Lai Journal of Inequalities and Applications (2015) 2015:357
DOI 10.1186/s13660-015-0874-2

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

The integer part of a nonlinear form with integer variables

Kai Lai*

*Correspondence:
k.ching.lai@foxmail.com
College of Computer and Information Engineering, Henan University of Economics and Law, Zhengzhou, 450011, P.R. China

Abstract

Using the Davenport-Heilbronn method, we show that if $\lambda_1, \lambda_2, \ldots, \lambda_9$ are positive real numbers, at least one of the ratios λ_i/λ_j $(1 \le i < j \le 9)$ is irrational, then the integer parts of $\lambda_1 x_1^3 + \lambda_2 x_2^3 + \lambda_3 x_3^4 + \lambda_4 x_4^4 + \lambda_5 x_5^5 + \cdots + \lambda_9 x_9^5$ are prime infinitely often for natural numbers $x_1, x_2, \ldots, x_9$.

Keywords: Davenport-Heilbronn method; integer variables; diophantine approximation

1 Introduction

In 2010, Brüdern *et al.* [1] proved that if $\lambda_1, \ldots, \lambda_s$ are positive real numbers, λ_1/λ_2 is irrational, all Dirichlet L-functions satisfy the Riemann hypothesis $s \ge \frac{8}{3}k + 2$, then the integer parts of

$$\lambda_1 x_1^k + \lambda_2 x_2^k + \cdots + \lambda_s x_s^k$$

are prime infinitely often for natural numbers x_j.

Motivated by [1], using the Davenport-Heilbronn method, we consider the integer part of a nonlinear form with integer variables and mixed powers 3, 4 and 5, and establish one result as follows.

Theorem 1.1 *Let $\lambda_1, \lambda_2, \ldots, \lambda_9$ be positive real numbers, at least one of the ratios λ_i/λ_j $(1 \le i < j \le 9)$ is irrational. Then the integer parts of*

$$\lambda_1 x_1^3 + \lambda_2 x_2^3 + \lambda_3 x_3^4 + \lambda_4 x_4^4 + \lambda_5 x_5^5 + \cdots + \lambda_9 x_9^5$$

are prime infinitely often for natural numbers $x_1, x_2, \ldots, x_9$.

It is noted that Theorem 1.1 holds without the Riemann hypothesis.

2 Notation

Throughout, we use p to denote a prime number and x_j to denote a natural number. We denote by δ a sufficiently small positive number and by ε an arbitrarily small positive number. Constants, both explicit and implicit, in Landau or Vinogradov symbols may depend on $\lambda_1, \lambda_2, \ldots, \lambda_9$. We write $e(x) = \exp(2\pi ix)$. We use $[x]$ to denote the integer part of real variable x. We take X to be the basic parameter, a large real integer. Since at least one of the ratios λ_i/λ_j $(1 \le i < j \le 9)$ is irrational, without loss of generality we may assume that λ_1/λ_2 is irrational. For the other cases, the only difference is in the following intermediate region, and we may deal with the same method in Section 4.

Since λ_1/λ_2 is irrational, then there are infinitely many pairs of integers q, a with $|\lambda_1/\lambda_2 - a/q| \le q^{-2}$, $(a,q) = 1$, $q > 0$ and $a \ne 0$. We choose q to be large in terms of $\lambda_1, \lambda_2, \ldots, \lambda_9$ and make the following definitions.

$$
\begin{aligned}
&N \asymp X, \qquad L = \log N, \qquad \bigl[N^{1-8\delta}\bigr] = q, \qquad \tau = N^{-1+\delta},\\
&Q = \bigl(|\lambda_1|^{-1} + |\lambda_2|^{-1}\bigr)N^{1-\delta}, \qquad P = N^{6\delta}, \qquad T = N^{\frac{1}{3}}.
\end{aligned}
$$

