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Generalized multi-valued mappings satisfy some inequalities conditions on metric spaces

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Abstract

In this paper, we prove a condition of the existence for generalized multi-valued mappings satisfying some inequalities in metric spaces. These results are improved versions of results of Boško Damjanović and Dragan Dorić (Filomat 25:125-131, 2011).

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1 Introduction and preliminaries

Let (X,d) be a metric space. We denote by $\mathrm{CB}(X)$ the family of all non-empty closed bounded subsets of X. Let $H(\cdot,\cdot)$ be the Hausdorff metric, *i.e.*,

$$H(A,B)=\max\Bigl\{\sup_{a\in A} d(a,B),\sup_{b\in B} d(A,b)\Bigr\},$$

for $A,B\in\mathrm{CB}(X)$, where

$$d(x,B)=\inf_{y\in B} d(x,y).$$

(i) Let T be a self-mapping on X. Then T is called a Banach contraction mapping if there exists $r\in[0,1)$ such that

$$d(Tx,Ty)\leq rd(x,y)$$

for all $x,y\in X$.

(ii) T is called a Kannan mapping if there exists $a\in[0,\frac{1}{2})$ such that

$$d(Tx,Ty)\leq ad(x,Tx)+ad(y,Ty)$$

for all $x,y\in X$.

(iii) If T is a mapping such that

$$d(Tx,Ty) \le r\max\{d(x,Tx),d(y,Ty)\},$$

such that $r \in [0,1)$ and all $x,y \in X$, then T is called a generalized Kannan mapping.

In 1973, Hardy and Rogers [1] introduced a condition as follows:

(iv) Let $x,y \in X$. Then there exists $a_i \ge 0$ such that

$$d(Tx,Ty) \le a_1 d(x,y) + a_2 d(x,Tx) + a_3 d(y,Ty) + a_4 d(x,Ty) + a_5 d(y,Tx),$$

where $\sum_{i=1}^{5} a_i < 1$.

(v) Ciric [2] defined the following condition which generalizes the Banach contraction and Kannan mapping, that is,

$$d(Tx,Ty) \le r\max\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\},$$

such that $r \in [0,1)$ and all $x,y \in X$.

If X is complete and at least one of (i), (ii), (iii), (iv), and (v) holds, then T has a unique fixed point (see [1–5]).

In 2008, Suzuki [6] introduced the condition C as follows. T is said to satisfy condition C if

$$\frac{1}{2}d(x,Tx) \le d(x,y) \quad \text{implies} \quad d(Tx,Ty) \le d(x,y),$$

for all $x,y \in C$.

In the same year, Kikkawa and Suzuki [7] generalized the Kannan mapping resulting in the following theorem.

Theorem 1.1 (Kikkawa and Suzuki [7]) *Let T be a mapping on complete metric space (X,d) and let φ be a non-increasing function from $[0,1)$ into $(\frac{1}{2},1]$ defined by*

$$\varphi(r) = \begin{cases} 1, & \text{if } 0 \le r < \frac{1}{\sqrt{2}}, \\ \frac{1}{1+r}, & \text{if } \frac{1}{\sqrt{2}} \le r < \frac{1}{2}. \end{cases}$$

Let $\alpha \in [0,\frac{1}{2})$ and put $r = \frac{\alpha}{1-\alpha} \in [0,1)$. Suppose that

$$\varphi(r)d(x,Tx) \le d(x,y) \quad \text{implies} \quad d(Tx,Ty) \le \alpha d(x,Tx) + \alpha d(y,Ty) \tag{1.1}$$

for all $x,y \in X$. Then T has a unique fixed point z and $\lim_{n\to\infty} T^n x = z$ holds for every $x \in X$.

Theorem 1.2 (Kikkawa and Suzuki [7]) *Let T be a mapping on a complete metric space (X,d) and let θ be a non-increasing function from $[0,1)$ into $(\frac{1}{2},1]$ defined by*

$$\theta(r) = \begin{cases} 1, & \text{if } 0 \le r < \frac{1}{2}(\sqrt{5}-1), \\ \frac{1-r}{r^2}, & \text{if } \frac{1}{2}(\sqrt{5}-1) \le r < \frac{1}{\sqrt{2}}, \\ \frac{1}{r+1}, & \text{if } \frac{1}{\sqrt{2}} \le r < 1. \end{cases}$$

Suppose that $r \in [0,1)$ *such that*

$$\theta(r)d(x,Tx) \leq d(x,y) \quad \textit{implies} \quad d(Tx,Ty) \leq r\max\bigl\{d(x,Tx),d(y,Ty)\bigr\} \tag{1.2}$$

for all $x,y \in X$. *Then* T *has a unique fixed point* z *and* $\lim_{n\to\infty} T^n x = z$ *holds for every* $x \in X$.

