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Embedding theorems for composition of homotopy and projection operators

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Abstract

We prove both local and global embedding theorems with L^φ -norms for the composition of the homotopy operator and projection operator applied to differential forms. We also establish some L^φ -norm inequalities for certain compositions of the related operators.

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1 Introduction

The homotopy operator T and projection operator H defined on differential forms are two critical operators which have been very well studied and used in recent years, see [1–7]. In many situations, we need to deal with the composition $T \circ H$ of the homotopy operator T and the projection operator H . For example, when we consider the decomposition of $H(u)$, we have to face the composition $T \circ H$. The study of the composition $T \circ H$ of homotopy and projection operators was initiated by Ding and Liu in 2009 in [5] and [6], respectively, where they investigated singular integrals of this composite operator and established some L^p inequalities for the composite operator $T \circ H$ with singular factors. Later, in 2011, Bi and Ding proved some L^φ -estimates for this composite operator $T \circ H$ in [7], where φ satisfies the $G(p, q, C)$ condition. The purpose of this paper is to establish the L^φ -embedding theorems for the composition $T \circ H$ applied to differential forms, here φ satisfies the $NG(p, q)$ condition. If we choose $\varphi(t) = t^p$, the L^φ -norm inequalities reduce to L^p -norm inequalities. Our main L^φ -embedding inequality for the composite operator can be simply stated as

$$\|T(H(u)) - (T(H(u)))\|_{W^{1,\varphi}(\Omega)} \leq C\|u\|_{L^\varphi(\Omega)}, \quad (1.1)$$

where Ω is any bounded domain in \mathbb{R}^n , $n \geq 2$, $\varphi : [0, \infty) \rightarrow [0, \infty)$ with $\varphi(0) = 0$ is a Young function satisfying certain conditions described later, and C is a constant independent of the differential form u . In order to establish the above main L^φ -embedding inequality, we also prove the Poincaré inequality and some inequalities with L^φ -norm for the related compositions of operators.

We keep using the traditional notations throughout this paper. Let B and σB be the balls with the same center and $\text{diam}(\sigma B) = \sigma \text{diam}(B)$. Let $|E|$ be the n -dimensional Lebesgue measure of a set $E \subseteq \mathbb{R}^n$. In this paper, we treat a ball same as a cube and use $u_B = \frac{1}{|B|} \int_B u \, dx$ to denote the average of a function u . Let $\wedge^l = \wedge^l(\mathbb{R}^n)$ be the set of all l -forms in \mathbb{R}^n , $D^l(\Omega, \wedge^l)$ be the space of all differential l -forms in Ω , and $L^p(\Omega, \wedge^l)$ be the l -forms $u(x) = \sum_I u_I(x) \, dx_I$ in Ω satisfying $\int_\Omega |u_I|^p < \infty$ for all ordered l -tuples $I, l = 1, 2, \dots, n$. We denote the exterior derivative by d and the Hodge star operator by \star .

The definition of the operator K_y with the case $y = 0$ and its generalized version can be found in [8, 9]. To each $y \in \Omega$ there corresponds a linear operator $K_y : C^\infty(\Omega, \wedge^l) \rightarrow C^\infty(\Omega, \wedge^{l-1})$ defined by $(K_y \omega)(x; \xi_1, \dots, \xi_{l-1}) = \int_0^1 t^{l-1} \omega(tx + y - ty; x - y, \xi_1, \dots, \xi_{l-1}) \, dt$ and the decomposition $\omega = d(K_y \omega) + K_y(d\omega)$. A homotopy operator $T : C^\infty(\Omega, \wedge^l) \rightarrow C^\infty(\Omega, \wedge^{l-1})$ is defined by averaging K_y over all points $y \in \Omega$: $T\omega = \int_\Omega \phi(y) K_y \omega \, dy$, where $\phi \in C_0^\infty(\Omega)$ is normalized so that $\int \phi(y) \, dy = 1$. For each differential form u , we have the decomposition

$$u = d(Tu) + T(du) \tag{1.2}$$

and

$$\|\nabla(Tu)\|_{p,B} \leq C|B|\|u\|_{p,B} \quad \text{and} \quad \|Tu\|_{p,B} \leq C|B| \text{diam}(B)\|u\|_{p,B}. \tag{1.3}$$

From [10], p.16, we know that any open subset Ω in \mathbb{R}^n is the union of a sequence of cubes Q_k , whose sides are parallel to the axes, whose interiors are mutually disjoint, and whose diameters are approximately proportional to their distances from F , where F is the complement of Ω in \mathbb{R}^n . Specifically, (i) $\Omega = \bigcup_{k=1}^\infty Q_k$, (ii) $Q_j \cap Q_k = \emptyset$ if $j \neq k$, (iii) there exist two constants $c_1, c_2 > 0$ (we can take $c_1 = 1$, and $c_2 = 4$), so that $c_1 \text{diam}(Q_k) \leq \text{distance}(Q_k, F) \leq c_2 \text{diam}(Q_k)$. Thus, the definition of the homotopy operator T can be generalized to any domain Ω in \mathbb{R}^n : For any $x \in \Omega, x \in Q_k$ for some k . Let T_{Q_k} be the homotopy operator defined on Q_k (each cube is bounded and convex). Thus, we can define the homotopy operator T_Ω on any domain Ω by $T_\Omega = \sum_{k=1}^\infty T_{Q_k} \chi_{Q_k}(x)$. The nonlinear partial differential equation for differential forms