Let ν be a positive real number, we define

$$
\begin{aligned}
&K_\nu(\alpha) = \nu\left(\frac{\sin \pi\nu\alpha}{\pi\nu\alpha}\right)^2, \quad \alpha \ne 0, \qquad K_\nu(0) = \nu, \qquad (2.1)\\
&F_i(\alpha) = \sum_{1\le x\le X^{\frac{1}{3}}} e\bigl(\alpha x^3\bigr), \quad i = 1,2,\\
&F_j(\alpha) = \sum_{1\le x\le X^{\frac{1}{4}}} e\bigl(\alpha x^4\bigr), \quad j = 3,4,\\
&F_k(\alpha) = \sum_{1\le x\le X^{\frac{1}{5}}} e\bigl(\alpha x^5\bigr), \quad k = 5,\ldots,9,\\
&G(\alpha) = \sum_{p\le N} (\log p) e(\alpha p),\\
&f_i(\alpha) = \int_1^{X^{\frac{1}{3}}} e\bigl(\alpha x^3\bigr)\,dx, \quad i = 1,2,\\
&f_j(\alpha) = \int_1^{X^{\frac{1}{4}}} e\bigl(\alpha x^4\bigr)\,dx, \quad j = 3,4,\\
&f_k(\alpha) = \int_1^{X^{\frac{1}{5}}} e\bigl(\alpha x^5\bigr)\,dx, \quad k = 5,\ldots,9,\\
&g(\alpha) = \int_1^{N} e(\alpha x)\,dx.
\end{aligned}
$$

It follows from (2.1) that

$$
K_\nu(\alpha) \ll \min\bigl(\nu, \nu^{-1}|\alpha|^{-2}\bigr), \qquad (2.2)
$$

$$
\int_{-\infty}^{+\infty} e(\alpha y)K_\nu(\alpha)\,d\alpha = \max\bigl(0, 1 - \nu^{-1}|y|\bigr). \qquad (2.3)
$$

From (2.3) it is clear that

$$
\begin{aligned}
J =: &\int_{-\infty}^{+\infty} \prod_{i=1}^{9} F_i(\lambda_i \alpha) G(-\alpha) e\left(-\frac{1}{2}\alpha\right) K_{\frac{1}{2}}(\alpha)\, d\alpha \\
&\le \log N \sum_{\substack{|\lambda_1 x_1^3+\lambda_2 x_2^3+\lambda_3 x_3^4+\lambda_4 x_4^4+\lambda_5 x_5^5+\cdots+\lambda_9 x_9^5-p-\frac{1}{2}|<\frac{1}{2} \\ 1\le x_1,x_2\le X^{1/3},1\le x_3,x_4\le X^{1/4},1\le x_5,\ldots,x_9\le X^{1/5},p\le N}} 1 \\
&=: (\log N)\mathcal{N}(X),
\end{aligned}
$$

thus

$$
\mathcal{N}(X) \ge (\log N)^{-1} J.
$$

To estimate J, we split the range of infinite integration into three sections, traditional named the neighborhood of the origin $\mathfrak{C} = \{\alpha \in \mathbb{R} : |\alpha| \le \tau\}$, the intermediate region $\mathfrak{D} = \{\alpha \in \mathbb{R} : \tau < |\alpha| \le P\}$ and the trivial region $\mathfrak{c} = \{\alpha \in \mathbb{R} : |\alpha| > P\}$.

3 The neighborhood of the origin

Lemma 3.1 *If* $\alpha = a/q + \beta$, *where* $(a, q) = 1$, *then*

$$
\sum_{1\le x\le N^{1/t}} e\left(\alpha x^t\right) = q^{-1} \sum_{m=1}^{q} e\left(am^t/q\right) \int_1^{N^{1/t}} e\left(\beta y^t\right) dy + O\left(q^{1/2+\varepsilon}\left(1 + N|\beta|\right)\right).
$$

Proof This is Theorem 4.1 of [2]. □

If $|\alpha| \in \mathfrak{C}$, by Lemma 3.1, taking $a = 0$, $q = 1$, then

$$
F_i(\alpha) = f_i(\alpha) + O\left(X^{\delta}\right), \quad i = 1, 2, \ldots, 9. \tag{3.1}
$$