In 2011, Karapinar and Tas [8] stated some new conditions which are modifications of Suzuki's condition C, as follows. T is said to satisfy condition SCC if

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \quad \text{implies} \quad d(Tx,Ty) \leq M(x,y)$$

for all $x,y \in K$, where

$$M(x,y) = \max\bigl\{d(x,y),d(x,Tx),d(y,Ty),d(y,Tx),d(x,Ty)\bigr\}.$$

In 1969, Nadler [9] proved a multi-valued extension of the Banach contraction theorem as follows.

Theorem 1.3 (Nadler [10]) *Let* (X,d) *be a complete metric space and let* T *be a mapping from* X *into* $\mathrm{CB}(X)$. *Assume that there exists* $r \in [0,1)$ *such that*

$$H(Tx,Ty) \leq rd(x,y)$$

for all $x,y \in X$. *Then there exists* $z \in X$ *such that* $z \in Tz$.

Next, the result of Kikkawa and Suzuki [9] is a generalization of Nadler.

Theorem 1.4 (Kikkawa and Suzuki [9]) *Let* (X,d) *be a complete metric space and let* T *be a mapping from* X *into* $\mathrm{CB}(X)$. *Define a strictly decreasing function* η *from* $[0,1)$ *onto* $(\frac{1}{2},1]$ *by*

$$\eta(r) = \frac{r}{1+r}$$

and assume that there exists $r \in [0,1)$ *such that*

$$\eta(r)d(x,Tx) \leq d(x,y) \quad \textit{implies} \quad H(Tx,Ty) \leq rd(x,y)$$

for all $x,y \in X$. *Then there exists* $z \in X$ *such that* $z \in Tz$.

In 2011, Damjanović and Dorić [11] generalized the result of Kannan (iii) and Nadler.

Theorem 1.5 (Damjanović and Dorić [11]) *Define a non-increasing function* φ *from* $[0,1)$ *into* $(0,1]$ *by*

$$\varphi(r) = \begin{cases} 1, & \textit{if } 0 \leq r < \frac{\sqrt{5}-1}{2}, \\ 1-r, & \textit{if } \frac{\sqrt{5}-1}{2} \leq r < 1. \end{cases}$$

Let (X, d) *be a complete metric space and let* T *be a mapping from* X *into* $\mathrm{CB}(X)$. *Assume that*

$$\varphi(r)d(x, Tx) \leq d(x, y) \quad \textit{implies} \quad H(Tx, Ty) \leq r \max\bigl\{d(x, Tx), d(y, Ty)\bigr\} \tag{1.3}$$

for all $x, y \in X$. *Then there exists* $z \in X$ *such that* $z \in Tz$.

Corollary 1.6 (Damjanović and Dorić [11]) *Let* (X, d) *be a complete metric space and let* T *be a mapping from* X *into* $\mathrm{CB}(X)$. *Let* $\alpha \in [0, \frac{1}{2})$ *and put* $r = 2\alpha$. *Suppose that*

$$\varphi(r)d(x, Tx) \leq d(x, y) \quad \textit{implies} \quad H(Tx, Ty) \leq \alpha d(x, Tx) + \alpha d(y, Ty) \tag{1.4}$$

for all $x, y \in X$, *where the function* φ *is defined as in Theorem* 1.5. *Then there exists* $z \in X$ *such that* $z \in Tz$.

In this paper, we prove a condition of the existence for generalized multi-valued mappings under SCC conditions in metric spaces. These results are improved versions of results of Boško Damjanović and Dragan Dorić [11].

2 Main results

Theorem 2.1 *Define a non-increasing function* φ *from* $[0, \frac{1}{2})$ *into* $(0, 1]$ *by*

$$\varphi(r) = \begin{cases} 1, & \textit{if } 0 \leq r < \frac{\sqrt{5}-1}{\sqrt{5}+1}, \\ \frac{1-2r}{1-r}, & \textit{if } \frac{\sqrt{5}-1}{\sqrt{5}+1} \leq r < \frac{1}{2}. \end{cases}$$

Let (X, d) *be a complete metric space and let* T *be a mapping from* X *into* $\mathrm{CB}(X)$. *Assume that*

$$\varphi(r)d(x, Tx) \leq d(x, y) \quad \textit{implies} \quad H(Tx, Ty) \leq rM(x, y) \tag{2.1}$$

where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$, *for all* $x, y \in X$. *Then there exists* $z \in X$ *such that* $z \in Tz$.