$$d^* A(x, du) = B(x, du) \tag{1.4}$$

is called a non-homogeneous A -harmonic equation, where $A : \Omega \times \wedge^l(\mathbb{R}^n) \rightarrow \wedge^l(\mathbb{R}^n)$ and $B : \Omega \times \wedge^l(\mathbb{R}^n) \rightarrow \wedge^{l-1}(\mathbb{R}^n)$ satisfy the conditions:

$$|A(x, \xi)| \leq a|\xi|^{p-1}, \quad A(x, \xi) \cdot \xi \geq |\xi|^p \quad \text{and} \quad |B(x, \xi)| \leq b|\xi|^{p-1} \tag{1.5}$$

for $x \in \Omega$ a.e. and all $\xi \in \wedge^l(\mathbb{R}^n)$. Here $p > 1$ is a constant related to the equation (1.4), and $a, b > 0$. See [11–17] for recent results on the A -harmonic equations and related topics. Assume that $\wedge^l \Omega$ is the l th exterior power of the cotangent bundle, $C^\infty(\wedge^l \Omega)$ is the space of smooth l -forms on Ω and $\mathcal{W}(\wedge^l \Omega) = \{u \in L^1_{\text{loc}}(\wedge^l \Omega) : u \text{ has generalized gradient}\}$. The harmonic l -fields are defined by $\mathcal{H}(\wedge^l \Omega) = \{u \in \mathcal{W}(\wedge^l \Omega) : du = d^*u = 0, u \in L^p \text{ for some } 1 < p < \infty\}$. The orthogonal complement of \mathcal{H} in L^1 is defined by $\mathcal{H}^\perp = \{u \in L^1 : \langle u, h \rangle = 0 \text{ for all } h \in \mathcal{H}\}$. Then the Green’s operator G is defined as $G : C^\infty(\wedge^l \Omega) \rightarrow \mathcal{H}^\perp \cap C^\infty(\wedge^l \Omega)$

by assigning $G(u)$ be the unique element of $\mathcal{H}^\perp \cap C^\infty(\wedge^l \Omega)$ satisfying Poisson’s equation $\Delta G(u) = u - H(u)$, where H is the harmonic projection operator that maps $C^\infty(\wedge^l \Omega)$ onto \mathcal{H} so that $H(u)$ is the harmonic part of u . See [18] for more properties of these operators.

2 Local embedding theorem

The purpose of this section is to prove the local L^φ -embedding theorem and some related L^φ -norm inequalities that will be used to prove the global embedding theorem in the next section. We first recall the following subclass of Young functions that can be found in [19–21].

Definition 2.1 A Young function $\varphi : [0, \infty) \rightarrow [0, \infty)$ is said to be in the class $NG(p, q)$ if φ satisfies the nonstandard growth condition

$$p\varphi(t) \leq t\varphi'(t) \leq q\varphi(t), \quad 1 < p \leq q < \infty. \tag{2.1}$$

The first inequality in (2.1) is equivalent to that $\frac{\varphi(t)}{t^p}$ is increasing, and the second inequality in (2.1) is equivalent to Δ_2 -condition, *i.e.*, for each $t > 0$, $\varphi(2t) \leq K\varphi(t)$, where $K > 1$, and $\frac{\varphi(t)}{t^q}$ is decreasing with t . Also, condition (2.1) implies that $\varphi(t)$ satisfies

$$c_1 t^p - c_2 \leq \varphi(t) \leq c_3 (t^q + 1). \tag{2.2}$$

Particularly, $\varphi(t) = t^p$ satisfies (2.1) because of $t\varphi'(t) = p\varphi(t)$, and this makes inequalities with the norm $\|\cdot\|_p$ become a special case of Theorem 2.5; for more details see [19] and [20].

An Orlicz function is a continuously increasing function $\varphi : [0, \infty) \rightarrow [0, \infty)$ with $\varphi(0) = 0$. The Orlicz space $L^\varphi(\Omega)$ consists of all measurable functions f on Ω such that $\int_\Omega \varphi\left(\frac{|f|}{\lambda}\right) dx < \infty$ for some $\lambda = \lambda(f) > 0$. $L^\varphi(\Omega)$ is equipped with the nonlinear Luxemburg functional

$$\|f\|_{L^\varphi(\Omega)} = \inf \left\{ \lambda > 0 : \int_\Omega \varphi\left(\frac{|f|}{\lambda}\right) dx \leq 1 \right\}.$$

A convex Orlicz function φ is often called a Young function. If φ is a Young function, then $\|\cdot\|_{L^\varphi(\Omega)}$ defines a norm in $L^\varphi(\Omega)$, which is called the Luxemburg norm or Orlicz norm. For any subset $E \subset \mathbb{R}^n$, we use $W^{1,\varphi}(E, \wedge^l)$ to denote the Orlicz-Sobolev space of l -forms which equals $L^\varphi(E, \wedge^l) \cap L_1^\varphi(E, \wedge^l)$ with norm

$$\|u\|_{W^{1,\varphi}(E)} = \|u\|_{W^{1,\varphi}(E, \wedge^l)} = \text{diam}(E)^{-1} \|u\|_{L^\varphi(E)} + \|\nabla u\|_{L^\varphi(E)}. \tag{2.3}$$

If we choose $\varphi(t) = t^p$, $p > 1$ in (2.3), we obtain the usual L^p -norm for $W^{1,p}(E, \wedge^l)$

$$\|u\|_{W^{1,p}(E)} = \|u\|_{W^{1,p}(E, \wedge^l)} = \text{diam}(E)^{-1} \|u\|_{p,E} + \|\nabla u\|_{p,E}. \tag{2.4}$$

Next, we recall some lemmas that will be used in this paper.

