Zhang et al. *Journal of Inequalities and Applications* (2015) 2015:340
DOI 10.1186/s13660-015-0871-5

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

On dual mixed quermassintegral quotient functions

Ping Zhang, Weidong Wang and Xiaohua Zhang*

*Correspondence:
zhangxiaohua07@163.com
Department of Mathematics, China Three Gorges University, Yichang, 443002, P.R. China

Abstract
We introduce the notion of dual mixed quermassintegral quotient functions and establish the Brunn-Minkowski inequalities for them in this paper.

MSC: 52A20; 52A40

Keywords: dual quermassintegral; dual mixed quermassintegral quotient function; Brunn-Minkowski inequality

1 Introduction and main results

The setting for this paper is Euclidean n-space $\mathbb{R}^n$. Let $\mathcal{S}_o^n$ denote the set of star bodies containing the origin in their interiors in $\mathbb{R}^n$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$, and let $V(K)$ denote the n-dimensional volume of body K. For the standard unit ball B in $\mathbb{R}^n$, we use $\omega_n = V(B)$ to denote its volume.

In 1975, Lutwak (see [1]) gave the notion of dual mixed volumes as follows: For $K_1, K_2, \ldots, K_n \in \mathcal{S}_o^n$, the dual mixed volume, $\widetilde{V}(K_1, K_2, \ldots, K_n)$, of $K_1, K_2, \ldots, K_n$ is defined by

$$\widetilde{V}(K_1, \ldots, K_n) = \frac{1}{n} \int_{S^{n-1}} \rho(K_1, u) \cdots \rho(K_n, u)\, dS(u). \tag{1.1}$$

Taking $K_1 = \cdots = K_{n-i} = K$, $K_{n-i+1} = \cdots = K_n = L$ in (1.1), we write $\widetilde{V}_i(K, L) = \widetilde{V}(K, n-i; L, i)$, where K appears $n-i$ times and L appears i times. Then

$$\widetilde{V}_i(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-i} \rho(L, u)^i\, dS(u). \tag{1.2}$$

Let $L = B$ in (1.2) and notice $\rho(B, \cdot) = 1$, and allow i is any real, then the dual quermassintegrals can be defined as follows: For $K \in \mathcal{S}_o^n$ and i is any real, the dual quermassintegrals, $\widetilde{W}_i(K)$, of K are given by (see [1])

$$\widetilde{W}_i(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-i}\, dS(u). \tag{1.3}$$

Associated with dual quermassintegrals, Zhao (see [2]) defined the dual quermassintegral quotient functions of a star body K by

$$Q_{\widetilde{W}_{i,j}(K)} = \frac{\widetilde{W}_i(K)}{\widetilde{W}_j(K)} \quad (i, j \in \mathbb{R}). \tag{1.4}$$

Further, in [2] the Brunn-Minkowski type inequalities for the dual quermassintegral quotient functions of star bodies were established as follows.

Theorem A *If $K, L \in \mathcal{S}_o^n$ and reals i, j satisfy $i \le n-1 \le j \le n$, then*

$$Q^{\frac{1}{j-i}}_{\widetilde{W}_{i,j}(K \tilde{+} L)} \le Q^{\frac{1}{j-i}}_{\widetilde{W}_{i,j}(K)} + Q^{\frac{1}{j-i}}_{\widetilde{W}_{i,j}(L)}.$$

Here $\tilde{+}$ is the radial Minkowski sum.

Theorem B *If $K, L \in \mathcal{S}_o^n$ and reals i, j satisfy $i \le 1 \le j \le n$, then*

$$Q^{\frac{n-1}{j-i}}_{\widetilde{W}_{i,j}(K \check{+} L)} \le Q^{\frac{n-1}{j-i}}_{\widetilde{W}_{i,j}(K)} + Q^{\frac{n-1}{j-i}}_{\widetilde{W}_{i,j}(L)}.$$

Here $\check{+}$ is the radial Blaschke sum.

Theorem C *If $K, L \in \mathcal{S}_o^n$ and reals i, j satisfy $i \le -1 \le j \le n$, then*

$$\frac{Q^{\frac{n+1}{j-i}}_{\widetilde{W}_{i,j}(K \hat{+} L)}}{V(K \hat{+} L)} \le \frac{Q^{\frac{n+1}{j-i}}_{\widetilde{W}_{i,j}(K)}}{V(K)} + \frac{Q^{\frac{n+1}{j-i}}_{\widetilde{W}_{i,j}(L)}}{V(L)}.$$

Here $\hat{+}$ is the harmonic Blaschke sum.

Motivated by the work of Zhao, we give the following definition of dual mixed quermassintegral quotient function.