Lemma 3.2 *Let* $\rho = \beta + i\gamma$ *be a typical zero of the Riemann zeta function*, C *be a positive constant*,

$$
I(\alpha) = \sum_{|\gamma|\le T, \beta\ge\frac{2}{3}} \sum_{n\le N} n^{\rho-1} e(n\alpha), \qquad J(\alpha) = O\left(\left(1 + |\alpha|N\right) N^{\frac{2}{3}} L^C\right),
$$

then

$$
G(\alpha) = g(\alpha) - I(\alpha) + J(\alpha), \tag{3.2}
$$

$$
\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|I(\alpha)\right|^2 d\alpha \ll N \exp\left(-L^{\frac{1}{5}}\right), \tag{3.3}
$$

$$
\int_{-\tau}^{\tau} \left|J(\alpha)\right|^2 d\alpha \ll N \exp\left(-L^{\frac{1}{5}}\right). \tag{3.4}
$$

Proof Equations (3.2), (3.3), (3.4) can be seen from Lemma 5, (29) and (33) given by Vaughan [3]. □

Lemma 3.3 *We have*

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|f_i(\alpha)\right|^2 d\alpha \ll X^{-\frac{1}{3}}, \quad i = 1, 2,$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|f_j(\alpha)\right|^2 d\alpha \ll X^{-\frac{1}{2}}, \quad j = 3, 4,$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|f_k(\alpha)\right|^2 d\alpha \ll X^{-\frac{3}{5}}, \quad k = 5, \ldots, 9.$$

Proof These results are from Lemma 5 of [3]. □

Lemma 3.4 *We have*

$$\int_{\mathfrak{C}} \left| \prod_{i=1}^{9} F_i(\lambda_i\alpha)G(-\alpha) - \prod_{i=1}^{9} f_i(\lambda_i\alpha)g(-\alpha) \right| K_{\frac{1}{2}}(\alpha)\, d\alpha \ll X^{\frac{13}{6}} L^{-1}.$$

Proof It is obvious that

$$F_i(\lambda_i\alpha) \ll X^{\frac{1}{3}}, \qquad f_i(\lambda_i\alpha) \ll X^{\frac{1}{3}}, \quad i = 1, 2,$$

$$F_j(\lambda_j\alpha) \ll X^{\frac{1}{4}}, \qquad f_j(\lambda_j\alpha) \ll X^{\frac{1}{4}}, \quad j = 3, 4,$$

$$F_k(\lambda_k\alpha) \ll X^{\frac{1}{5}}, \qquad f_k(\lambda_k\alpha) \ll X^{\frac{1}{5}}, \quad k = 5, \ldots, 9,$$

$$G(-\alpha) \ll N, \qquad g(-\alpha) \ll N,$$

$$\begin{aligned}
&\prod_{i=1}^{9} F_i(\lambda_i\alpha)G(-\alpha) - \prod_{i=1}^{9} f_i(\lambda_i\alpha)g(-\alpha) \\
&\quad = \big(F_1(\lambda_1\alpha) - f_1(\lambda_1\alpha)\big) \prod_{i=2}^{9} F_i(\lambda_i\alpha)G(-\alpha) \\
&\qquad + \big(F_2(\lambda_2\alpha) - f_2(\lambda_2\alpha)\big) \prod_{\substack{i=1 \\ i\neq 2}}^{9} F_i(\lambda_i\alpha)G(-\alpha) + \cdots \\
&\qquad + \big(F_9(\lambda_9\alpha) - f_9(\lambda_9\alpha)\big) \prod_{i=1}^{8} f_i(\lambda_i\alpha)G(-\alpha) + \prod_{i=1}^{9} f_i(\lambda_i\alpha)\big(G(-\alpha) - g(-\alpha)\big).
\end{aligned}$$