Proof Let r_1 be a real number such that $0 \leq r < r_1 < \frac{1}{2}$. Let $u_1 \in X$ and $u_2 \in Tu_1$ be arbitrary. Since $u_2 \in Tu_1$, we have $d(u_2, Tu_2) \leq H(Tu_1, Tu_2)$ and

$$\varphi(r)d(u_1, Tu_1) \leq d(u_1, Tu_1) \leq d(u_1, u_2).$$

Thus from the assumption (2.1),

$$d(u_2, Tu_2) \leq H(Tu_1, Tu_2) \leq rM(u_1, u_2)$$

where $M(u_1, u_2) = \max\{d(u_1, u_2), d(u_1, Tu_1), d(u_2, Tu_2), d(u_1, Tu_2), d(u_2, Tu_1)\}$. Consider

$$\begin{aligned} d(u_2, Tu_2) &\leq r \max\bigl\{d(u_1, u_2), d(u_1, Tu_1), d(u_2, Tu_2), d(u_1, Tu_2), d(u_2, Tu_1)\bigr\} \\ &= r \max\bigl\{d(u_1, u_2), d(u_1, Tu_2)\bigr\}. \end{aligned}$$

If $\max\{d(u_1,u_2), d(u_1,Tu_2)\} = d(u_1,Tu_2)$, then

$$\begin{aligned} d(u_2,Tu_2) &\le rd(u_1,Tu_2) \\ &\le rd(u_1,u_2) + rd(u_2,Tu_2) \end{aligned}$$

and then

$$d(u_2,Tu_2) \le \left(\frac{r}{1-r}\right) d(u_1,u_2).$$

If $\max\{d(u_1,u_2), d(u_1,Tu_2)\} = d(u_1,u_2)$, then

$$d(u_2,Tu_2) \le rd(u_1,u_2) \le \left(\frac{r}{1-r}\right) d(u_1,u_2).$$

So

$$d(u_2,Tu_2) \le \left(\frac{r}{1-r}\right) d(u_1,u_2).$$

So there exists $u_3 \in Tu_2$ such that $d(u_2,u_3) \le (\frac{r_1}{1-r_1}) d(u_1,u_2)$. Thus, we can construct a sequence $\{x_n\}$ in X such that $u_{n+1} \in Tu_n$ and

$$d(u_{n+1},u_{n+2}) \le \left(\frac{r_1}{1-r_1}\right) d(u_n,u_{n+1}).$$

Hence, by induction,

$$d(u_n,u_{n+1}) \le \left(\frac{r_1}{1-r_1}\right)^{n-1} d(u_1,u_2).$$

Then by the triangle inequality, we have

$$\sum_{n=1}^{\infty} d(u_n,u_{n+1}) \le \sum_{n=1}^{\infty} \left(\frac{r_1}{1-r_1}\right)^{n-1} d(u_1,u_2) < \infty.$$

Hence we conclude that $\{u_n\}$ is a Cauchy sequence. Since X is complete, there is some point $z \in X$ such that

$$\lim_{n\to\infty} u_n = z.$$

Now, we will show that $d(z,Tx) \le rd(x,Tx)$ for all $x \in X \setminus \{z\}$.

Let $x \in X \setminus \{z\}$. Since $u_n \to z$, there exists $n_0 \in N$ such that $d(z,u_n) \le (\frac{1}{3}) d(z,x)$ for all $n \ge n_0$. Then we have

$$\begin{aligned} \varphi(r)d(u_n,Tu_n) &\le d(u_n,Tu_n) \\ &\le d(u_n,u_{n+1}) \\ &\le d(u_n,z) + d(z,u_{n+1}) \end{aligned}$$

$$\begin{aligned}
&\leq \left(\frac{2}{3}\right)d(z,x)\\
&= d(z,x) - \frac{1}{3}d(z,x)\\
&\leq d(z,x) - d(z,u_n)\\
&\leq d(x,u_n).
\end{aligned} \tag{2.2}$$

Then from (2.1) we have

$$H(Tu_n, Tx) \leq r\max\bigl\{d(u_n,x), d(u_n, Tu_n), d(x, Tx), d(u_n, Tx), d(x, Tu_n)\bigr\}.$$

Since $u_{n+1} \in Tu_n$, $d(u_{n+1}, Tx) \leq H(Tu_n, Tx)$, so that

$$d(u_{n+1}, Tx) \leq r\max\bigl\{d(u_n,x), d(u_n, u_{n+1}), d(x, Tx), d(u_n, Tx), d(x, u_{n+1})\bigr\}$$

for all $n \geq n_0$. Letting $n \to \infty$, we obtain

$$d(z, Tx) \leq r\max\bigl\{d(z,x), d(x, Tx), d(z, Tx)\bigr\}.$$

It follows that

$$d(z, Tx) \leq \left(\frac{r}{1-r}\right)d(x, Tx) \quad \text{for all } x \in X \setminus \{z\}. \tag{2.3}$$

Next, we show that $z \in Tz$. Suppose that z is not an element in Tz.