Lemma 2.2 [9] *Let $u \in D'(B, \wedge^l)$ and $du \in L^p(B, \wedge^{l+1})$. Then $u - u_B \in L^{\frac{np}{n-p}}(B, \wedge^l)$, and*

$$\left(\int_B |u - u_B|^{\frac{np}{n-p}} dx \right)^{\frac{n-p}{np}} \leq c(n,p) \left(\int_B |du|^p dx \right)^{\frac{1}{p}} \tag{2.5}$$

for B is a ball or cube in Ω , $l = 0, 1, \dots, n - 1$ and $1 < p < n$.

Lemma 2.3 [19] *Suppose φ is a continuous function in the class $NG(p, q)$ with $q(n - p) < np$, $1 < p \leq q < \infty$. For any $t > 0$, setting*

$$A(t) = \int_0^t \left(\frac{\varphi(s^{1/q})}{s} \right)^{\frac{n+q}{q}} ds, \quad K(t) = \frac{(\varphi(t^{1/q}))^{\frac{n+q}{q}}}{t^{n/q}}. \tag{2.6}$$

Then $A(t)$ is a concave function, and there exists a constant C , such that

$$K(t) \leq A(t) \leq CK(t), \quad \forall t > 0. \tag{2.7}$$

Lemma 2.4 [1] *Let u be a differential form satisfying the non-homogeneous A -harmonic equation (1.4) in Ω , $\sigma > 1$ and $0 < s, t < \infty$. Then, there exists a constant C , independent of u , such that $\|u\|_{s,B} \leq C|B|^{(t-s)/st} \|u\|_{t,\sigma B}$ for all balls or cubes B with $\sigma B \subset \Omega$.*

We are ready to state our main local L^φ -embedding theorem as follows, which will be used to prove the global L^φ -embedding theorem in the next section.

Theorem 2.5 *Let φ be a Young function in the class $NG(p, q)$ with $q(n - p) < np$, $1 < p \leq q < \infty$. Ω be a bounded domain, $u \in L^p(\Omega, \wedge^l)$ be a solution of the non-homogeneous A -harmonic equation, T be the homotopy operator and H be the projection operator. If $\varphi(|u|) \in L^1_{loc}(\Omega)$, then there exists a constant C , independent of u , such that*

$$\|T(H(u)) - (T(H(u)))_B\|_{W^{1,\varphi}(B, \wedge^l)} \leq C|B| \|u\|_{L^\varphi(\sigma B)} \tag{2.8}$$

for all balls B with $\sigma B \subset \Omega$, where $\sigma > 1$ is a constant.

In order to prove the above local L^φ -embedding theorem, we need to prove some local L^φ -norm inequalities. We begin with the following Poincaré-type inequality with L^φ -norm first.

Theorem 2.6 *Let φ be a Young function in the class $NG(p, q)$ with $q(n - p) < np$, $1 < p \leq q < \infty$, Ω be a bounded domain, $u \in L^p(\Omega, \wedge^l)$, T be the homotopy operator and H be the projection operator. If $\varphi(|u|) \in L^1_{loc}(\Omega)$, then there exists a constant C , independent of u , such that*

$$\|T(H(u)) - (T(H(u)))_B\|_{L^\varphi(B)} \leq C|B| \|u\|_{L^\varphi(B)} \tag{2.9}$$

for all balls $B \subset \Omega$.

Proof First, we consider the case $1 < p < n$. By assumption, we have $q < \frac{np}{n-p}$. Using the Poincaré-type inequality, Lemma 2.2 to differential forms $T(H(u))$

$$\left(\int_B |T(H(u)) - (T(H(u)))_B|^{np/(n-p)} dx \right)^{(n-p)/np} \leq C_1 \left(\int_B |d(T(H(u)))|^p dx \right)^{1/p}, \tag{2.10}$$

we find that

$$\left(\int_B |T(H(u)) - (T(H(u)))_B|^q dx \right)^{1/q} \leq C_2 \left(\int_B |d(T(H(u)))|^p dx \right)^{1/p}. \tag{2.11}$$

It is well known that, for any differential form u , $d(T(u)) = u_B$ and $\|u_B\|_{p,B} \leq C_3 \|u\|_{p,B}$. Hence,

$$\left(\int_B |d(T(H(u)))|^p dx \right)^{1/p} \leq C_4 \left(\int_B |H(u)|^p dx \right)^{1/p}. \tag{2.12}$$

Note that

$$\|\Delta G(u)\|_{p,B} = \|(d^*d + dd^*)G(u)\|_{p,B} \leq C_5 \|u\|_{p,B}.$$

We have

$$\begin{aligned} \|H(u)\|_{p,B} &= \|u - \Delta G(u)\|_{p,B} \\ &\leq \|u\|_{p,B} + \|\Delta G(u)\|_{p,B} \\ &\leq C_6 \|u\|_{p,B}. \end{aligned} \tag{2.13}$$