Then by (3.1), Lemmas 3.2 and 3.3, we have

$$\int_{\mathfrak{C}} \left| \big(F_1(\lambda_1\alpha) - f_1(\lambda_1\alpha)\big) \prod_{i=2}^{9} F_i(\lambda_i\alpha)G(-\alpha) \right| K_{\frac{1}{2}}(\alpha)\, d\alpha \ll N^{-1+\delta} X^{\delta} X^{\frac{11}{6}} N \ll X^{\frac{11}{6}+2\delta},$$

$$\begin{aligned}
&\int_{\mathfrak{C}} \left| \prod_{i=1}^{9} f_i(\lambda_i\alpha)\big(G(-\alpha) - g(-\alpha)\big) \right| K_{\frac{1}{2}}(\alpha)\, d\alpha \\
&\quad \ll X^{\frac{11}{6}} \left(\int_{\mathfrak{C}} \left|f_1(\lambda_1\alpha)\right|^2 K_{\frac{1}{2}}(\alpha)\, d\alpha \right)^{\frac{1}{2}} \left(\int_{\mathfrak{C}} \left|J(-\alpha) - I(-\alpha)\right|^2 K_{\frac{1}{2}}(\alpha)\, d\alpha \right)^{\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
&\ll X^{\frac{11}{6}}\left(\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|f_1(\lambda_1\alpha)\right|^2 d\alpha\right)^{\frac{1}{2}}\left(\int_{\mathfrak{C}}\left|J(\alpha)\right|^2 d\alpha+\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|I(\alpha)\right|^2 d\alpha\right)^{\frac{1}{2}}\\
&\ll X^{\frac{11}{6}}X^{-\frac{1}{6}}\left(N\exp\left(-L^{\frac{1}{5}}\right)\right)^{\frac{1}{2}}\\
&\ll X^{\frac{13}{6}}L^{-1}.
\end{aligned}$$

The other cases are similar, and the proof of Lemma 3.4 is completed. □

Lemma 3.5 *We have*

$$\int_{|\alpha|>N^{-1+\delta}}\left|\prod_{i=1}^{9}f_i(\lambda_i\alpha)g(-\alpha)\right|K_{\frac{1}{2}}(\alpha)\,d\alpha\ll X^{\frac{13}{6}-\frac{13}{6}\delta}.$$

Proof It follows from Vaughan [2] that for $\alpha\neq 0$,

$$\begin{aligned}
&f_i(\lambda_i\alpha)\ll|\alpha|^{-\frac{1}{3}},\quad i=1,2,\qquad f_j(\lambda_j\alpha)\ll|\alpha|^{-\frac{1}{4}},\quad j=3,4,\\
&f_k(\lambda_k\alpha)\ll|\alpha|^{-\frac{1}{5}},\quad k=5,\dots,9,\qquad g(-\alpha)\ll|\alpha|^{-1}.
\end{aligned}$$

Thus

$$\int_{|\alpha|>N^{-1+\delta}}\left|\prod_{i=1}^{9}f_i(\lambda_i\alpha)g(-\alpha)\right|K_{\frac{1}{2}}(\alpha)\,d\alpha\ll\int_{|\alpha|>N^{-1+\delta}}|\alpha|^{-\frac{19}{6}}\,d\alpha\ll X^{\frac{13}{6}-\frac{13}{6}\delta}.$$ □

Lemma 3.6 *We have*

$$\int_{-\infty}^{+\infty}\prod_{i=1}^{9}f_i(\lambda_i\alpha)g(-\alpha)e\left(-\frac{1}{2}\alpha\right)K_{\frac{1}{2}}(\alpha)\,d\alpha\gg X^{\frac{13}{6}}.$$