Let $K_1 = \cdots = K_{n-i-1} = K$, $K_{n-i} = \cdots = K_{n-1} = B$, $K_n = L$ in (1.1), then we write $\widetilde{W}_i(K, L) = \widetilde{V}(K, n-i-1; B, i; L, 1)$, where K appears $n-i-1$ times, B appears i times and L appears 1 time. Here, we allow i to be any real and define as follows: For $K, L \in \mathcal{S}_o^n$ and i any real, the dual mixed quermassintegrals, $\widetilde{W}_i(K, L)$, of K and L are given by

$$\widetilde{W}_i(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-i-1} \rho(L, u)\, dS(u). \tag{1.5}$$

Obviously, from (1.3) and (1.5), we have $\widetilde{W}_i(K, K) = \widetilde{W}_i(K)$. According to (1.5), we define the following.

Definition 1.1 Let $K, L \in \mathcal{S}_o^n$ and $i, j \in \mathbb{R}$, the dual mixed quermassintegral quotient function, $Q_{\widetilde{W}_{i,j}(K,L)}$, of K and L can be defined by

$$Q_{\widetilde{W}_{i,j}(K,L)} = \frac{\widetilde{W}_i(K, L)}{\widetilde{W}_j(K, L)}. \tag{1.6}$$

Obviously, if $L = K$, then (1.6) is just (1.4).

The aim of this paper is to establish the following Brunn-Minkowski type inequalities for dual mixed quermassintegral quotient functions of star bodies.

Theorem 1.1 *For $K, K', L \in \mathcal{S}_o^n$, if $i \le n-2 \le j < n-1$, then*

$$Q^{\frac{1}{j-i}}_{\widetilde{W}_{i,j}(K \tilde{+} K', L)} \le Q^{\frac{1}{j-i}}_{\widetilde{W}_{i,j}(K,L)} + Q^{\frac{1}{j-i}}_{\widetilde{W}_{i,j}(K',L)}; \tag{1.7}$$

if $n-2 \le i < n-1 < j$, *then*

$$Q^{\frac{1}{j-i}}_{\widetilde{W}_{i,j}(K \tilde{+} K',L)} \ge Q^{\frac{1}{j-i}}_{\widetilde{W}_{i,j}(K,L)} + Q^{\frac{1}{j-i}}_{\widetilde{W}_{i,j}(K',L)}. \tag{1.8}$$

In each case, equality holds if and only if K *and* K' *are dilates. Here* $\tilde{+}$ *is the radial Minkowski sum.*

Theorem 1.2 *For* $K, K', L \in \mathcal{S}^n_o$, *if* $i \le 0 \le j < n-1$, *then*

$$Q^{\frac{n-1}{j-i}}_{\widetilde{W}_{i,j}(K \check{+} K',L)} \le Q^{\frac{n-1}{j-i}}_{\widetilde{W}_{i,j}(K,L)} + Q^{\frac{n-1}{j-i}}_{\widetilde{W}_{i,j}(K',L)}; \tag{1.9}$$

if $0 \le i < n-1 < j$, *then*

$$Q^{\frac{n-1}{j-i}}_{\widetilde{W}_{i,j}(K \check{+} K',L)} \ge Q^{\frac{n-1}{j-i}}_{\widetilde{W}_{i,j}(K,L)} + Q^{\frac{n-1}{j-i}}_{\widetilde{W}_{i,j}(K',L)}. \tag{1.10}$$

In each case, equality holds if and only if K *and* K' *are dilates. Here* $\check{+}$ *is the radial Blaschke sum.*

Theorem 1.3 *For* $K, K', L \in \mathcal{S}^n_o$, *if* $i \le -2 \le j < n-1$, *then*

$$\frac{Q^{\frac{n+1}{j-i}}_{\widetilde{W}_{i,j}(K \hat{+} K',L)}}{V(K \hat{+} K')} \le \frac{Q^{\frac{n+1}{j-i}}_{\widetilde{W}_{i,j}(K,L)}}{V(K)} + \frac{Q^{\frac{n+1}{j-i}}_{\widetilde{W}_{i,j}(K',L)}}{V(K')}; \tag{1.11}$$

if $-2 \le i < n-1 < j$, *then*

$$\frac{Q^{\frac{n+1}{j-i}}_{\widetilde{W}_{i,j}(K \hat{+} K',L)}}{V(K \hat{+} K')} \ge \frac{Q^{\frac{n+1}{j-i}}_{\widetilde{W}_{i,j}(K,L)}}{V(K)} + \frac{Q^{\frac{n+1}{j-i}}_{\widetilde{W}_{i,j}(K',L)}}{V(K')}. \tag{1.12}$$

In each case, equality holds if and only if K *and* K' *are dilates. Here* $\hat{+}$ *is the harmonic Blaschke sum.*

