averaging domains.

2 Main results

Before the main results given, we will make some restrictions to the Young function $\varphi(x)$. Here, we let the Young function $\varphi(x)$ belong to the $G(p,q,C)$-class ($1 \le p < q < \infty$, $C \ge 1$), that is, for any $t > 0$, the Young function $\varphi(x)$ satisfies:

(1) $1/C \le \varphi(t^{1/p})/f(t) \le C$;

(2) $1/C \le \varphi(t^{1/q})/g(t) \le C$,

where f and g are increasingly convex and concave functions defined on $[0,\infty]$, respectively.

Now, we establish four important theorems based on the above-mentioned conditions.

Theorem 2.1 *Let T be the homotopy operator, D be the Dirac operator, and G be the Green's operator. Meanwhile, we assume that $\varphi(|u|) \in L^1_{\mathrm{loc}}(E)$, $u \in C^{\infty}(\Lambda^l E)$ is a solution of the non-homogeneous A-harmonic equation, the Young function imposed a doubling property $\varphi(x)$ belongs to the $G(p,q,C)$-class, and the bounded subset $E \subseteq \mathbb{R}^n$ is the L^{φ}-averaging domain. Then, for any ball $B \subseteq E$, we get*

$$\bigl\|TDG(u) - \bigl(TDG(u)\bigr)_B\bigr\|_{L_B^{\varphi}} \le C\operatorname{diam}(B)\|u\|_{L_{\sigma B}^{\varphi}},$$

where $\sigma B \subseteq E$ and $\sigma > 1$ is a constant.

Theorem 2.2 *Let T be the homotopy operator, D be the Dirac operator, and G be the Green's operator. Meanwhile, we assume that $\varphi(|u|) \in L^1_{\mathrm{loc}}(E)$, $u \in C^{\infty}(\Lambda^l E)$, is a solution of the non-homogeneous A-harmonic equation, the Young function $\varphi(x)$ having imposed a doubling property belongs to the $G(p,q,C)$-class and the bounded subset $E \subseteq \mathbb{R}^n$ is the L^{φ}-averaging domain. Then, for any ball $B \subseteq E$, we get*

$$\bigl\|TDG(u) - \bigl(TDG(u)\bigr)_B\bigr\|_{W_B^{1,\varphi}} \le C\operatorname{diam}(B)\|u\|_{L_{\sigma B}^{1,\varphi}},$$

where $\sigma B \subseteq E$ and $\sigma > 1$ is a constant.

Based on the above theorem, we cannot only establish the following global embedding inequality on L^{φ}-averaging domains, but we also get the global Poincaré-type inequality fortunately.

Theorem 2.3 *Let T be the homotopy operator, D be the Dirac operator, and G be the Green's operator. Additionally, we assume that $\varphi(|u|) \in L^1_{\mathrm{loc}}(E)$, $u \in C^{\infty}(\Lambda^l E)$, is a solution of the non-homogeneous A-harmonic equation, the Young function $\varphi(x)$ having imposed a doubling property belongs to the $G(p,q,C)$-class and the bounded subset $E \subseteq \mathbb{R}^n$ is the L^{φ}-averaging domain. Then, for any ball $B \subseteq E$, we have*

$$\|TDG(u) - (TDG(u))_E\|_{W^{1,\varphi}_E} \leq C\|u\|_{L^{1,\varphi}_E}.$$

Theorem 2.4 *Let T be the homotopy operator, D be the Dirac operator, and G be the Green's operator. Additionally, we assume that $\varphi(|u|) \in L^1_{\mathrm{loc}}(E)$, $u \in C^{\infty}(\Lambda^l E)$, is a solution of the non-homogeneous A-harmonic equation, the Young function $\varphi(x)$ having imposed a doubling property belongs to the $G(p,q,C)$-class and the bounded subset $E \subseteq \mathbb{R}^n$ is the L^{φ}-averaging domain. Then, for any ball $B \subseteq E$, we have*

$$\|TDG(u) - (TDG(u))_{B_0}\|_{L^{\varphi}_E} \leq C\|u\|_{L^{\varphi}_E},$$

where $B_0 \subseteq B$ is a fixed ball.

3 Preliminary results

For proving the theorems in Section 2, we shall show and demonstrate some lemmas in this section.