Case (i): $0 \leq r < \frac{\sqrt{5}-1}{\sqrt{5}+1}$. Let $a \in Tz$. Then $a \neq z$ and so by (2.3), we have

$$d(z, Ta) \leq \left(\frac{r}{1-r}\right)d(a, Ta).$$

On the other hand, since $\varphi(r)d(z, Tz) = d(z, Tz) \leq d(z,a)$, from (2.1) we have

$$H(Tz, Ta) \leq r\max\bigl\{d(z,a), d(z, Tz), d(a, Ta), d(z, Ta), d(a, Tz)\bigr\}.$$

So

$$d(a, Ta) \leq H(Tz, Ta) \leq r\max\bigl\{d(z,a), d(z, Tz), d(z, Ta)\bigr\}. \tag{2.4}$$

It implies that

$$d(a, Ta) \leq r\max\bigl\{d(z,a), d(z, Tz), d(z, Ta)\bigr\}.$$

Since $d(z,a) \leq d(z, Tz) + d(Tz,a) = d(z, Tz)$, we have

$$d(a, Ta) \leq \left(\frac{r}{1-r}\right)d(z, Tz). \tag{2.5}$$

Using (2.3), (2.4), and (2.5), we have

$$\begin{aligned}
d(z,Tz) &\le d(z,Ta) + H(Ta,Tz) \\
&\le \left(\frac{r}{1-r}\right) d(a,Ta) + r\max\bigl\{d(z,a), d(z,Tz), d(z,Ta)\bigr\} \\
&\le \left(\frac{r}{1-r}\right) d(a,Ta) + r\max\left\{d(z,a), d(z,Tz), \left(\frac{r}{1-r}\right) d(a,Ta)\right\} \\
&\le \left(\frac{r}{1-r}\right) d(a,Ta) + r\max\bigl\{d(z,a), d(z,Tz)\bigr\} \\
&\le \left(\frac{r}{1-r}\right) d(a,Ta) + rd(z,Tz) \\
&\le \left(\frac{r}{1-r}\right)^2 d(z,Tz) + rd(z,Tz) \\
&\le \left(\frac{r}{1-r}\right)^2 d(z,Tz) + \left(\frac{r}{1-r}\right) d(z,Tz) \\
&\le \left[\left(\frac{r}{1-r}\right)^2 + \left(\frac{r}{1-r}\right)\right] d(z,Tz) \\
&\le \bigl[k^2 + k\bigr] d(z,Tz),
\end{aligned}$$

where $k = \frac{r}{1-r}$. Since $r < \frac{\sqrt{5}-1}{\sqrt{5}+1}$, we have $k^2 + k < 1$ and so $d(z,Tz) < d(z,Tz)$, a contradiction. Thus $z \in Tz$.

Case (ii): $\frac{\sqrt{5}-1}{\sqrt{5}+1} \le r < \frac{1}{2}$. Let $x \in X$. If $x = z$, then $H(Tx,Tz) \le r\max\{d(x,z), d(x,Tx), d(z,Tz), d(x,Tz), d(z,Tx)\}$ holds. If $x \neq z$, then for all $n \in \mathbb{N}$, there exists $y_n \in Tx$ such that

$$d(z,y_n) \le d(z,Tx) + \left(\frac{1}{n}\right) d(x,z).$$

We consider

$$\begin{aligned}
d(x,Tx) &\le d(x,y_n) \\
&\le d(x,z) + d(z,y_n) \\
&\le d(x,z) + d(z,Tx) + \left(\frac{1}{n}\right) d(x,z) \\
&\le d(x,z) + \left(\frac{r}{1-r}\right) d(x,Tx) + \left(\frac{1}{n}\right) d(x,z).
\end{aligned}$$

Thus, $(\frac{1-2r}{1-r}) d(x,Tx) \le (1 + \frac{1}{n}) d(x,z)$. Take $n \to \infty$, we obtain

$$\left(\frac{1-2r}{1-r}\right) d(x,Tx) \le d(x,z),$$

by using (2.1), implies $H(Tx,Tz) \le r\max\{d(x,z), d(x,Tx), d(z,Tz), d(x,Tz), d(z,Tx)\}$. Hence, as $u_{n+1} \in Tu_n$, it follows that with $x = u_n$

$$\begin{aligned}
d(z,Tz) &= \lim_{n\to\infty} d(u_{n+1},Tz) \\
&\le H(Tu_n,Tz)
\end{aligned}$$

$$\begin{aligned}
&\leq \lim_{n\to\infty} r\max\bigl\{d(u_n,z),d(u_n,Tu_n),d(z,Tz),d(u_n,Tz),d(z,Tu_n)\bigr\}\\
&\leq \lim_{n\to\infty} r\max\bigl\{d(u_n,z),d(u_n,u_{n+1}),d(z,Tz),d(u_n,Tz),d(z,u_{n+1})\bigr\}\\
&\leq rd(z,Tz).
\end{aligned}$$

Therefore, $(1-r)d(z,Tz) \leq 0$, which implies $d(z,Tz) = 0$. Since Tz is closed, we have $z \in Tz$. This completes the proof. □

Example 2.2 Let $X = [0,\infty)$ be endowed with the usual metric d. Define $T : X \to \mathrm{CB}(X)$ by

$$T(x) = \begin{cases} [0,x^2], & 0 \leq x \leq \frac{1}{2}, \\ [0,\frac{x}{3}], & \frac{1}{2} < x < 1, \\ [0,\log(x)], & 1 \leq x. \end{cases} \tag{2.6}$$

Proof We show that T satisfies (2.1). Let $x,y \in X$. We prove by cases.