2 Preliminaries

For a compact set K in $\mathbb{R}^n$ which is star shaped with respect to the origin, we define the radial function $\rho_K(u) = \rho(K,u)$ of K by

$$\rho(K,u) = \max\{\lambda \ge 0 : \lambda u \in K\}, \quad u \in S^{n-1}.$$

If ρ_K is positive and continuous, K will be called a star body (about the origin). Two star bodies K and L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

For $K_1, K_2 \in \mathcal{S}^n_o$, and $\lambda_1, \lambda_2 \ge 0$ (not both 0), the radial function of the radial Minkowski linear combination $\lambda_1 K_1 \tilde{+} \lambda_2 K_2$ is given by Zhang (see [3]):

$$\rho(\lambda_1 K_1 \tilde{+} \lambda_2 K_2, u) = \lambda_1 \rho(K_1,u) + \lambda_2 \rho(K_2,u). \tag{2.1}$$

For $K_1, K_2 \in \mathcal{S}_o^n$, and $\lambda_1, \lambda_2 \geq 0$ (not both 0), the radial Blaschke linear combination $\lambda_1 \cdot K_1 \check{+} \lambda_2 \cdot K_2$ is a star body whose radial function is given by Lutwak (see [4]):

$$\rho(\lambda_1 \cdot K_1 \check{+} \lambda_2 \cdot K_2, u)^{n-1} = \lambda_1 \rho(K_1, u)^{n-1} + \lambda_2 \rho(K_2, u)^{n-1}. \tag{2.2}$$

For $K_1, K_2 \in \mathcal{S}_o^n$, and $\lambda_1, \lambda_2 \geq 0$ (not both 0), the harmonic Blaschke linear combination $\lambda_1 \circ K_1 \hat{+} \lambda_2 \circ K_2$ is a star body whose radial function is given by Lutwak (see [5]):

$$\frac{\rho(\lambda_1 \circ K_1 \hat{+} \lambda_2 \circ K_2, u)^{n+1}}{V(\lambda_1 \circ K_1 \hat{+} \lambda_2 \circ K_2)} = \lambda_1 \frac{\rho(K_1, u)^{n+1}}{V(K_1)} + \lambda_2 \frac{\rho(K_2, u)^{n+1}}{V(K_2)}. \tag{2.3}$$

3 Proofs of theorems

According to a generalization of the Dresher inequality (see [6]), we get the reverse Dresher inequality.

Lemma 3.1 (Dresher's inequality) *Let functions $f_1, f_2, g_1, g_2 \geq 0$, E is a bounded measurable subset in $\mathbb{R}^n$. If $p \geq 1 \geq r \geq 0$, then*

$$\left(\frac{\int_E (f_1 + f_2)^p \, dx}{\int_E (g_1 + g_2)^r \, dx}\right)^{\frac{1}{p-r}} \leq \left(\frac{\int_E f_1^p \, dx}{\int_E g_1^r \, dx}\right)^{\frac{1}{p-r}} + \left(\frac{\int_E f_2^p \, dx}{\int_E g_2^r \, dx}\right)^{\frac{1}{p-r}}, \tag{3.1}$$

equality holds if and only if $f_1/f_2 = g_1/g_2$.

Lemma 3.2 (Reverse Dresher's inequality) *Let functions $f_1, f_2, g_1, g_2 \geq 0$, E is a bounded measurable subset in $\mathbb{R}^n$. If $1 \geq p > 0 > r$, then*

$$\left(\frac{\int_E (f_1 + f_2)^p \, dx}{\int_E (g_1 + g_2)^r \, dx}\right)^{\frac{1}{p-r}} \geq \left(\frac{\int_E f_1^p \, dx}{\int_E g_1^r \, dx}\right)^{\frac{1}{p-r}} + \left(\frac{\int_E f_2^p \, dx}{\int_E g_2^r \, dx}\right)^{\frac{1}{p-r}}, \tag{3.2}$$

equality holds if and only if $f_1/f_2 = g_1/g_2$.

Proof of Lemma 3.2 If $f_1, f_2, g_1, g_2 \geq 0$, and $1 \geq p > 0 > r$, according to the Minkowski inequality,

$$\left(\int_E (f_1 + f_2)^p \, dx\right)^{\frac{1}{p}} \geq \left(\int_E f_1^p \, dx\right)^{\frac{1}{p}} + \left(\int_E f_2^p \, dx\right)^{\frac{1}{p}},$$

$$\left(\int_E (g_1 + g_2)^r \, dx\right)^{\frac{1}{r}} \geq \left(\int_E g_1^r \, dx\right)^{\frac{1}{r}} + \left(\int_E g_2^r \, dx\right)^{\frac{1}{r}}.$$