Combining (2.11), (2.12), and (2.13), we obtain

$$\left(\int_B |T(H(u)) - (T(H(u)))_B|^q dx \right)^{1/q} \leq C_7 \left(\int_B |u|^p dx \right)^{1/p} \tag{2.14}$$

for $1 < p < n$. Next, for the case of $p \geq n$, since the L^p -norm of $|T(H(u)) - (T(H(u)))_B|$ increases with p and $\frac{np}{n-p} \rightarrow \infty$ as $p \rightarrow n$. Then, there exists $1 < p_0 < n$ such that $q < \frac{np_0}{n-p_0}$. Hence, it follows that

$$\begin{aligned} &\left(\int_B |T(H(u)) - (T(H(u)))_B|^q dx \right)^{1/q} \\ &\leq \left(\int_B |T(H(u)) - (T(H(u)))_B|^{np_0/(n-p_0)} dx \right)^{(n-p_0)/np_0} \\ &\leq C_8 \left(\int_B |d(T(H(u)))|^{p_0} dx \right)^{1/p_0} \\ &\leq C_9 \left(\int_B |d(T(H(u)))|^p dx \right)^{1/p} \\ &\leq C_{10} \left(\int_B |u|^p dx \right)^{1/p}. \end{aligned} \tag{2.15}$$

Hence, from (2.14) and (2.15), we obtain

$$\left(\int_B |T(H(u)) - (T(H(u)))_B|^q dx\right)^{1/q} \leq C_{10} \left(\int_B |u|^p dx\right)^{1/p} \tag{2.16}$$

for any $p > 1$. Using the Hölder inequality with $1 = \frac{q}{n+q} + \frac{n}{n+q}$, we obtain

$$\begin{aligned} &\int_B \varphi(|T(H(u)) - (T(H(u)))_B|) dx \\ &= \int_B \frac{\varphi(|T(H(u)) - (T(H(u)))_B|)}{|T(H(u)) - (T(H(u)))_B|^{\frac{nq}{n+q}}} |T(H(u)) - (T(H(u)))_B|^{\frac{nq}{n+q}} dx \\ &\leq \left(\int_B \frac{\varphi(|T(H(u)) - (T(H(u)))_B|^{\frac{n+q}{q}})}{|T(H(u)) - (T(H(u)))_B|^n} dx\right)^{\frac{q}{n+q}} \\ &\quad \times \left(\int_B |T(H(u)) - (T(H(u)))_B|^q dx\right)^{\frac{n}{n+q}}. \end{aligned}$$

Applying Lemma 2.3 and noticing $A(t)$ is a concave function, we obtain

$$\begin{aligned} &\int_B \varphi(|T(H(u)) - (T(H(u)))_B|) dx \\ &\leq \left(\int_B K(|T(H(u)) - (T(H(u)))_B|^q) dx\right)^{\frac{q}{n+q}} \left(\int_B |T(H(u)) - (T(H(u)))_B|^q dx\right)^{\frac{n}{n+q}} \\ &\leq \left(\int_B A(|T(H(u)) - (T(H(u)))_B|^q) dx\right)^{\frac{q}{n+q}} \left(\int_B |T(H(u)) - (T(H(u)))_B|^q dx\right)^{\frac{n}{n+q}} \\ &\leq A^{\frac{q}{n+q}} \left(\int_B (|T(H(u)) - (T(H(u)))_B|^q) dx\right) \left(\int_B |T(H(u)) - (T(H(u)))_B|^q dx\right)^{\frac{n}{n+q}} \\ &\leq C_1(n, q) K^{\frac{q}{n+q}} \left(\int_B (|T(H(u)) - (T(H(u)))_B|^q) dx\right) \\ &\quad \times \left(\int_B |T(H(u)) - (T(H(u)))_B|^q dx\right)^{\frac{n}{n+q}} \\ &= C_1(n, q) \frac{\varphi\left(\left(\int_B (|T(H(u)) - (T(H(u)))_B|^q) dx\right)^{1/q}\right)}{\left(\int_B (|T(H(u)) - (T(H(u)))_B|^q) dx\right)^{\frac{n}{n+q}}} \\ &\quad \times \left(\int_B |T(H(u)) - (T(H(u)))_B|^q dx\right)^{\frac{n}{n+q}} \\ &= C_1(n, q) \varphi\left(\left(\int_B (|T(H(u)) - (T(H(u)))_B|^q) dx\right)^{1/q}\right). \end{aligned} \tag{2.17}$$

Note that φ is increasing and satisfies Δ_2 -condition, substituting (2.16) into (2.17) gives

$$\int_B \varphi(|T(H(u)) - (T(H(u)))_B|) dx \leq C_{11} \varphi\left(\left(\int_B |u|^p dx\right)^{1/p}\right). \tag{2.18}$$

Let $h(t) = \int_0^t \frac{\varphi(s)}{s} ds$. From (2.1) we know that $\varphi(t)/t^q$ is decreasing with t , thus,

$$h(t) = \int_0^t \frac{\varphi(s)}{s} ds = \int_0^t \frac{\varphi(s)}{s^q} s^{q-1} ds \geq \varphi(t)/t^q \int_0^t \frac{1}{s^q} ds = \frac{1}{q} \varphi(t).$$