Case (i): Suppose that $x,y \in [0,\frac{1}{2}]$. Thus, if $x^2 \leq y$, then

$$\varphi\left(\frac{1}{4}\right)d(x,Tx) = \left|x-x^2\right| \geq |x-y| = d(x,y).$$

But if $x^2 > y$, then

$$\varphi\left(\frac{1}{4}\right)d(x,Tx) = \left|x-x^2\right| \leq |x-y| = d(x,y)$$

and

$$\begin{aligned}
H(Tx,Ty) &= \left|x^2-y^2\right|\\
&\leq \frac{1}{4}\left|(2x)^2-(2y)^2\right|\\
&\leq \frac{1}{4}\left|x-2y^2\right|\\
&\leq \frac{1}{4}\left|x-y^2\right|\\
&= \frac{1}{4}\max\bigl\{|x-y|,\left|x-x^2\right|,\left|y-y^2\right|,\left|x-y^2\right|,\left|y-x^2\right|\bigr\}\\
&= \frac{1}{4}\max\bigl\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\bigr\}\\
&= rM(x,y),
\end{aligned} \tag{2.7}$$

where $r = \frac{1}{4}$. Hence T satisfies (2.1).

Case (ii): Suppose that $x,y \in (\frac{1}{2},1)$. Thus, if $\frac{x}{3} \leq y$, then

$$\varphi\left(\frac{1}{3}\right)d(x,Tx) = \left|x-\frac{x}{3}\right| \geq |x-y| = d(x,y).$$

But if $\frac{x}{3} > y$, then

$$\varphi\left(\frac{1}{3}\right)d(x,Tx) = \left|x - \frac{x}{3}\right| \leq |x-y| = d(x,y)$$

and

$$\begin{aligned}
H(Tx,Ty) &= \frac{1}{3}|x-y| \\
&\leq \frac{1}{3}\left|x - \frac{y}{3}\right| \\
&= \frac{1}{3}\max\left\{|x-y|, \left|x-\frac{x}{3}\right|, \left|y-\frac{y}{3}\right|, \left|x-\frac{y}{3}\right|, \left|y-\frac{x}{3}\right|\right\} \\
&= \frac{1}{3}\max\bigl\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\bigr\} \\
&= rM(x,y), \qquad (2.8)
\end{aligned}$$

where $r = \frac{1}{3}$. Hence T satisfies (2.1).

Case (iii): Suppose that $x, y \in [1,\infty]$. Thus, if $\log(x) \leq y$, then

$$\varphi\left(\frac{1}{3}\right)d(x,Tx) = \bigl|x - \log(x)\bigr| \geq |x-y| = d(x,y).$$

But if $\log(x) > y$, then

$$\varphi\left(\frac{1}{3}\right)d(x,Tx) = \bigl|x - \log(x)\bigr| \leq |x-y| = d(x,y)$$

and

$$\begin{aligned}
H(Tx,Ty) &= \bigl|\log(x) - \log(y)\bigr| \\
&= \frac{1}{3}\bigl(3\log(x) - 3\log(y)\bigr) \\
&\leq \frac{1}{3}\bigl|x - \log(y)\bigr| \\
&= \frac{1}{3}\max\bigl\{|x-y|, \bigl|x-\log(x)\bigr|, \bigl|y-\log(y)\bigr|, \bigl|x-\log(y)\bigr|, \bigl|y-\log(x)\bigr|\bigr\} \\
&= \frac{1}{3}\max\bigl\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\bigr\} \\
&= rM(x,y), \qquad (2.9)
\end{aligned}$$

where $r = \frac{1}{3}$. Hence T satisfies (2.1).

Case (iv): Suppose that $x \in [0, \frac{1}{2}]$ and $y \in (\frac{1}{2}, 1)$. Then $x^2 < x < y$. Thus, $\varphi(\frac{1}{3})d(x,Tx) = |x - x^2| \geq |x-y| = d(x,y)$. Hence T satisfies (2.1).