For $1 \geq p > 0 > r$, we have

$$\int_E (f_1 + f_2)^p \, dx \geq \left(\left(\int_E f_1^p \, dx\right)^{\frac{1}{p}} + \left(\int_E f_2^p \, dx\right)^{\frac{1}{p}}\right)^p, \tag{3.3}$$

$$\int_E (g_1 + g_2)^r \, dx \leq \left(\left(\int_E g_1^r \, dx\right)^{\frac{1}{r}} + \left(\int_E g_2^r \, dx\right)^{\frac{1}{r}}\right)^r. \tag{3.4}$$

According to the Hölder inequality, $\frac{p-r}{p} > 1$, and (3.3), (3.4),

$$\begin{aligned}
\left(\frac{\int_E (f_1+f_2)^p\,dx}{\int_E (g_1+g_2)^r\,dx}\right)^{\frac{1}{p-r}} &\geq \left[\frac{((\int_E f_1^p\,dx)^{\frac{1}{p}} + (\int_E f_2^p\,dx)^{\frac{1}{p}})^p}{((\int_E g_1^r\,dx)^{\frac{1}{r}} + (\int_E g_2^r\,dx)^{\frac{1}{r}})^r}\right]^{\frac{1}{p-r}} \\
&= \frac{[(\int_E f_1^p\,dx)^{\frac{1}{p}} + (\int_E f_2^p\,dx)^{\frac{1}{p}}]^{\frac{p}{p-r}}}{[(\int_E g_1^r\,dx)^{\frac{1}{r}} + (\int_E g_2^r\,dx)^{\frac{1}{r}}]^{\frac{r}{p-r}}} \\
&= \left[\left(\left(\int_E f_1^p\,dx\right)^{\frac{1}{p-r}}\right)^{\frac{p-r}{p}} + \left(\left(\int_E f_2^p\,dx\right)^{\frac{1}{p-r}}\right)^{\frac{p-r}{p}}\right]^{\frac{p}{p-r}} \\
&\quad \times \left[\left(\left(\int_E g_1^r\,dx\right)^{\frac{-1}{p-r}}\right)^{\frac{-(p-r)}{r}} + \left(\left(\int_E g_2^r\,dx\right)^{\frac{-1}{p-r}}\right)^{\frac{-(p-r)}{r}}\right]^{\frac{-r}{p-r}} \\
&\geq \left(\int_E f_1^p\,dx\right)^{\frac{1}{p-r}}\left(\int_E g_1^r\,dx\right)^{\frac{-1}{p-r}} + \left(\int_E f_2^p\,dx\right)^{\frac{1}{p-r}}\left(\int_E g_2^r\,dx\right)^{\frac{-1}{p-r}} \\
&= \left(\frac{\int_E f_1^p\,dx}{\int_E g_1^r\,dx}\right)^{\frac{1}{p-r}} + \left(\frac{\int_E f_2^p\,dx}{\int_E g_2^r\,dx}\right)^{\frac{1}{p-r}}.
\end{aligned}$$

According to the equality condition of the Minkowski inequality and the Hölder inequality, equality holds in (3.2) if and only if $f_1/f_2 = g_1/g_2$. □

Proof of Theorem 1.1 From (2.1), for $K, K', L \in \mathcal{S}_o^n$,

$$\begin{aligned}
\widetilde{W}_{n-p-1}\big(K \tilde{+} K', L\big) &= \frac{1}{n}\int_{S^{n-1}} \rho^p_{K\tilde{+}K'}(u)\rho_L(u)\,dS(u) \\
&= \frac{1}{n}\int_{S^{n-1}} \big(\rho_K(u) + \rho_{K'}(u)\big)^p \rho_L(u)\,dS(u) \\
&= \frac{1}{n}\int_{S^{n-1}} \big(\rho_K(u)\rho_L^{\frac{1}{p}}(u) + \rho_{K'}(u)\rho_L^{\frac{1}{p}}(u)\big)^p\,dS(u)
\end{aligned} \tag{3.5}$$

and

$$\widetilde{W}_{n-r-1}\big(K \tilde{+} K', L\big) = \frac{1}{n}\int_{S^{n-1}} \big(\rho_K(u)\rho_L^{\frac{1}{r}}(u) + \rho_{K'}(u)\rho_L^{\frac{1}{r}}(u)\big)^r\,dS(u). \tag{3.6}$$