Similarly, using the fact that $\varphi(t)/t^p$ is increasing with t , we have $h(t) \leq \frac{1}{p} \varphi(t)$. Therefore,

$$\frac{1}{q} \varphi(t) \leq h(t) \leq \frac{1}{p} \varphi(t). \tag{2.19}$$

Let $g(t) = h(t^{1/p})$, then $(h(t^{1/p}))' = \frac{1}{p} \frac{\varphi(t^{1/p})}{t}$ is increasing. Hence, g is a convex function. From the definitions of g and h and using Jensen's inequality to g , we have

$$h\left(\left(\int_B |u|^p dx\right)^{1/p}\right) = g\left(\int_B |u|^p dx\right) \leq \int_B g(|u|^p) dx = \int_B h(|u|) dx. \tag{2.20}$$

Combining (2.18), (2.19), and (2.20), we have

$$\begin{aligned} & \int_B \varphi(|T(H(u)) - (T(H(u)))_B|) dx \\ & \leq C_{11} \varphi\left(\left(\int_B |u|^p dx\right)^{1/p}\right) \\ & \leq C_{12} h\left(\left(\int_B |u|^p dx\right)^{1/p}\right) \\ & \leq C_{13} \int_B h(|u|) dx \\ & \leq C_{14} \int_B \varphi(|u|) dx, \end{aligned}$$

which indicates (2.9) holds. We have completed the proof of Theorem 2.6. □

Theorem 2.7 *Let φ be a Young function in the class $NG(p, q)$ with $q(n - p) < np$, $1 < p \leq q < \infty$, Ω be a bounded domain, $u \in L^p(\Omega, \wedge^l)$ be a solution of the non-homogeneous A -harmonic equation, T be the homotopy operator and H be the projection operator. If $\varphi(|u|) \in L^1_{loc}(\Omega)$, then there exists a constant C , independent of u , such that*

$$\|TdTH(u)\|_{L^\varphi(B)} \leq C|B|\|u\|_{L^\varphi(\sigma B)}$$

for all balls B with $\sigma B \subset \Omega$, where $\sigma > 1$ is a constant.

Proof For any differential form u , we have $\|dT(u)\|_{q,B} = \|u_B\|_{q,B} \leq C_1 \|u\|_{q,B}$. From (1.3) and (2.13), it follows that

$$\begin{aligned} \|TdTH(u)\|_{q,B} & \leq C_2 |B| \text{diam}(B) \|dTH(u)\|_{q,B} \\ & = C_2 |B| \text{diam}(B) \|H(u)\|_{q,B} \\ & = C_3 |B| \text{diam}(B) \|u\|_{q,B}. \end{aligned} \tag{2.21}$$

By Lemma 2.4 and $p, q > 0$, we have

$$\|u\|_{q,B} \leq C_4 |B|^{(p-q)/pq} \|u\|_{p,\sigma B}, \tag{2.22}$$

where $\sigma > 1$. Combining (2.21) and (2.22) yields

$$\|T_dTH(u)\|_{q,B} \leq C_5 |B|^{(p-q)/pq} \|u\|_{p,\sigma B}. \tag{2.23}$$

From the Hölder inequality with $1 = \frac{q}{n+q} + \frac{n}{n+q}$, we find that

$$\begin{aligned} & \int_B \varphi(|T_dTH(u)|) \, dx \\ &= \int_B \varphi(|T_dTH(u)|) (|T_dTH(u)|)^{-\frac{nq}{n+q}} (|T_dTH(u)|)^{\frac{nq}{n+q}} \, dx \\ &\leq \left(\int_B \frac{\varphi(|T_dTH(u)|)^{\frac{n+q}{q}}}{|T_dTH(u)|^n} \, dx \right)^{\frac{q}{n+q}} \left(\int_B |T_dTH(u)|^q \, dx \right)^{\frac{n}{n+q}}. \end{aligned}$$

Using Lemma 2.3 and noticing $A(t)$ is a concave function, we obtain

$$\begin{aligned} & \int_B \varphi(|T_dTH(u)|) \, dx \\ &\leq \left(\int_B K(|T_dTH(u)|^q) \, dx \right)^{\frac{q}{n+q}} \left(\int_B |T_dTH(u)|^q \, dx \right)^{\frac{n}{n+q}} \\ &\leq \left(\int_B A(|T_dTH(u)|^q) \, dx \right)^{\frac{q}{n+q}} \left(\int_B |T_dTH(u)|^q \, dx \right)^{\frac{n}{n+q}} \\ &\leq A^{\frac{q}{n+q}} \left(\int_B (|T_dTH(u)|^q) \, dx \right) \left(\int_B |T_dTH(u)|^q \, dx \right)^{\frac{n}{n+q}} \\ &\leq C_6 K^{\frac{q}{n+q}} \left(\int_B (|T_dTH(u)|^q) \, dx \right) \left(\int_B |T_dTH(u)|^q \, dx \right)^{\frac{n}{n+q}} \\ &= C_6 \frac{\varphi\left(\left(\int_B (|T_dTH(u)|^q) \, dx\right)^{1/q}\right)}{\left(\int_B (|T_dTH(u)|^q) \, dx\right)^{\frac{n}{n+q}}} \left(\int_B |T_dTH(u)|^q \, dx \right)^{\frac{n}{n+q}} \\ &= C_6 \varphi\left(\left(\int_B (|T_dTH(u)|^q) \, dx\right)^{1/q}\right). \end{aligned} \tag{2.24}$$