Lemma 3.1 [1] *Let $0 < p, q < \infty$, and $\frac{1}{t} = \frac{1}{p} + \frac{1}{q}$, if f and g are the measurable functions defined on $\mathbb{R}^n$, then*

$$\|fg\|_{t,I} \leq \|f\|_{p,I} \cdot \|g\|_{q,I}$$

for any $I \subseteq \mathbb{R}^n$.

The inequality in Lemma 3.1 is actually the generalized Hölder inequality. Specifically, if $t = 1$ and $1 < p, q < \infty$, the inequality above is the classical Hölder inequality.

Lemma 3.2 [1] *Let u be a solution of the non-homogeneous A-harmonic equation in a domain Ω, and $0 < s, t < \infty$, then there exists a constant C, independent of u, such that*

$$\|u\|_{s,B} \leq C|B|^{\frac{t-s}{st}}\|u\|_{t,\sigma B}$$

for all balls B with $\sigma B \subset E$, where $\sigma > 1$ is some constant.

In fact, we can get a valuable result; if u satisfies the inequality in Lemma 3.2, we say u belongs to the WRH-class.

Lemma 3.3 [2] *Suppose that the differential form $u \in L^p(E, \Lambda^l)$, $l = 1, \dots, n$, and G is the Green's operator, then there exists a constant C, independent of u, such that*

$$\left\| dd^* G(u) + d^* dG(u) + \left(d + d^*\right)G(u) + G(u) \right\|_{p,B} \le C\|u\|_{p,B}$$

for any $B \subset \Omega$.

Because $\triangle = d^*d + dd^*$ and $D = d^* + d$, by using Lemma 3.3, it implies that

$$\left\| \triangle G(u) \right\|_{p,B} \le C\|u\|_{p,B} \quad \text{and} \quad \left\| DG(u) \right\|_{p,B} \le C\|u\|_{p,B}.$$

Lemma 3.4 [10] *If $u \in L^p(D, \Lambda^l)$, $1 < p < \infty$, then $u_D \in L^p(D, \Lambda^l)$ and*

$$\|u_D\|_{p,D} \le A_p(n)\mu(D)\|u\|_D.$$

According to the above results, we can prove a very useful lemma as follows.

Lemma 3.5 *Let T be the homotopy operator, D be the Dirac operator, and G be the Green's operator. Additionally, we assume that $u \in C^\infty(\Lambda^l E)$ is a solution of the non-homogeneous A-harmonic equation, the Young function $\varphi(x)$ belongs to the $G(p,q,C)$-class, and the bounded subset $E \subseteq \mathbb{R}^n$ is the L^φ-averaging domain. Then, for any ball $B \subseteq E$, we have*

$$\left\| TDG(u) - \left(TDG(u)\right)_B \right\|_{s,B} \le C|B| \operatorname{diam}(B)\|u\|_{s,B}$$

for any $B \subset E$, where $s > 1$.

Proof Notice that $TDG(u)$ is at least differential 1-form, therefore, by using (1.1) and replacing u by $TDG(u)$, we have

$$TDG(u) = Td\left(TDG(u)\right) + dT\left(TDG(u)\right).$$

It follows from (1.2) that $dT(TDG(u)) = (TDG(u))_B$, that is,

$$\left\| TDG(u) - \left(TDG(u)\right)_B \right\|_{s,B} = \left\| Td\left(TDG(u)\right) \right\|_{s,B}.$$

Applying (1.3), Lemma 3.3, and Lemma 3.4, we obtain

$$\begin{aligned}
\left\| TDG(u) - \left(TDG(u)\right)_B \right\|_{s,B} &= \left\| Td\left(TDG(u)\right) \right\|_{s,B} \\
&\le C_1|B| \operatorname{diam}(B) \left\| d\left(TDG(u)\right) \right\|_{s,B} \\
&\le C_1|B| \operatorname{diam}(B) \left\| \left(DG(u)\right)_B \right\|_{s,B} \\
&\le C_2|B| \operatorname{diam}(B) \left\| DG(u) \right\|_{s,B} \\
&\le C_3|B| \operatorname{diam}(B)\|u\|_{s,B}.
\end{aligned}$$