From (3.1), (3.5), and (3.6), for $p \geq 1 \geq r > 0$, we have

$$\begin{aligned}
\left(\frac{\widetilde{W}_{n-p-1}(K\tilde{+}K', L)}{\widetilde{W}_{n-r-1}(K\tilde{+}K', L)}\right)^{\frac{1}{p-r}} &= \left(\frac{\int_{S^{n-1}}(\rho_K(u)\rho_L^{\frac{1}{p}}(u) + \rho_{K'}(u)\rho_L^{\frac{1}{p}}(u))^p\,dS(u)}{\int_{S^{n-1}}(\rho_K(u)\rho_L^{\frac{1}{r}}(u) + \rho_{K'}(u)\rho_L^{\frac{1}{r}}(u))^r\,dS(u)}\right)^{\frac{1}{p-r}} \\
&\leq \left(\frac{\int_{S^{n-1}}(\rho_K(u)\rho_L^{\frac{1}{p}}(u))^p\,dS(u)}{\int_{S^{n-1}}(\rho_K(u)\rho_L^{\frac{1}{r}}(u))^r\,dS(u)}\right)^{\frac{1}{p-r}} \\
&\quad + \left(\frac{\int_{S^{n-1}}(\rho_{K'}(u)\rho_L^{\frac{1}{p}}(u))^p\,dS(u)}{\int_{S^{n-1}}(\rho_{K'}(u)\rho_L^{\frac{1}{r}}(u))^r\,dS(u)}\right)^{\frac{1}{p-r}} \\
&= \left(\frac{\int_{S^{n-1}}\rho_K^p(u)\rho_L(u)\,dS(u)}{\int_{S^{n-1}}\rho_K^r(u)\rho_L(u)\,dS(u)}\right)^{\frac{1}{p-r}}
\end{aligned}$$

$$+\left(\frac{\int_{S^{n-1}}\rho_{K'}^{p}(u)\rho_L(u)\,dS(u)}{\int_{S^{n-1}}\rho_{K'}^{r}(u)\rho_L(u)\,dS(u)}\right)^{\frac{1}{p-r}}$$
$$=\left(\frac{\widetilde{W}_{n-p-1}(K,L)}{\widetilde{W}_{n-r-1}(K,L)}\right)^{\frac{1}{p-r}}+\left(\frac{\widetilde{W}_{n-p-1}(K',L)}{\widetilde{W}_{n-r-1}(K',L)}\right)^{\frac{1}{p-r}}. \tag{3.7}$$

According to the equality condition of inequality (3.1), we see that equality holds in (3.7) if and only if K and L, K', and L are dilates, respectively. So K and K' are dilates.

Let $i=n-p-1$, $j=n-r-1$, then $p\geq 1\geq r>0$ and $i\leq n-2\leq j<n-1$ are equivalent. This and (3.7) yield inequality (1.7) and its equality condition.

Similarly, if $1\geq p>0>r$, according to (3.2), (3.5), and (3.6), we have

$$\left(\frac{\widetilde{W}_{n-p-1}(K\tilde{+}K',L)}{\widetilde{W}_{n-r-1}(K\tilde{+}K',L)}\right)^{\frac{1}{p-r}}\geq\left(\frac{\widetilde{W}_{n-p-1}(K,L)}{\widetilde{W}_{n-r-1}(K,L)}\right)^{\frac{1}{p-r}}+\left(\frac{\widetilde{W}_{n-p-1}(K',L)}{\widetilde{W}_{n-r-1}(K',L)}\right)^{\frac{1}{p-r}}, \tag{3.8}$$

and equality holds if and only if K and K' are dilates.

Let $i=n-p-1$, $j=n-r-1$, then (3.8) gives inequality (1.8) and its equality condition. □

Proof of Theorem 1.2 From (2.2), for $K,K',L\in\mathcal{S}_o^n$, we have

$$\begin{aligned}\widetilde{W}_{n-p-1}\big(K\check{+}K',L\big)&=\frac{1}{n}\int_{S^{n-1}}\rho_{K\check{+}K'}^{p}(u)\rho_L(u)\,dS(u)\\&=\frac{1}{n}\int_{S^{n-1}}\big(\rho_K^{n-1}(u)+\rho_{K'}^{n-1}(u)\big)^{\frac{p}{n-1}}\rho_L(u)\,dS(u)\\&=\frac{1}{n}\int_{S^{n-1}}\big(\rho_K^{n-1}(u)\rho_L^{\frac{n-1}{p}}(u)+\rho_{K'}^{n-1}(u)\rho_L^{\frac{n-1}{p}}(u)\big)^{\frac{p}{n-1}}\,dS(u)\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\widetilde{W}_{n-r-1}\big(K\check{+}K',L\big)&=\frac{1}{n}\int_{S^{n-1}}\rho_{K\check{+}K'}^{r}(u)\rho_L(u)\,dS(u)\\&=\frac{1}{n}\int_{S^{n-1}}\big(\rho_K^{n-1}(u)\rho_L^{\frac{n-1}{r}}(u)+\rho_{K'}^{n-1}(u)\rho_L^{\frac{n-1}{r}}(u)\big)^{\frac{r}{n-1}}\,dS(u).\end{aligned} \tag{3.10}$$