Case (v): Suppose that $x \in (\frac{1}{2}, 1)$ and $y \in [0, \frac{1}{2}]$. So $x > y$. Thus, if $\frac{x}{3} \leq y$, then

$$\varphi\left(\frac{1}{3}\right)d(x,Tx) = \left|x - \frac{x}{3}\right| \geq |x-y| = d(x,y).$$

But if $\frac{x}{3} > y$, then

$$\varphi\left(\frac{1}{3}\right)d(x,Tx) = \left|x - \frac{x}{3}\right| \leq |x-y| = d(x,y)$$

and

$$\begin{aligned} H(Tx,Ty) &= \left|\frac{x}{3} - y^2\right| \\ &\leq \frac{1}{3}\left|x - 3y^2\right| \\ &\leq \frac{1}{3}\left|x - y^2\right| \\ &= \frac{1}{3}\max\left\{|x-y|, \left|x - \frac{x}{3}\right|, \left|y - y^2\right|, \left|x - y^2\right|, \left|y - \frac{x}{3}\right|\right\} \\ &= \frac{1}{3}\max\left\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\right\} \\ &= rM(x,y), \end{aligned} \tag{2.10}$$

where $r = \frac{1}{3}$. Hence T satisfies (2.1).

Case (vi): Suppose that $x \in [0, \frac{1}{2}]$ and $y \in [1, \infty]$.

$$\varphi\left(\frac{1}{3}\right)d(x,Tx) = \left|x - x^2\right| \leq |x-y| = d(x,y)$$

and

$$\begin{aligned} H(Tx,Ty) &= \left|x^2 - \log(y)\right| \\ &= \frac{1}{3}\left|3x^2 - 3\log(y)\right| = \frac{1}{3}\left|3\log(y) - 3x^2\right| \\ &\leq \frac{1}{3}\max\left\{\left|y - \log(y)\right|, \left|y - x^2\right|\right\} \\ &= \frac{1}{3}\max\left\{|x-y|, \left|x - x^2\right|, \left|y - \log(y)\right|, \left|x - \log(y)\right|, \left|y - x^2\right|\right\} \\ &= \frac{1}{3}\max\left\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\right\} \\ &= rM(x,y), \end{aligned} \tag{2.11}$$

where $r = \frac{1}{3}$. Hence T satisfies (2.1).

Case (vii): Suppose that $x \in [1, \infty]$ and $y \in [0, \frac{1}{2}]$. Thus, if $\log(x) \leq y$, then

$$\varphi\left(\frac{1}{4}\right)d(x,Tx) = \left|x - \log(x)\right| \geq |x-y| = d(x,y).$$

But if $\log(x) > y$, then

$$\varphi\left(\frac{1}{4}\right)d(x,Tx) = \left|x - \log(x)\right| \leq |x-y| = d(x,y)$$

and

$$\begin{aligned}
H(Tx,Ty) &= \left|\log(x) - y^2\right| \\
&= \frac{1}{4}\left|4\log(x) - 4y^2\right| \\
&\le \frac{1}{4}\left|x - y^2\right| \\
&= \frac{1}{4}\max\left\{|x-y|, \left|x-\log(x)\right|, \left|y-y^2\right|, \left|x-y^2\right|, \left|y-\log(x)\right|\right\} \\
&= \frac{1}{4}\max\left\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\right\} \\
&= rM(x,y), \qquad (2.12)
\end{aligned}$$

where $r = \frac{1}{4}$. Hence T satisfies (2.1).

Case (viii): Suppose that $x \in (\frac{1}{2}, 1)$ and $y \in [1, \infty]$.

$$\varphi\left(\frac{1}{3}\right)d(x,Tx) = \left|x - \frac{x}{3}\right| \le |x-y| = d(x,y)$$

and

$$\begin{aligned}
H(Tx,Ty) &= \left|\frac{x}{3} - \log(y)\right| \\
&= \frac{1}{3}\left|x - 3\log(y)\right| = \frac{1}{3}\left|3\log(y) - x\right| \\
&\le \frac{1}{3}\max\left\{\left|y-\log(y)\right|, \left|y - \frac{x}{3}\right|\right\} \\
&= \frac{1}{3}\max\left\{|x-y|, \left|x-\frac{x}{3}\right|, \left|y-\log(y)\right|, \left|x-\log(y)\right|, \left|y-\frac{x}{3}\right|\right\} \\
&= \frac{1}{3}\max\left\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\right\} \\
&= rM(x,y), \qquad (2.13)
\end{aligned}$$

where $r = \frac{1}{3}$. Hence T satisfies (2.1).