Thus, we finish the proof of this lemma. □

Note By observing the proving process, we can get a valuable result as follows:

$$\left\| Td\bigl(TDG(u)\bigr)\right\|_{s,B} \le C|B| \operatorname{diam}(B)\|u\|_{s,B}.$$

Lemma 3.6 *Let T be the homotopy operator, D be the Dirac operator, and G be the Green's operator. Additionally, we assume that $u \in C^{\infty}(\Lambda^l E)$ is a solution of the non-homogeneous A-harmonic equation, the Young function $\varphi(x)$ having imposed a doubling property belongs to the $G(p,q,C)$-class and the bounded subset $E \subseteq \mathbb{R}^n$ is the L^{φ}-averaging domain. Then, for any ball $B \subseteq E$, we have*

$$\left\| TDG(u)\right\|_{L^{\varphi}_B} \le C \operatorname{diam}(B)\|u\|_{L^{\varphi}_{\sigma B}},$$

where the constant $\sigma > 1$ and $\sigma B \subseteq E$.

Proof From (1.3), we have

$$\begin{aligned}\left\| TDG(u)\right\|_{s,B} &\le C_1 \operatorname{diam}(B)|B| \left\| DG(u)\right\|_{s,B} \\ &\le C_2 \operatorname{diam}(B)|B| \|u\|_{s,B}.\end{aligned} \tag{3.1}$$

Because φ belongs to the $G(p,q,C)$-class, we have

$$\begin{aligned}&\int_B \varphi\bigl(\left|TDG(u)\right|\bigr)\,dx \\ &\quad= g \cdot g^{-1}\left(\int_B \varphi\bigl(\left|TDG(u)\right|\bigr)\,dx\right) \\ &\quad\le g\left(\int_B g^{-1}\varphi\bigl(\left|TDG(u)\right|\bigr)\,dx\right) \\ &\quad\le g\left(\int_B C_1\left|TDG(u)\right|^q\,dx\right) \\ &\quad\le C_2\varphi\left(\int_B C_1\left|TDG(u)\right|^q\,dx\right)^{\frac{1}{q}}.\end{aligned}$$

Since φ is doubling, according to (3.1) and Lemma 3.2, we have

$$\begin{aligned}&\int_B \varphi\bigl(\left|TDG(u)\right|\bigr)\,dx \\ &\quad\le C_2\varphi\bigl(C_3|B|^{1+\frac{1}{n}}\|u\|_{q,B}\bigr) \\ &\quad\le C_4\varphi\left(|B|^{1+\frac{1}{n}}|B|^{\frac{1}{p}-\frac{1}{q}}\left(\int_{\sigma B}|u|^p\,dx\right)^{\frac{1}{p}}\right) \\ &\quad\le C_5 f\left(|B|^{p(1+\frac{1}{n})}|B|^{\frac{p}{q}-1}\left(\int_{\sigma B}|u|^p\,dx\right)\right) \\ &\quad\le C_6\left(\int_{\sigma B} f\bigl(|B|^{p-1+p(\frac{1}{q}+\frac{1}{n})}+|u|^p\bigr)\,dx\right) \\ &\quad\le C_7\left(\int_{\sigma B} C_8\varphi\bigl(|B|^{1+\frac{1}{q}+\frac{1}{n}-\frac{1}{p}}|u|\bigr)\,dx\right)\end{aligned}$$

$$\leq C_9|B|^{1+\frac{1}{q}+\frac{1}{n}-\frac{1}{p}} \int_{\sigma B} \varphi(|u|)\,dx$$
$$\leq C_{10}|B|^{\frac{1}{n}} \int_{\sigma B} \varphi(|u|)\,dx. \tag{3.2}$$

Therefore, by using $|B|^{\frac{1}{n}} = C_{11}\operatorname{diam}(B)$, we can get the following result:

$$\|TDG(u)\|_{L_B^{\varphi}} \leq C\operatorname{diam}(B)\|u\|_{L_{\sigma B}^{\varphi}},$$

where $\sigma > 1$.