Proof From (2.3) one has

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\prod_{i=1}^{9}f_i(\lambda_i\alpha)g(-\alpha)e\left(-\frac{1}{2}\alpha\right)K_{\frac{1}{2}}(\alpha)\,d\alpha\\
&\quad=\int_1^{X^{\frac{1}{3}}}\int_1^{X^{\frac{1}{3}}}\int_1^{X^{\frac{1}{4}}}\int_1^{X^{\frac{1}{4}}}\int_1^{X^{\frac{1}{5}}}\cdots\int_1^{X^{\frac{1}{5}}}\int_1^{N}\int_{-\infty}^{+\infty}e\Bigg(\alpha\Bigg(\lambda_1x_1^3+\lambda_2x_2^3+\lambda_3x_3^4+\lambda_4x_4^4\\
&\qquad+\lambda_5x_5^5+\cdots+\lambda_9x_9^5-x-\frac{1}{2}\Bigg)\Bigg)K_{\frac{1}{2}}(\alpha)\,d\alpha\,dx\,dx_9\cdots dx_5\,dx_4\,dx_3\,dx_2\,dx_1\\
&\quad=\frac{1}{450{,}000}\int_1^{X}\cdots\int_1^{X}\int_1^{N}\int_{-\infty}^{+\infty}x_1^{-\frac{2}{3}}x_2^{-\frac{2}{3}}x_3^{-\frac{3}{4}}x_4^{-\frac{3}{4}}x_5^{-\frac{4}{5}}\cdots x_9^{-\frac{4}{5}}e\left(\alpha\left(\sum_{i=1}^{9}\lambda_ix_i-x-\frac{1}{2}\right)\right)\\
&\qquad\cdot K_{\frac{1}{2}}(\alpha)\,d\alpha\,dx\,dx_9\cdots dx_1\\
&\quad=\frac{1}{450{,}000}\int_1^{X}\cdots\int_1^{X}\int_1^{N}x_1^{-\frac{2}{3}}x_2^{-\frac{2}{3}}x_3^{-\frac{3}{4}}x_4^{-\frac{3}{4}}x_5^{-\frac{4}{5}}\cdots x_9^{-\frac{4}{5}}\\
&\qquad\cdot\max\left(0,\frac{1}{2}-\left|\sum_{i=1}^{9}\lambda_ix_i-x-\frac{1}{2}\right|\right)dx\,dx_9\cdots dx_1.
\end{aligned}$$

Let $|\sum_{i=1}^{9}\lambda_i x_i - x - \frac{1}{2}| \leq \frac{1}{4}$, then $\sum_{i=1}^{9}\lambda_i x_i - \frac{3}{4} \leq x \leq \sum_{i=1}^{9}\lambda_i x_i - \frac{1}{4}$. Based on

$$\sum_{i=1}^{9}\lambda_i x_i - \frac{3}{4} > 1, \qquad \sum_{i=1}^{9}\lambda_i x_i - \frac{1}{4} < N,$$

one may take

$$\lambda_j X\left(8\sum_{i=1}^{9}\lambda_i\right)^{-1} \leq x_j \leq \lambda_j X\left(4\sum_{i=1}^{9}\lambda_i\right)^{-1}, \quad j = 1,\dots,9,$$

hence

$$\int_{-\infty}^{+\infty}\prod_{i=1}^{9} f_i(\lambda_i\alpha)g(-\alpha)e\left(-\frac{1}{2}\alpha\right)K_{\frac{1}{2}}(\alpha)\,d\alpha \geq \frac{1}{3{,}600{,}000}\prod_{j=1}^{9}\lambda_j\left(8\sum_{i=1}^{9}\lambda_i\right)^{-9}X^{\frac{13}{6}}.$$

This completes the proof of Lemma 3.6. □

4 The intermediate region

Lemma 4.1 *We have*

$$\int_{-\infty}^{+\infty}\left|F_i(\lambda_i\alpha)\right|^8 K_{\frac{1}{2}}(\alpha)\,d\alpha \ll X^{\frac{5}{3}+\frac{1}{3}\varepsilon}, \quad i = 1,2, \tag{4.1}$$

$$\int_{-\infty}^{+\infty}\left|F_j(\lambda_j\alpha)\right|^{16} K_{\frac{1}{2}}(\alpha)\,d\alpha \ll X^{3+\frac{1}{4}\varepsilon}, \quad j = 3,4, \tag{4.2}$$

$$\int_{-\infty}^{+\infty}\left|F_k(\lambda_k\alpha)\right|^{32} K_{\frac{1}{2}}(\alpha)\,d\alpha \ll X^{\frac{27}{5}+\frac{1}{5}\varepsilon}, \quad k = 5,\dots,9, \tag{4.3}$$

$$\int_{-\infty}^{+\infty}\left|G(-\alpha)\right|^2 K_{\frac{1}{2}}(\alpha)\,d\alpha \ll NL. \tag{4.4}$$