Since φ is increasing and satisfies Δ_2 -condition, substituting (2.23) into (2.24), we have

$$\int_B \varphi(|T_dTH(u)|) \, dx \leq C_7 \varphi\left(\left(\int_{\sigma B} |u|^p \, dx\right)^{1/p}\right). \tag{2.25}$$

Starting from (2.25) and using the same discussion as we did in the proof of Theorem 2.6, we obtain

$$\int_B \varphi(|T_dTH(u)|) \, dx \leq C \int_{\sigma B} \varphi(|u|) \, dx.$$

We have completed the proof of Theorem 2.7. □

From (1.3) and (2.13), we have

$$\begin{aligned} \|\nabla TdTH(u)\|_{q,B} &\leq C_1|B|\|dTH(u)\|_{q,B} \\ &= C_1|B|\|(H(u))_B\|_{q,B} \\ &\leq C_2|B|\|H(u)\|_{q,B} \\ &\leq C_3|B|\|u\|_{q,B}. \end{aligned} \tag{2.26}$$

Using (2.26) and the similar techniques to the ones developed in the proof of Theorem 2.7, we obtain the following L^φ -norm estimate.

Theorem 2.8 *Let φ be a Young function in the class $NG(p, q)$ with $q(n - p) < np$, $1 < p \leq q < \infty$, Ω be a bounded domain, and $u \in L^p(\Omega, \wedge^l)$ be a solution of the non-homogeneous A -harmonic equation, T be the homotopy operator and H be the projection operator. If $\varphi(|u|) \in L^1_{loc}(\Omega)$, then there exists a constant C , independent of u , such that*

$$\|\nabla TdTH(u)\|_{L^\varphi(B)} \leq C|B|\|u\|_{L^\varphi(\sigma B)} \tag{2.27}$$

for all balls B with $\sigma B \subset \Omega$, where $\sigma > 1$ is a constant.

Proof of Theorem 2.5 By definition (2.3), Theorem 2.7 and Theorem 2.8, we have

$$\begin{aligned} &\|T(H(u)) - (T(H(u)))_B\|_{W^{1,\varphi}(B,\wedge^l)} \\ &= \|TdTH(u)\|_{W^{1,\varphi}(B,\wedge^l)} \\ &= \text{diam}(B)^{-1}\|TdTH(u)\|_{L^\varphi(B)} + \|\nabla TdTH(u)\|_{L^\varphi(B)} \\ &\leq \text{diam}(B)^{-1}C_1|B|\text{diam}(B)\|u\|_{L^\varphi(\sigma_1 B)} + C_2|B|\|u\|_{L^\varphi(\sigma_2 B)} \\ &\leq C_3|B|\|u\|_{L^\varphi(\sigma B)}, \end{aligned} \tag{2.28}$$

where $\sigma = \max\{\sigma_1, \sigma_2\}$. We have completed the proof of Theorem 2.5. □

3 Global embedding theorem

In this section, we prove our main result, the global L^φ -embedding theorem for the solutions of the non-homogeneous A -harmonic equation. We will use the following well-known covering lemma.

Lemma 3.1 *Each domain Ω has a modified Whitney cover of cubes $\mathcal{V} = \{Q_i\}$ such that*

$$\bigcup_i Q_i = \Omega, \quad \sum_{Q_i \in \mathcal{V}} \chi_{\sqrt{\frac{5}{4}}Q_i} \leq N\chi_\Omega$$

and some $N > 1$, and if $Q_i \cap Q_j \neq \emptyset$, then there exists a cube R (this cube need not be a member of \mathcal{V}) in $Q_i \cap Q_j$ such that $Q_i \cup Q_j \subset NR$. Moreover, if Ω is δ -John, then there is a distinguished cube $Q_0 \in \mathcal{V}$ which can be connected with every cube $Q \in \mathcal{V}$ by a chain of cubes $Q_0, Q_1, \dots, Q_k = Q$ from \mathcal{V} and such that $Q \subset \rho Q_i, i = 0, 1, 2, \dots, k$, for some $\rho = \rho(n, \delta)$.

We are ready to prove the following global L^φ -embedding theorem with the L^φ -norm now.

Theorem 3.2 *Let φ be a Young function in the class $NG(p, q)$ with $q(n - p) < np$, $1 < p \leq q < \infty$, $u \in L^p(\Omega, \wedge^l)$ be a solution of the non-homogeneous A -harmonic equation, T be the homotopy operator and H be the projection operator. If $\varphi(|u|) \in L^1(\Omega)$, then there exists a constant C , independent of u , such that*

$$\|T(H(u)) - (T(H(u)))_\Omega\|_{W^{1,\varphi}(\Omega, \wedge^l)} \leq C \|u\|_{L^\varphi(\Omega)} \tag{3.1}$$

for a bounded domain $\Omega \subset \mathbb{R}^n$.