This is the end of the proof of Lemma 3.6. □

Lemma 3.7 *Let T be the homotopy operator, D be the Dirac operator, and G be the Green's operator. Additionally, we assume that $\varphi(|u|) \in L^1_{\mathrm{loc}}(E)$ and $u \in C^{\infty}(\Lambda^l E)$ is a solution of the non-homogeneous A-harmonic equation, the Young function $\varphi(x)$ having imposed a doubling property belongs to the $G(p,q,C)$-class and E is a bounded domain. Then we get*

$$\|\nabla Td(TDG(u))\|_{L_B^{\varphi}} \leq C|B|\operatorname{diam}(B)\|u\|_{L_{\sigma B}^{\varphi}}$$

for all balls B with $\sigma B \subset E$, where $\sigma > 1$ is some constant.

Proof According to the elementary inequality (1.3) of the homotopy operator T and Lemma 3.4, we have

$$\|\nabla Td(TDG(u))\|_{p,B} \leq C_1|B|\|d(TDG(u))\|_{p,B} \leq C_2|B|\|DG(u)\|_{p,B}. \tag{3.3}$$

Applying (3.3) and Lemma 3.3, we get

$$\|\nabla Td(TDG(u))\|_{p,B} \leq C_2|B|\operatorname{diam}(B)\|u\|_{p,B}.$$

Using the same method as used in Lemma 3.6, the following inequality holds:

$$\|\nabla Td(TDG(u))\|_{L_B^{\varphi}} \leq C\operatorname{diam}(B)\|u\|_{L_{\sigma B}^{\varphi}}.$$

The proof of Lemma 3.7 is completed. □

Lemma 3.8 (Covering lemma) [1] *Each domain Ω has a modified Whitney cover of cubes $\mathcal{V} = \{Q_i\}$ such that*

$$\bigcup_i Q_i = \Omega, \qquad \sum_{Q_i \in \mathcal{V}} \chi_{\sqrt{\frac{5}{4}}Q_i} \leq N\chi_{\Omega}$$

and some $N > 1$, and if $Q_i \cap Q_j \neq \emptyset$, then there exists a cube R (this cube does not need to be a member of $\mathcal{V}$) in $Q_i \cap Q_j$ such that $Q_i \cup Q_j \subset NR$. Moreover, if Ω is δ-John, then there is a distinguished cube $Q_0 \in \mathcal{V}$ which can be connected with every cube $Q \in \mathcal{V}$ by a chain of cubes $Q_0, Q_1, \ldots, Q_k = Q$ from $\mathcal{V}$ and such that $Q \subset \rho Q_i$, $i = 0, 1, 2, \ldots, k$, for some $\rho = \rho(n, \delta)$.

4 Demonstration of main results

According to the above definitions and lemmas, we will prove four theorems in detail. First of all, let us prove Theorem 2.1.

Proof of Theorem 2.1 First of all, similar to the proof of Lemma 3.5, we have

$$\left\| TDG(u) - \left(TDG(u)\right)_B \right\|_{L^{\varphi}_B} = \left\| Td\left(TDG(u)\right) \right\|_{L^{\varphi}_B}. \tag{4.1}$$

This means that the key point is to prove the following inequality:

$$\left\| Td\left(TDG(u)\right) \right\|_{L^{\varphi}_B} \le C\|u\|_{L^{\varphi}_{\sigma B}}.$$

According to the properties of the $G(p,q,C)$-class, we have

$$\begin{aligned}
&\int_B \varphi\left(\left|Td\left(TDG(u)\right)\right|\right) dx \\
&\quad = g \cdot g^{-1}\left(\int_B \varphi\left(\left|Td\left(TDG(u)\right)\right|\right) dx\right) \\
&\quad \le g\left(\int_B g^{-1}\varphi\left(\left|Td\left(TDG(u)\right)\right|\right) dx\right) \\
&\quad \le g\left(\int_B C_1\left|Td\left(TDG(u)\right)\right|^q dx\right) \\
&\quad \le C_2\varphi\left(\int_B C_1\left|Td\left(TDG(u)\right)\right|^q dx\right)^{\frac{1}{q}}.
\end{aligned}$$