Proof By (2.2) and Hua's inequality, for $i = 1,2$, we have

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\left|F_i(\lambda_i\alpha)\right|^8 K_{\frac{1}{2}}(\alpha)\,d\alpha \\
&\quad\ll \sum_{m=-\infty}^{+\infty}\int_m^{m+1}\left|F_i(\lambda_i\alpha)\right|^8 K_{\frac{1}{2}}(\alpha)\,d\alpha \\
&\quad\ll \sum_{m=0}^{1}\int_m^{m+1}\left|F_i(\lambda_i\alpha)\right|^8 d\alpha + \sum_{m=2}^{+\infty}m^{-2}\int_m^{m+1}\left|F_i(\lambda_i\alpha)\right|^8 d\alpha \\
&\quad\ll X^{\frac{5}{3}+\frac{1}{3}\varepsilon} + X^{\frac{5}{3}+\frac{1}{3}\varepsilon}\sum_{m=2}^{+\infty}m^{-2} \\
&\quad\ll X^{\frac{5}{3}+\frac{1}{3}\varepsilon}.
\end{aligned}$$

The proofs of (4.2)-(4.4) are similar to (4.1). □

Lemma 4.2 *Suppose that* $(a,q)=1$, $|\alpha-a/q|\le q^{-2}$, $\phi(x)=\alpha x^k+\alpha_1x^{k-1}+\cdots+\alpha_{k-1}x+\alpha_k$, *then*

$$\sum_{x=1}^{M}e\big(\phi(x)\big)\ll M^{1+\varepsilon}\big(q^{-1}+M^{-1}+qM^{-k}\big)^{2^{1-k}}.$$

Proof This is Lemma 2.4 (Weyl's inequality) of Vaughan [2]. □

Lemma 4.3 *For every real number* $\alpha\in\mathfrak{D}$, *let* $W(\alpha)=\min(|F_1(\lambda_1\alpha)|,|F_2(\lambda_2\alpha)|)$, *then*

$$W(\alpha)\ll X^{\frac{1}{3}-\frac{1}{4}\delta+\frac{1}{3}\varepsilon}.$$

Proof For $\alpha\in\mathfrak{D}$ and $i=1,2$, we choose a_i, q_i such that

$$|\lambda_i\alpha-a_i/q_i|\le q_i^{-1}Q^{-1} \tag{4.5}$$

with $(a_i,q_i)=1$ and $1\le q_i\le Q$.

Firstly, we note that $a_1a_2\neq 0$. Secondly, if $q_1,q_2\le P$, then

$$\left|a_2q_1\frac{\lambda_1}{\lambda_2}-a_1q_2\right|\le\left|\frac{a_2/q_2}{\lambda_2\alpha}q_1q_2\left(\lambda_1\alpha-\frac{a_1}{q_1}\right)\right|+\left|\frac{a_1/q_1}{\lambda_2\alpha}q_1q_2\left(\lambda_2\alpha-\frac{a_2}{q_2}\right)\right|\ll PQ^{-1}<\frac{1}{2q}.$$

We recall that q was chosen as the denominator of a convergent to the continued fraction for λ_1/λ_2. Thus, by Legendre's law of best approximation, we have $|q'\frac{\lambda_1}{\lambda_2}-a'|>\frac{1}{2q}$ for all integers a', q' with $1\le q'<q$, thus $|a_2q_1|\ge q=[N^{1-8\delta}]$. However, from (4.5) we have $|a_2q_1|\ll q_1q_2P\ll N^{18\delta}$, this is a contradiction. We have thus established that for at least one i, $P<q_i\ll Q$. Hence Lemma 4.2 gives the desired inequality for $W(\alpha)$. □

Lemma 4.4 *We have*

$$\int_{\mathfrak{D}}\prod_{i=1}^{9}F_i(\lambda_i\alpha)G(-\alpha)e\left(-\frac{1}{2}\alpha\right)K_{\frac{1}{2}}(\alpha)\,d\alpha\ll X^{\frac{13}{6}-\frac{1}{16}\delta+\varepsilon}.$$