According to (3.1), (3.9), and (3.10), for $p\geq n-1\geq r>0$,

$$\begin{aligned}Q^{\frac{n-1}{p-r}}_{\widetilde{W}_{n-p-1,n-r-1}(K\check{+}K',L)}&=\left[\frac{\widetilde{W}_{n-p-1}(K\check{+}K',L)}{\widetilde{W}_{n-r-1}(K\check{+}K',L)}\right]^{\frac{n-1}{p-r}}\\&=\left[\frac{\int_{S^{n-1}}(\rho_K^{n-1}(u)\rho_L^{\frac{n-1}{p}}(u)+\rho_{K'}^{n-1}(u)\rho_L^{\frac{n-1}{p}}(u))^{\frac{p}{n-1}}\,dS(u)}{\int_{S^{n-1}}(\rho_K^{n-1}(u)\rho_L^{\frac{n-1}{r}}(u)+\rho_{K'}^{n-1}(u)\rho_L^{\frac{n-1}{r}}(u))^{\frac{r}{n-1}}\,dS(u)}\right]^{\frac{n-1}{p-r}}\\&\leq\left[\frac{\int_{S^{n-1}}(\rho_K^{n-1}(u)\rho_L^{\frac{n-1}{p}}(u))^{\frac{p}{n-1}}\,dS(u)}{\int_{S^{n-1}}(\rho_K^{n-1}(u)\rho_L^{\frac{n-1}{r}}(u))^{\frac{r}{n-1}}\,dS(u)}\right]^{\frac{n-1}{p-r}}\\&\quad+\left[\frac{\int_{S^{n-1}}(\rho_{K'}^{n-1}(u)\rho_L^{\frac{n-1}{p}}(u))^{\frac{p}{n-1}}\,dS(u)}{\int_{S^{n-1}}(\rho_{K'}^{n-1}(u)\rho_L^{\frac{n-1}{r}}(u))^{\frac{r}{n-1}}\,dS(u)}\right]^{\frac{n-1}{p-r}}\end{aligned}$$

$$
\begin{aligned}
&= \left[\frac{\int_{S^{n-1}} \rho_K^p(u)\rho_L(u)\,dS(u)}{\int_{S^{n-1}} \rho_K^r(u)\rho_L(u)\,dS(u)}\right]^{\frac{n-1}{p-r}} + \left[\frac{\int_{S^{n-1}} \rho_{K'}^p(u)\rho_L(u)\,dS(u)}{\int_{S^{n-1}} \rho_{K'}^r(u)\rho_L(u)\,dS(u)}\right]^{\frac{n-1}{p-r}} \\
&= \left[\frac{\widetilde{W}_{n-p-1}(K,L)}{\widetilde{W}_{n-r-1}(K,L)}\right]^{\frac{n-1}{p-r}} + \left[\frac{\widetilde{W}_{n-p-1}(K',L)}{\widetilde{W}_{n-r-1}(K',L)}\right]^{\frac{n-1}{p-r}} \\
&= Q^{\frac{n-1}{p-r}}_{\widetilde{W}_{n-p-1,n-r-1}(K,L)} + Q^{\frac{n-1}{p-r}}_{\widetilde{W}_{n-p-1,n-r-1}(K',L)}.
\end{aligned}
$$

Then

$$
Q^{\frac{n-1}{p-r}}_{\widetilde{W}_{n-p-1,n-r-1}(K\tilde{+}K',L)} \le Q^{\frac{n-1}{p-r}}_{\widetilde{W}_{n-p-1,n-r-1}(K,L)} + Q^{\frac{n-1}{p-r}}_{\widetilde{W}_{n-p-1,n-r-1}(K',L)}. \tag{3.11}
$$

According to the equality condition of inequality (3.1), we see that equality holds in (3.11) if and only if K and K' are dilates.

Let $i = n-p-1$ and $j = n-r-1$, then $p \ge n-1 \ge r > 0$ and $i \le 0 \le j < n-1$ are equivalent. This and (3.11) yield inequality (1.9) and its equality condition.

Similarly, if $n-1 \ge p > 0 > r$, according to (3.2), (3.9), and (3.10), we have

$$
\left(\frac{\widetilde{W}_{n-p-1}(K\tilde{+}K',L)}{\widetilde{W}_{n-r-1}(K\tilde{+}K',L)}\right)^{\frac{1}{p-r}} \ge \left(\frac{\widetilde{W}_{n-p-1}(K,L)}{\widetilde{W}_{n-r-1}(K,L)}\right)^{\frac{1}{p-r}} + \left(\frac{\widetilde{W}_{n-p-1}(K',L)}{\widetilde{W}_{n-r-1}(K',L)}\right)^{\frac{1}{p-r}}, \tag{3.12}
$$

and equality holds if and only if K and K' are dilates.