Because φ is doubling, using Lemma 3.5, we have

$$\begin{aligned}
&\int_B \varphi\left(\left|Td\left(TDG(u)\right)\right|\right) dx \\
&\quad \le C_2\varphi\left(C_3|B|^{1+\frac{1}{n}}\|u\|_{q,B}\right) \\
&\quad \le C_4\varphi\left(|B|^{1+\frac{1}{n}}|B|^{\frac{1}{p}-\frac{1}{q}}\left(\int_{\sigma B}|u|^p dx\right)^{\frac{1}{p}}\right) \\
&\quad \le C_5 f\left(|B|^{p(1+\frac{1}{n})}|B|^{\frac{p}{q}-1}\left(\int_{\sigma B}|u|^p dx\right)\right) \\
&\quad \le C_6\left(\int_{\sigma B} f\left(|B|^{p-1+p(\frac{1}{q}+\frac{1}{n})}|u|^p\right) dx\right) \\
&\quad \le C_7\left(\int_{\sigma B} C_8\varphi\left(|B|^{1+\frac{1}{q}+\frac{1}{n}-\frac{1}{p}}|u|\right) dx\right) \\
&\quad \le C_9|B|^{1+\frac{1}{q}+\frac{1}{n}-\frac{1}{p}}\int_{\sigma B}\varphi\left(|u|\right) dx \\
&\quad \le C_{10}|B|^{\frac{1}{n}}\int_{\sigma B}\varphi\left(|u|\right) dx.
\end{aligned} \tag{4.2}$$

Combining (4.1) with (4.2) and using $|B|^{\frac{1}{n}} = C_{11}\operatorname{diam}(B)$ yield

$$\int_B \varphi\left(\left|TDG(u) - \left(TDG(u)\right)_B\right|\right) dx \le C_{12}\operatorname{diam}(B)\int_{\sigma B}\varphi\left(|u|\right) dx.$$

As a result, we obtain

$$\int_B \varphi\left(\frac{|TDG(u)-(TDG(u))_B|}{\lambda}\right)dx \le C\operatorname{diam}(B)\int_{\sigma B}\varphi\left(\frac{|u|}{\lambda}\right)dx$$

for any B with $\sigma B \subset E$ and any constant $\lambda > 0$. It means that the following inequality on L_E^{φ}-averaging domains holds:

$$\left\|TDG(u)-\bigl(TDG(u)\bigr)_B\right\|_{L_B^{\varphi}} \le C\operatorname{diam}(B)\|u\|_{L_{\sigma B}^{\varphi}}.$$

The proof of this theorem is finished. □

Remark From the proof of Theorem 2.1, it is easy to derive the following inequality:

$$\left\|Td\bigl(TDG(u)\bigr)\right\|_{L_B^{\varphi}} \le C\operatorname{diam}(B)\|u\|_{L_{\sigma B}^{\varphi}}. \tag{4.3}$$

We shall prove Theorem 2.2 by using Theorem 2.1 and Lemma 3.7.

Proof for Theorem 2.2 Using (1.1) repeatedly, we have

$$\left\|TDG(u)-\bigl(TDG(u)\bigr)_B\right\|_{W_B^{1,\varphi}} = \left\|Td\bigl(TDG(u)\bigr)\right\|_{W_B^{1,\varphi}}. \tag{4.4}$$

Combining (4.4) with (1.5), and applying Theorem 2.1 and Lemma 3.7, we have

$$\begin{aligned}\left\|TDG(u)-\bigl(TDG(u)\bigr)_B\right\|_{W_B^{1,\varphi}} &= \operatorname{diam}(B)^{-1}\left\|Td\bigl(TDG(u)\bigr)\right\|_{L_B^{\varphi}} + \left\|\nabla Td\bigl(TDG(u)\bigr)\right\|_{L_B^{\varphi}}\\ &\le \bigl(\operatorname{diam}(B)\bigr)^{-1}\bigl(C_1\operatorname{diam}(B)\|u\|_{L_{\sigma_1 B}^{\varphi}}\bigr) + C_2\|u\|_{L_{\sigma_2 B}^{\varphi}}\\ &\le C_1\|u\|_{L_{\sigma_1 B}^{\varphi}} + C_1\|u\|_{L_{\sigma_2 B}^{\varphi}}\\ &\le C\|u\|_{L_{\sigma B}^{\varphi}},\end{aligned}$$

where $\sigma = \max\{\sigma_1,\sigma_2\}$ and $\sigma > 1$, for all balls B with $\sigma B \subset E$.