Proof By Lemmas 4.1, 4.3 and Hölder's inequality, we have

$$\begin{aligned}
&\int_{\mathfrak{D}}\prod_{i=1}^{9}\big|F_i(\lambda_i\alpha)G(-\alpha)\big|K_{\frac{1}{2}}(\alpha)\,d\alpha\\
&\quad\ll\max_{\alpha\in\mathfrak{D}}\big|W(\alpha)\big|^{\frac{1}{4}}\left(\left(\int_{-\infty}^{+\infty}\big|F_1(\lambda_1\alpha)\big|^8\right)^{\frac{1}{8}}\left(\int_{-\infty}^{+\infty}\big|F_2(\lambda_2\alpha)\big|^8\right)^{\frac{3}{32}}\right.\\
&\qquad\left.+\left(\int_{-\infty}^{+\infty}\big|F_1(\lambda_1\alpha)\big|^8\right)^{\frac{3}{32}}\left(\int_{-\infty}^{+\infty}\big|F_2(\lambda_2\alpha)\big|^8\right)^{\frac{1}{8}}\right)\\
&\qquad\cdot\left(\prod_{j=3}^{4}\int_{-\infty}^{+\infty}\big|F_j(\lambda_j\alpha)\big|^{16}K_{\frac{1}{2}}(\alpha)\,d\alpha\right)^{\frac{1}{16}}\left(\prod_{k=5}^{9}\int_{-\infty}^{+\infty}\big|F_k(\lambda_k\alpha)\big|^{32}K_{\frac{1}{2}}(\alpha)\,d\alpha\right)^{\frac{1}{32}}\\
&\qquad\cdot\left(\int_{-\infty}^{+\infty}\big|G(-\alpha)\big|^2K_{\frac{1}{2}}(\alpha)\,d\alpha\right)^{\frac{1}{2}}
\end{aligned}$$

$$\ll \left(X^{\frac{1}{3}-\frac{1}{4}\delta+\frac{1}{3}\varepsilon}\right)^{\frac{1}{4}}\left(X^{\frac{5}{3}+\frac{1}{3}\varepsilon}\right)^{\frac{7}{32}}\left(X^{3+\frac{1}{4}\varepsilon}\right)^{\frac{1}{8}}\left(X^{\frac{27}{5}+\frac{1}{5}\varepsilon}\right)^{\frac{5}{32}}(NL)^{\frac{1}{2}}$$
$$\ll X^{\frac{13}{6}-\frac{1}{16}\delta+\varepsilon}. \qquad \square$$

5 The trivial region

Lemma 5.1 (Lemma 2 of [4]) *Let* $V(\alpha) = \sum e(\alpha f(x_1,\dots,x_m))$, *where* f *is any real function and the summation is over any finite set of values of* $x_1,\dots,x_m$. *Then, for any* $A > 4$, *we have*

$$\int_{|\alpha|>A}\left|V(\alpha)\right|^2 K_\nu(\alpha)\,d\alpha \le \frac{16}{A}\int_{-\infty}^{\infty}\left|V(\alpha)\right|^2 K_\nu(\alpha)\,d\alpha.$$

Lemma 5.2 *We have*

$$\int_{\mathfrak{c}}\prod_{i=1}^{9}F_i(\lambda_i\alpha)G(-\alpha)e\left(-\frac{1}{2}\alpha\right)K_{\frac{1}{2}}(\alpha)\,d\alpha \ll X^{\frac{13}{6}-6\delta+\varepsilon}.$$