Case (ix): Suppose that $x \in [1, \infty]$ and $y \in (\frac{1}{2}, 1)$. Thus, if $\log(x) \le y$, then

$$\varphi\left(\frac{1}{3}\right)d(x,Tx) = \left|x - \log(x)\right| \ge |x-y| = d(x,y).$$

But if $\log(x) > y$, then

$$\varphi\left(\frac{1}{3}\right)d(x,Tx) = \left|x - \log(x)\right| \le |x-y| = d(x,y)$$

and

$$\begin{aligned}
H(Tx,Ty) &= \left|\log(x) - \frac{y}{3}\right| \\
&= \frac{1}{3}\left|3\log(x) - y\right|
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{3}|x-y| \\
&= \frac{1}{3}\max\left\{|x-y|, \left|x-\log(x)\right|, \left|y-\frac{y}{3}\right|, \left|x-\frac{y}{3}\right|, \left|y-\log(x)\right|\right\} \\
&= \frac{1}{3}\max\left\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\right\} \\
&= rM(x,y),
\end{aligned} \tag{2.14}$$

where $r = \frac{1}{3}$. Hence T satisfies (2.1).

Thus we see that T satisfies condition (2.1) and satisfies Theorem 2.1. So there exists $z \in X$ such that $z \in Tz$. Moreover, $0 \in T(0)$. □

Theorem 2.3 *Define a non-increasing function* φ *from* $[0, \frac{1}{5})$ *into* $(0,1]$ *by*

$$\varphi(r) = \begin{cases} 1, & \text{if } 0 \leq r < \frac{\sqrt{5}-1}{4+2\sqrt{5}}, \\ \frac{1-5r}{1-2r}, & \text{if } \frac{\sqrt{5}-1}{4+2\sqrt{5}} \leq r < \frac{1}{5}. \end{cases}$$

Let (X,d) *be a complete metric space and let* T *be a mapping from* X *into* $\mathrm{CB}(X)$. *Assume that*

$$\varphi(r)d(x,Tx) \leq d(x,y) \quad \textit{implies} \quad H(Tx,Ty) \leq S(x,y) \tag{2.15}$$

where $S(x,y) = rd(x,y) + rd(x,Tx) + rd(y,Ty) + rd(x,Ty) + rd(y,Tx)$ *for all* $x, y \in X$. *Then there exists* $z \in X$ *such that* $z \in Tz$.

Proof Let r_1 be a real number such that $0 \leq r < r_1 < 1$. Let $u_1 \in X$ and $u_2 \in Tu_1$ be arbitrary. Since $u_2 \in Tu_1$, we have $d(u_2, Tu_2) \leq H(Tu_1, Tu_2)$ and

$$\varphi(r)d(u_1,Tu_1) \leq d(u_1,Tu_1) \leq d(u_1,u_2).$$

Thus, from the assumption (2.15),

$$d(u_2,Tu_2) \leq H(Tu_1,Tu_2) \leq S(u_1,u_2)$$

where $S(u_1,u_2) = rd(u_1,u_2) + rd(u_1,Tu_1) + rd(u_2,Tu_2) + rd(u_1,Tu_2) + rd(u_2,Tu_1)$. Consider

$$\begin{aligned}
d(u_2,Tu_2) &\leq rd(u_1,u_2) + rd(u_1,Tu_1) + rd(u_2,Tu_2) + rd(u_1,Tu_2) + rd(u_2,Tu_1) \\
&\leq 3rd(u_1,u_2) + 2rd(u_2,Tu_2).
\end{aligned}$$

So

$$d(u_2,Tu_2) \leq \left(\frac{3r}{1-2r}\right)d(u_1,u_2).$$

So there exists $u_3 \in Tu_2$ such that $d(u_2,u_3) \leq (\frac{3r_1}{1-2r_1})d(u_1,u_2)$. Thus, we can construct a sequence $\{x_n\}$ in X such that $u_{n+1} \in Tu_n$ and

$$d(u_{n+1},u_{n+2}) \leq \left(\frac{3r_1}{1-2r_1}\right)d(u_n,u_{n+1}).$$

Hence, by induction,

$$d(u_n, u_{n+1}) \leq \left(\frac{3r_1}{1-2r_1}\right)^{n-1} d(u_1, u_2).$$

Then by the triangle inequality, we have

$$\sum_{n=1}^{\infty} d(u_n, u_{n+1}) \leq \sum_{n=1}^{\infty}\left(\frac{3r_1}{1-2r_1}\right)^{n-1} d(u_1, u_2) < \infty.$$

Hence we conclude that $\{u_n\}$ is a Cauchy sequence. Since X is complete, there is some point $z \in X$ such that

$$\lim_{n\to\infty} u_n = z.$$

Now, we will show that $d(z, Tx) \leq (\frac{3r}{1-2r})d(x, Tx)$ for all $x \in X \setminus \{z\}$.