Proof From the Lemma 3.1 and Theorem 2.8, we have

$$\begin{aligned} \|\nabla Td(T(H(u)))\|_{L^\varphi(\Omega)} &\leq \sum_{B \in \mathcal{V}} \|\nabla Td(T(H(u)))\|_{L^\varphi(B)} \\ &\leq \sum_{B \in \mathcal{V}} (C_1 |B| \|u\|_{L^\varphi(\sigma B)}) \\ &\leq C_2 N \|u\|_{L^\varphi(\Omega)} \\ &\leq C_3 \|u\|_{L^\varphi(\Omega)}. \end{aligned} \tag{3.2}$$

Similarly, from Theorem 2.7 and Lemma 3.1, it follows that

$$\begin{aligned} \|Td(T(H(u)))\|_{L^\varphi(\Omega)} &\leq \sum_{B \in \mathcal{V}} \|Td(T(H(u)))\|_{L^\varphi(B)} \\ &\leq \sum_{B \in \mathcal{V}} (C_4 \text{diam}(B) \|u\|_{L^\varphi(\sigma B)}) \\ &\leq C_5 \text{diam}(\Omega) N \|u\|_{L^\varphi(\Omega)} \\ &\leq C_6 \text{diam}(\Omega) \|u\|_{L^\varphi(\Omega)}. \end{aligned} \tag{3.3}$$

Using (2.3), (3.2), and (3.3), we find that

$$\begin{aligned} &\|T(H(u)) - (T(H(u)))_\Omega\|_{W^{1,\varphi}(\Omega)} \\ &= \|Td(T(H(u)))\|_{W^{1,\varphi}(\Omega)} \\ &= (\text{diam}(\Omega))^{-1} \|Td(T(H(u)))\|_{L^\varphi(\Omega)} + \|\nabla Td(T(H(u)))\|_{L^\varphi(\Omega)} \\ &\leq (\text{diam}(\Omega))^{-1} (C_6 \text{diam}(\Omega) \|u\|_{L^\varphi(\Omega)}) + C_3 \|u\|_{L^\varphi(\Omega)} \\ &\leq C_7 \|u\|_{L^\varphi(\Omega)}. \end{aligned} \tag{3.4}$$

We have completed the proof of Theorem 3.2. □

Choosing $\varphi(t) = t^p \log_+^\alpha t$ in Theorems 3.2, we have the following embedding inequality with the $L^p(\log_+^\alpha L)$ -norms.

Corollary 3.3 *Let $\varphi(t) = t^p \log_+^\alpha t$, $p \geq 1$, $\alpha \in \mathbb{R}$, $u \in L^p(\Omega, \wedge^l)$ be a solution of the non-homogeneous A -harmonic equation, T be the homotopy operator, and H be the projection operator. If $\varphi(|u|) \in L^1(\Omega)$, then there exists a constant C , independent of u , such that*

$$\|T(H(u)) - (T(H(u)))_\Omega\|_{W^{1,p} \log_+^\alpha t(\Omega)} \leq C \|u\|_{L^{t^p \log_+^\alpha t}(\Omega)} \tag{3.5}$$

for any bounded domain Ω .

Let $\varphi(t) = t^p$ in Theorem 3.2. Then, we obtain the following version of the embedding inequality with L^p -norms.

Corollary 3.4 *Let $\varphi(t) = t^p$, $p \geq 1$, $u \in L^p(\Omega, \wedge^l)$ be a solution of the non-homogeneous A -harmonic equation, T be the homotopy operator, and H be the projection operator. If $\varphi(|u|) \in L^1(\Omega)$, then there exists a constant C , independent of u , such that*

$$\|T(H(u)) - (T(H(u)))_\Omega\|_{W^{1,p}(\Omega)} \leq C \|u\|_{p,\Omega}$$

holds for any bounded domain Ω .

Similarly, Theorem 2.6 can be extended into the following global Poincaré-type inequality with L^φ -norm.

Theorem 3.5 *Let φ be a Young function in the class $NG(p, q)$ with $q(n - p) < np$, $1 < p \leq q < \infty$, $u \in L^p(\Omega, \wedge^l)$ be a solution of the non-homogeneous A -harmonic equation, T be the homotopy operator and H be the projection operator. If $\varphi(|u|) \in L^1(\Omega)$, then there exists a constant C , independent of u , such that*

$$\|T(H(u)) - (T(H(u)))_\Omega\|_{L^\varphi(\Omega)} \leq C \text{diam}(\Omega) \|u\|_{L^\varphi(\Omega)} \tag{3.6}$$

for any bounded domain Ω .

Proof From (2.3) and Theorem 3.2, we have

$$\begin{aligned} & \|T(H(u)) - (T(H(u)))_\Omega\|_{L^\varphi(\Omega)} \\ &= \text{diam}(\Omega) \left((\text{diam}(\Omega))^{-1} \|T(H(u)) - (T(H(u)))_\Omega\|_{L^\varphi(\Omega)} \right) \\ &\leq \text{diam}(\Omega) \left((\text{diam}(\Omega))^{-1} \|T(H(u)) - (T(H(u)))_\Omega\|_{L^\varphi(\Omega)} \right) \\ &\quad + \|\nabla(T(H(u)) - (T(H(u)))_\Omega)\|_{L^\varphi(\Omega)} \\ &= C_1 \text{diam}(\Omega) \|T(H(u)) - (T(H(u)))_\Omega\|_{W^{1,\varphi}(\Omega, \wedge^l)} \\ &\leq C_2 \text{diam}(\Omega) \|u\|_{L^\varphi(\Omega)}. \end{aligned}$$

We have completed the proof of Theorem 3.5. □

4 Applications

As applications of our main results established in the previous sections, we consider the following examples.