Let $i = n-p-1$ and $j = n-r-1$, then (3.12) gives inequality (1.10) and its equality condition. □

Proof of Theorem 1.3 From (2.3), for $K, K', L \in \mathcal{S}_o^n$,

$$
\begin{aligned}
\frac{\widetilde{W}_{n-p-1}(K\hat{+}K',L)}{V(K\hat{+}K')^{p/(n+1)}} &= \frac{1}{n}\int_{S^{n-1}} \frac{\rho^p_{K\hat{+}K'}(u)\rho_L(u)}{V(K\hat{+}K')^{p/(n+1)}}\,dS(u) \\
&= \frac{1}{n}\int_{S^{n-1}} \left(\frac{\rho^{n+1}_{K\hat{+}K'}(u)}{V(K\hat{+}K')}\right)^{\frac{p}{n+1}} \rho_L(u)\,dS(u) \\
&= \frac{1}{n}\int_{S^{n-1}} \left(\frac{\rho^{n+1}_{K}(u)}{V(K)} + \frac{\rho^{n+1}_{K'}(u)}{V(K')}\right)^{\frac{p}{n+1}} \rho_L(u)\,dS(u) \\
&= \frac{1}{n}\int_{S^{n-1}} \left(\frac{\rho^{n+1}_{K}(u)\rho_L^{\frac{n+1}{p}}(u)}{V(K)} + \frac{\rho^{n+1}_{K'}(u)\rho_L^{\frac{n+1}{p}}(u)}{V(K')}\right)^{\frac{p}{n+1}} dS(u)
\end{aligned} \tag{3.13}
$$

and

$$
\frac{\widetilde{W}_{n-r-1}(K\hat{+}K',L)}{V(K\hat{+}K')^{r/(n+1)}} = \frac{1}{n}\int_{S^{n-1}} \left(\frac{\rho^{n+1}_{K}(u)\rho_L^{\frac{n+1}{r}}(u)}{V(K)} + \frac{\rho^{n+1}_{K'}(u)\rho_L^{\frac{n+1}{r}}(u)}{V(K')}\right)^{\frac{r}{n+1}} dS(u). \tag{3.14}
$$

According to (3.1), (3.13), and (3.14), for $p \ge n+1 > r > 0$,

$$
\begin{aligned}
Q^{\frac{n+1}{p-r}}_{\widetilde{W}_{n-p-1,n-r-1}(K\hat{+}K',L)} &= \left(\frac{\widetilde{W}_{n-p-1}(K\hat{+}K',L)}{\widetilde{W}_{n-r-1}(K\hat{+}K',L)}\right)^{\frac{n+1}{p-r}} \\
&= V\bigl(K\hat{+}K'\bigr)\left[\frac{\int_{S^{n-1}}(\frac{\rho_K^{n+1}(u)}{V(K)}\rho_L^{\frac{n+1}{p}}(u) + \frac{\rho_{K'}^{n+1}(u)}{V(K')}\rho_L^{\frac{n+1}{p}}(u))^{\frac{p}{n+1}}\,dS(u)}{\int_{S^{n-1}}(\frac{\rho_K^{n+1}(u)}{V(K)}\rho_L^{\frac{n+1}{r}}(u) + \frac{\rho_{K'}^{n+1}(u)}{V(K')}\rho_L^{\frac{n+1}{r}}(u))^{\frac{r}{n+1}}\,dS(u)}\right]^{\frac{n+1}{p-r}}
\end{aligned}
$$