This is the end of the proof of Theorem 2.2. □

Proof for Theorem 2.3 From the covering lemma, Lemma 3.8, and Lemma 3.7, we have

$$\begin{aligned}\left\|\nabla Td\bigl(TDG(u)\bigr)\right\|_{L_E^{\varphi}} &\le \sum_{B\in\mathcal{V}}\left\|\nabla Td\bigl(TDG(u)\bigr)\right\|_{L_B^{\varphi}}\\ &\le \sum_{B\in\mathcal{V}}\bigl(C_1|B|\|u\|_{L_{\sigma B}^{\varphi}}\bigr)\\ &\le C_2 N\|u\|_{L_E^{\varphi}}\\ &\le C_3\|u\|_{L_E^{\varphi}}.\end{aligned} \tag{4.5}$$

Similarly, using the covering lemma, Lemma 3.8, and (4.3) implies

$$\begin{aligned}\left\|Td\bigl(TDG(u)\bigr)\right\|_{L_E^{\varphi}} &\le \sum_{B\in\mathcal{V}}\left\|Td\bigl(TDG(u)\bigr)\right\|_{L_B^{\varphi}}\\ &\le \sum_{B\in\mathcal{V}}\bigl(C_4\operatorname{diam}(B)\|u\|_{L_{\sigma B}^{\varphi}}\bigr)\end{aligned}$$

$$\begin{aligned}&\le C_5 \operatorname{diam}(E) N \|u\|_{L_E^\varphi}\\ &\le C_6 \operatorname{diam}(E)\|u\|_{L_E^\varphi}.\end{aligned} \tag{4.6}$$

Thus, from (1.5), (4.5), and (4.6), we obtain

$$\begin{aligned}\big\|TDG(u) - \big(TDG(u)\big)_E\big\|_{W_E^{1,\varphi}} &= \big\|Td\big(TDG(u)\big)\big\|_{W_E^{1,\varphi}}\\ &= \big(\operatorname{diam}(E)\big)^{-1}\big\|Td\big(TDG(u)\big)\big\|_{L_E^\varphi} + \big\|\nabla Td\big(TDG(u)\big)\big\|_{L_E^\varphi}\\ &\le \big(\operatorname{diam}(E)\big)^{-1}\big(C_6 \operatorname{diam}(E)\|u\|_{L_E^\varphi}\big) + C_3\|u\|_{L_E^\varphi}\\ &\le C_7\|u\|_{L_E^\varphi}.\end{aligned}$$

We have completed the proof of Theorem 2.3. □

Next, we will prove Theorem 2.4 by using Definition 1.1 and Theorem 2.1.

Proof of Theorem 2.4 According to Definition 1.1, we have

$$\begin{aligned}|E|^{-1}\int_E \varphi\big(\big|TDG(u) - \big(TDG(u)\big)_{B_0}\big|\big)\,dx &\le \sup_{4B\subset E}|B|^{-1}\int_B \varphi\big(\big|TDG(u) - \big(TDG(u)\big)_B\big|\big)\,dx\\ &\le \sup_{4B\subset E}|B|^{-1}C\operatorname{diam}(B)\int_{\sigma B}\varphi\big(|u|\big)\,dx.\end{aligned}$$

Because $\sup_{B\subseteq E}\int_E \varphi(|u|)\,dx$ does not depend on B, we obtain

$$|E|^{-1}\int_E \varphi\big(\big|TDG(u) - \big(TDG(u)\big)_{B_0}\big|\big)\,dx \le \sup_{4B\subset E}|B|^{-1}C\operatorname{diam}(B)\int_E \varphi\big(|u|\big)\,dx.$$

Therefore, we have

$$\int_E \varphi\big(\lambda^{-1}\big|TDG(u) - \big(TDG(u)\big)_{B_0}\big|\big)\,dx \le C\int_E \varphi\big(\lambda^{-1}|u|\big)\,dx.$$

We finish the proof of Theorem 2.4. □

In addition, we can obtain a global estimate about the composite operators using the same method as of Theorem 2.1.