Example 4.1 Assume that $r > 0$ and $k > 0$ are any constants and $\Omega = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 \leq r^2\} \subset \mathbb{R}^3$. Consider the 1-form

$$u(x_1, x_2, x_3) = \frac{x_1}{k + x_1^2 + x_2^2 + x_3^2} dx_1 + \frac{x_2}{k + x_1^2 + x_2^2 + x_3^2} dx_2 + \frac{x_3}{k + x_1^2 + x_2^2 + x_3^2} dx_3 \tag{4.1}$$

defined in Ω . It is easy to check that $du = 0$. Hence, u is a solution of the non-homogeneous A -harmonic equation (1.4) for any operators A and B satisfying (1.5). Also, it can be calculated that

$$\begin{aligned} |u| &= \left(\left(\frac{x_1}{k + x_1^2 + x_2^2 + x_3^2} \right)^2 + \left(\frac{x_2}{k + x_1^2 + x_2^2 + x_3^2} \right)^2 + \left(\frac{x_3}{k + x_1^2 + x_2^2 + x_3^2} \right)^2 \right)^{1/2} \\ &= \left(\frac{x_1^2 + x_2^2 + x_3^2}{(k + x_1^2 + x_2^2 + x_3^2)^2} \right)^{1/2} < 1. \end{aligned} \tag{4.2}$$

Using (4.2) and (3.1), we find that

$$\|T(H(u)) - (T(H(u)))_\Omega\|_{W^{1,\varphi}(\Omega)} \leq C_1 \|u\|_{L^\varphi(\Omega)} \leq C_2 \|1\|_{L^\varphi(\Omega)},$$

that is,

$$\|T(H(u)) - (T(H(u)))_\Omega\|_{W^{1,\varphi}(\Omega)} \leq C_2 \int_\Omega \varphi\left(\frac{1}{\lambda}\right) dx \leq C_3 r^3.$$

We should notice that the above example can be extended to the case of \mathbb{R}^n . Specifically, we can check that the 1-form defined in \mathbb{R}^n

$$u(x_1, \dots, x_n) = \sum_{i=1}^n \frac{x_i}{k + x_1^2 + \dots + x_n^2} dx_i, \quad k > 0 \tag{4.3}$$

is a solution of the non-homogeneous A -harmonic equation (1.4) for any operators A and B satisfying (1.5). Hence, Theorem 3.2 is applicable to $u(x_1, \dots, x_n)$ as we did in Example 4.1. Finally, we consider the following example in \mathbb{R}^3 .

Example 4.2 Let $\Omega = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\} \subset \mathbb{R}^3$ and $u(x, y, z)$ be defined in \mathbb{R}^3 by

$$u(x, y, z) = e^{x^2+y^2+z^2} (x dx + y dy + z dz).$$

It is easy to check that $du = 0$. Hence, u is a solution of the non-homogeneous A -harmonic equation (1.4) for any operators A and B satisfying (1.5) in \mathbb{R}^3 . For any bounded domain Ω in \mathbb{R}^3 , it would be very complicated if we calculate the integral $\|T(H(u)) - (T(H(u)))_\Omega\|_{W^{1,\varphi}(\Omega)}$ directly. However, using the embedding inequality (3.1), we can easily obtain the upper bound of the norm $\|T(H(u)) - (T(H(u)))_\Omega\|_{W^{1,\varphi}(\Omega)}$ as follows. We notice

that

$$\begin{aligned}
 |u(x, y, z)| &= \left((e^{x^2+y^2+z^2} x)^2 + (e^{x^2+y^2+z^2} y)^2 + (e^{x^2+y^2+z^2} z)^2 \right)^{1/2} \\
 &= \left(e^{2(x^2+y^2+z^2)} (x^2 + y^2 + z^2) \right)^{1/2} \\
 &\leq (e^2 \cdot 1)^{1/2} = e.
 \end{aligned}
 \tag{4.4}$$

From (4.4) and (3.1), we have

$$\|T(H(u)) - (T(H(u)))_{\Omega}\|_{W^{1,\varphi}(\Omega)} \leq C_1 \|u\|_{L^{\varphi}(\Omega)} \leq C_2 \|e\|_{L^{\varphi}(\Omega)},$$

which is equal to

$$\|T(H(u)) - (T(H(u)))_{\Omega}\|_{W^{1,\varphi}(\Omega)} \leq C_2 \int_{\Omega} \varphi\left(\frac{e}{\lambda}\right) dx \leq C_3.$$

Remark (i) From inequalities (3.1), (3.2), and (3.3), we find that the compositions TH , $TdTH$ and $\nabla TdTH$ are bounded operators on $L^p(\Omega, \wedge^l)$. (ii) Note that our global embedding theorem holds on any bounded domains. Hence, the theorem is true if Ω is one of the bounded John domains, L^p -averaging domains or $L^{\varphi}(\mu)$ -averaging domains. See [1, 22, 23] for more properties of these kinds of domains.

Competing interests

The authors declare that there is no conflict of interests regarding the publication of this article.

Authors' contributions

All authors worked together on the research and writing of this manuscript. JN carried out the proofs of all research results in this manuscript, and wrote its draft. SD and YX proposed the study, participated in its design and revised its final version. All authors read and approved the final manuscript.

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