$$\begin{aligned}
&\leq V\big(K \hat{+} K'\big)\left[\frac{\int_{S^{n-1}}(V(K)^{-1}\rho_K^{n+1}(u)\rho_L^{\frac{n+1}{p}}(u))^{\frac{p}{n+1}}\,dS(u)}{\int_{S^{n-1}}(V(K)^{-1}\rho_K^{n+1}(u)\rho_L^{\frac{n+1}{r}}(u))^{\frac{r}{n+1}}\,dS(u)}\right]^{\frac{n+1}{p-r}} \\
&\quad + V\big(K \hat{+} K'\big)\left[\frac{\int_{S^{n-1}}(V(K')^{-1}\rho_{K'}^{n+1}(u)\rho_L^{\frac{n+1}{p}}(u))^{\frac{p}{n+1}}\,dS(u)}{\int_{S^{n-1}}(V(K')^{-1}\rho_{K'}^{n+1}(u)\rho_L^{\frac{n+1}{r}}(u))^{\frac{r}{n+1}}\,dS(u)}\right]^{\frac{n+1}{p-r}} \\
&= V\big(K \hat{+} K'\big)\left[\frac{\int_{S^{n-1}}(V(K)^{-1})^{\frac{p}{n+1}}\rho_K^p(u)\rho_L(u)\,dS(u)}{\int_{S^{n-1}}(V(K)^{-1})^{\frac{r}{n+1}}\rho_K^r(u)\rho_L(u)\,dS(u)}\right]^{\frac{n+1}{p-r}} \\
&\quad + V\big(K \hat{+} K'\big)\left[\frac{\int_{S^{n-1}}(V(K')^{-1})^{\frac{p}{n+1}}\rho_{K'}^p(u)\rho_L(u)\,dS(u)}{\int_{S^{n-1}}(V(K')^{-1})^{\frac{r}{n+1}}\rho_{K'}^r(u)\rho_L(u)\,dS(u)}\right]^{\frac{n+1}{p-r}} \\
&= \frac{V(K \hat{+} K')}{V(K)}\left(\frac{\int_{S^{n-1}}\rho_K^p(u)\rho_L(u)\,dS(u)}{\int_{S^{n-1}}\rho_K^r(u)\rho_L(u)\,dS(u)}\right)^{\frac{n+1}{p-r}} \\
&\quad + \frac{V(K \hat{+} K')}{V(K')}\left(\frac{\int_{S^{n-1}}\rho_{K'}^p(u)\rho_L(u)\,dS(u)}{\int_{S^{n-1}}\rho_{K'}^r(u)\rho_L(u)\,dS(u)}\right)^{\frac{n+1}{p-r}} \\
&= \frac{V(K \hat{+} K')}{V(K)}\left(\frac{\widetilde{W}_{n-p-1}(K,L)}{\widetilde{W}_{n-r-1}(K,L)}\right)^{\frac{n+1}{p-r}} \\
&\quad + \frac{V(K \hat{+} K')}{V(K')}\left(\frac{\widetilde{W}_{n-p-1}(K',L)}{\widetilde{W}_{n-r-1}(K',L)}\right)^{\frac{n+1}{p-r}} \\
&= V\big(K \hat{+} K'\big)\left(\frac{Q^{\frac{n+1}{p-r}}_{\widetilde{W}_{n-p-1,n-r-1}(K,L)}}{V(K)} + \frac{Q^{\frac{n+1}{p-r}}_{\widetilde{W}_{n-p-1,n-r-1}(K',L)}}{V(K')}\right),
\end{aligned}$$

i.e.,

$$\frac{Q^{\frac{n+1}{p-r}}_{\widetilde{W}_{n-p-1,n-r-1}(K\hat{+}K',L)}}{V(K \hat{+} K')} \leq \frac{Q^{\frac{n+1}{p-r}}_{\widetilde{W}_{n-p-1,n-r-1}(K,L)}}{V(K)} + \frac{Q^{\frac{n+1}{p-r}}_{\widetilde{W}_{n-p-1,n-r-1}(K',L)}}{V(K')}. \tag{3.15}$$

According to the equality condition of inequality (3.1), we see that equality holds in (3.15) if and only if K and K' are dilates.

Let $i = n-p-1$ and $j = n-r-1$, then $p \geq n+1 \geq r > 0$ and $i \leq -2 \leq j < n-1$ are equivalent. This and (3.15) yield inequality (1.11) and its equality condition.

If $n+1 \geq p > 0 > r$, according to (3.2), (3.13), and (3.14), we have

$$\frac{Q^{\frac{n+1}{p-r}}_{\widetilde{W}_{n-p-1,n-r-1}(K\hat{+}K',L)}}{V(K \hat{+} K')} \geq \frac{Q^{\frac{n+1}{p-r}}_{\widetilde{W}_{n-p-1,n-r-1}(K,L)}}{V(K)} + \frac{Q^{\frac{n+1}{p-r}}_{\widetilde{W}_{n-p-1,n-r-1}(K',L)}}{V(K')}, \tag{3.16}$$

with equality if and only if K and K' are dilates.

Let $i = n-p-1$ and $j = n-r-1$, then (3.16) gives inequality (1.12) and its equality condition. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The main idea of this paper was proposed by the second author. All authors contributed equally to the writing of the paper. All authors read and approved the final manuscript.

Acknowledgements
This work was supported by the Natural Science Foundation of China (Grant Nos. 11371224 and 11102101).

Received: 25 March 2015 Accepted: 16 October 2015 Published online: 26 October 2015

References

1. Lutwak, E: Dual mixed volumes. Pac. J. Math. **58**, 531-538 (1975)
2. Zhao, CJ: On volume quotient functions. Indag. Math. **24**, 57-67 (2013)
3. Zhang, GY: Centered bodies and dual mixed volumes. Trans. Am. Math. Soc. **345**, 777-801 (1994)
4. Lutwak, E: Intersection bodies and dual mixed volumes. Adv. Math. **71**, 232-261 (1988)
5. Lutwak, E: Centroid bodies and dual mixed volumes. Proc. Lond. Math. Soc. **60**, 365-391 (1990)
6. Pečarić, JE, Beeasck, PR: On Jessen's inequality for convex functions. J. Math. Anal. Appl. **118**, 125-144 (1986)