Corollary 4.1 *Let T be the homotopy operator, D be the Dirac operator, and G be the Green's operator. Additionally, we assume that $u \in C^\infty(\Lambda^l E)$ is a solution of the non-homogeneous A-harmonic equation, the Young function $\varphi(x)$ belongs to the $G(p,q,C)$-class, and the bounded subset $E \subseteq \mathbb{R}^n$ is the L^φ-averaging domain. Then, for any ball $B \subseteq E$, we have*

$$\big\|TDG(u)\big\|_{L_E^\varphi} \le C\operatorname{diam}(B)\|u\|_{L_E^\varphi}.$$

Remark If we choose $\varphi(x) = x^s$, we have

$$\big\|TDG(u)\big\|_{s,E} \le C\operatorname{diam}(B)\|u\|_{s,E}.$$

5 Applications

In this section, we will discuss the applications of the results obtained.

Example 5.1 Let $r > 0$ and $\Omega = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 \leq r^2\} \subset \mathbb{R}^3$. Consider the 1-form

$$u(x_1, x_2, x_3) = \frac{x_1}{1 + (x_1^2 + x_2^2 + x_3^2)^2}\,dx_1 + \frac{x_2}{1 + (x_1^2 + x_2^2 + x_3^2)^2}\,dx_2 + \frac{x_3}{1 + (x_1^2 + x_2^2 + x_3^2)^2}\,dx_3$$

defined in Ω. Then u is a solution of the non-homogeneous A-harmonic equation for any operators A and B satisfying the required conditions.

As the application of the obtained theorems, our goal is to get the upper bound of $TDG(u)$ satisfying the above conditions. Normally, we would consider calculating the integral of $TDG(u)$, however, one will see that $TDG(u)$ with the L^φ-norm is very complicated. In this case, we can use Theorem 2.3 to get the estimate of $TDG(u)$ with the L^φ-norm in $W_\Omega^{1,\varphi}$.

Initially, according to the condition and the expression of u in Example 5.1, we see that

$$\left|u(x_1, x_2, x_3)\right|^2 = \frac{x_1^2}{(1 + (x_1^2 + x_2^2 + x_3^2)^2)^2} + \frac{x_2^2}{(1 + (x_1^2 + x_2^2 + x_3^2)^2)^2} + \frac{x_3^2}{(1 + (x_1^2 + x_2^2 + x_3^2)^2)^2}.$$

Furthermore, because u is defined in Ω and $(x_1^2 + x_2^2 + x_3^2)^2 > 0$, we get

$$\left|u(x_1, x_2, x_3)\right| \leq 1. \tag{5.1}$$

As a result, by using Theorem 2.3 and (5.1), we have

$$\left\| TDG(u) - \left(TDG(u)\right)_\Omega \right\|_{W_\Omega^{1,\varphi}} \leq C_1 \|u\|_{L_\Omega^\varphi} \leq C_1 \|1\|_{L_\Omega^\varphi} \leq C_2 r^3.$$

The above example can be extended to the case of $\mathbb{R}^n$. Particularly, we can check that the 1-form defined in $\mathbb{R}^n$,

$$u(x_1, \dots, x_n) = \sum_{i=1}^{n} \frac{x_i}{1 + (x_1^2 + \cdots + x_n^2)^2}\,dx_i,$$

is a solution of the non-homogeneous A-harmonic equation for any operators A and B satisfying the required conditions. So, we can also apply Theorem 2.3 to this extended case as we did in Example 5.1.

To end this section, we take a 3-form in $\mathbb{R}^3$ as an example.

Example 5.2 Let $k, m > 0$ be constants. Consider a differential 3-form in $\mathbb{R}^3$,

$$u(x, y, z) = \frac{k}{m + x^2 + y^2 + z^2}\,dx \wedge dy \wedge dz.$$

Then u is a solution of the non-homogeneous A-harmonic equation for any operators A and B satisfying the required conditions.

Similarly, applying the same method, we can obtain the following result as regards $TDG(u)$ in any bounded set $\Omega \subset \mathbb{R}^3$:

$$\left|u(x,y,z)\right| \le \frac{k}{m}$$

and

$$\left\| TDG(u) - \left(TDG(u)\right)_\Omega \right\|_{W^{1,\varphi}_\Omega} \le C\|u\|_{L^\varphi_\Omega} \le C\|k/m\|_{L^\varphi_\Omega}.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All results and investigations of this manuscript were due to the joint efforts of all authors. GS mainly studied the theoretical proofs and drafted the manuscript. SD and YX conceived of the initial idea of this manuscript and improved the final version. All authors read and approved the final manuscript.

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