Proof By Lemmas 5.1, 4.1 and Schwarz's inequality, we have

$$\begin{aligned}
&\int_{\mathfrak{c}}\prod_{i=1}^{9}F_i(\lambda_i\alpha)G(-\alpha)e\left(-\frac{1}{2}\alpha\right)K_{\frac{1}{2}}(\alpha)\,d\alpha\\
&\quad\ll \int_{\mathfrak{c}}\left|\prod_{i=1}^{9}F_i(\lambda_i\alpha)G(-\alpha)\right|K_{\frac{1}{2}}(\alpha)\,d\alpha\\
&\quad\ll \frac{1}{P}\int_{-\infty}^{+\infty}\left|\prod_{i=1}^{9}F_i(\lambda_i\alpha)G(-\alpha)\right|K_{\frac{1}{2}}(\alpha)\,d\alpha\\
&\quad\ll N^{-6\delta}\left|F_1(\lambda_1\alpha)\right|^{\frac{1}{4}}\left(\left(\int_{-\infty}^{+\infty}\left|F_1(\lambda_1\alpha)\right|^8\right)^{\frac{3}{32}}\left(\int_{-\infty}^{+\infty}\left|F_2(\lambda_2\alpha)\right|^8\right)^{\frac{1}{8}}\right)\\
&\qquad\cdot\left(\prod_{j=3}^{4}\int_{-\infty}^{+\infty}\left|F_j(\lambda_j\alpha)\right|^{16}K_{\frac{1}{2}}(\alpha)\,d\alpha\right)^{\frac{1}{16}}\left(\prod_{k=5}^{9}\int_{-\infty}^{+\infty}\left|F_k(\lambda_k\alpha)\right|^{32}K_{\frac{1}{2}}(\alpha)\,d\alpha\right)^{\frac{1}{32}}\\
&\qquad\cdot\left(\int_{-\infty}^{+\infty}\left|G(-\alpha)\right|^2K_{\frac{1}{2}}(\alpha)\,d\alpha\right)^{\frac{1}{2}}\\
&\quad\ll N^{-6\delta}\left(X^{\frac{1}{3}}\right)^{\frac{1}{4}}\left(X^{\frac{5}{3}+\frac{1}{3}\varepsilon}\right)^{\frac{7}{32}}\left(X^{3+\frac{1}{4}\varepsilon}\right)^{\frac{1}{8}}\left(X^{\frac{27}{5}+\frac{1}{5}\varepsilon}\right)^{\frac{5}{32}}(NL)^{\frac{1}{2}}\\
&\quad\ll X^{\frac{13}{6}-6\delta+\varepsilon}. \qquad \square
\end{aligned}$$

6 The proof of Theorem 1.1

From Lemmas 3.4, 3.5 and 3.6 we conclude that $J(\mathfrak{C}) \gg X^{\frac{13}{6}}$. From Lemma 4.4 it follows that $J(\mathfrak{D}) = o(X^{\frac{13}{6}})$. From Lemma 5.2 we have $J(\mathfrak{c}) = o(X^{\frac{13}{6}})$. Thus

$$J \gg X^{\frac{13}{6}}, \qquad \mathcal{N}(X) \gg X^{\frac{13}{6}}L^{-1},$$

namely, under conditions of Theorem 1.1,

$$\left|\lambda_1x_1^3+\lambda_2x_2^3+\lambda_3x_3^4+\lambda_4x_4^4+\lambda_5x_5^5+\cdots+\lambda_9x_9^5-p-\frac{1}{2}\right|<\frac{1}{2} \tag{6.1}$$

has infinitely many solutions in positive integers $x_1, x_2, \ldots, x_9$ and prime p. It is evident from (6.1) that

$$p < \lambda_1 x_1^3 + \lambda_2 x_2^3 + \lambda_3 x_3^4 + \lambda_4 x_4^4 + \lambda_5 x_5^5 + \cdots + \lambda_9 x_9^5 < p + 1,$$

and hence

$$\left[\lambda_1 x_1^3 + \lambda_2 x_2^3 + \lambda_3 x_3^4 + \lambda_4 x_4^4 + \lambda_5 x_5^5 + \cdots + \lambda_9 x_9^5\right] = p.$$

The proof of Theorem 1.1 is complete.

Competing interests
The author declares that he has no competing interests.

Received: 30 August 2015 Accepted: 28 October 2015 Published online: 11 November 2015

References

1. Brüdern, J, Kawada, K, Wooley, TD: Additive representation in thin sequences, VIII: Diophantine inequalities in review. In: Number Theory: Dreaming in Dreams. Series on Number Theory and Its Applications, vol. 6, pp. 20-79 (2010)
2. Vaughan, R: The Hardy-Littlewood Method, 2nd edn. Cambridge Tracts in Mathematics, vol. 125. Cambridge University Press, Cambridge (1997)
3. Vaughan, R: Diophantine approximation by prime numbers I. Proc. Lond. Math. Soc. **28**, 373-384 (1974)
4. Davenport, H, Roth, KF: The solubility of certain Diophantine inequalities. Mathematika **2**, 81-96 (1955)