Let $x \in X \setminus \{z\}$. Since $u_n \to z$, there exists $n_0 \in N$ such that $d(z, u_n) \leq (\frac{1}{3})d(z, x)$ for all $n \geq n_0$. By using (2.2), we get

$$\varphi(r)d(u_n, Tu_n) \leq d(x, u_n).$$

Then from (2.15) we have

$$H(Tu_n, Tx) \leq r\bigl[d(u_n, x) + d(u_n, Tu_n) + d(x, Tx) + d(u_n, Tx) + d(x, Tu_n)\bigr].$$

Since $u_{n+1} \in Tu_n$, $d(u_{n+1}, Tx) \leq H(Tu_n, Tx)$, so that

$$d(u_{n+1}, Tx) \leq r\bigl[d(u_n, x) + d(u_n, u_{n+1}) + d(x, Tx) + d(u_n, Tx) + d(x, u_{n+1})\bigr]$$

for all $n \geq n_0$. Letting $n \to \infty$, we obtain

$$\begin{aligned} d(z, Tx) &\leq r\bigl[2d(z, x) + d(x, Tx) + d(z, Tx)\bigr] \\ &\leq r3d(z, x) + r2d(z, Tx). \end{aligned}$$

It follows that

$$d(z, Tx) \leq \left(\frac{3r}{1-2r}\right)d(x, Tx) \quad \text{for all } x \in X \setminus \{z\}. \tag{2.16}$$

Next, we show that $z \in Tz$. Suppose that z is not an element in Tz.

Case (i): $0 \leq r < \frac{\sqrt{5}-1}{4+2\sqrt{5}}$. Let $a \in Tz$. Then $a \neq z$ and so by (2.16), we have

$$d(z, Ta) \leq \left(\frac{3r}{1-2r}\right)d(a, Ta).$$

On the other hand, since $\varphi(r)d(z, Tz) = d(z, Tz) \leq d(z, a)$, from (2.15) we have

$$H(Tz, Ta) \leq r\bigl[d(z, a) + d(z, Tz) + d(a, Ta) + d(z, Ta) + d(a, Tz)\bigr].$$

So

$$\begin{aligned} d(a, Ta) &\leq H(Tz, Ta) \leq r\bigl[2d(z,a) + d(a, Ta) + d(z, Ta)\bigr] \\ &\leq r\bigl[3d(z,a) + 2d(a, Ta)\bigr]. \end{aligned} \tag{2.17}$$

Since $d(z,a) \leq d(z, Tz) + d(Tz, a) = d(z, Tz)$, we have

$$d(a, Ta) \leq \left(\frac{3r}{1-2r}\right)d(z, Tz).$$

Using (2.15), (2.16), and (2.17), we have

$$\begin{aligned} d(z, Tz) &\leq d(z, Ta) + H(Ta, Tz) \\ &\leq \left(\frac{3r}{1-2r}\right)d(a, Ta) + S(a,z) \\ &\leq \left(\frac{3r}{1-2r}\right)d(a, Ta) + r\bigl[d(z,a) + d(z, Tz) + d(a, Ta) + d(z, Ta) + d(a, Tz)\bigr] \\ &\leq \left(\frac{3r}{1-2r}\right)d(a, Ta) + 3rd(z,a) \\ &\leq \left(\frac{3r}{1-2r}\right)^2 d(z, Tz) + \left(\frac{3r}{1-2r}\right)d(z, Tz) \\ &\leq \bigl(k^2 + k\bigr)d(z, Tz), \end{aligned}$$

where $k = \frac{3r}{1-2r}$.

Since $0 \leq r < \frac{\sqrt{5}-1}{4+2\sqrt{5}}$, we have $0 \leq k^2 + k < 1$ and so, $d(z, Tz) < d(z, Tz)$, a contradiction. Thus $z \in Tz$.

Case (ii): $\frac{\sqrt{5}-1}{4+2\sqrt{5}} \leq r < \frac{1}{5}$. Let $x \in X$.

If $x = z$, then $H(Tx, Tz) \leq r[d(x,z) + d(x, Tx) + d(z, Tz) + d(x, Tz) + d(z, Tx)]$ holds. If $x \neq z$, then for all $n \in \mathbb{N}$, there exists $y_n \in Tx$ such that

$$d(z, y_n) \leq d(z, Tx) + \left(\frac{1}{n}\right)d(x,z).$$

We consider

$$\begin{aligned} d(x, Tx) &\leq d(x, y_n) \\ &\leq d(x,z) + d(z, y_n) \\ &\leq d(x,z) + d(z, Tx) + \left(\frac{1}{n}\right)d(x,z) \\ &\leq d(x,z) + \left(\frac{3r}{1-2r}\right)d(x, Tx) + \left(\frac{1}{n}\right)d(x,z). \end{aligned}$$

Thus, $(\frac{1-5r}{1-2r})d(x, Tx) \leq (1 + \frac{1}{n})d(x,z)$. Take $n \to \infty$, we obtain

$$\left(\frac{1-5r}{1-2r}\right)d(x, Tx) \leq d(x,z),$$

by using (2.15), implies $H(Tx, Tz) \leq S(x,z)$, where $S(x,z) = r[d(x,z) + d(x,Tx) + d(z,Tz) + d(x,Tz) + d(z,Tx)]$.

Hence, as $u_{n+1} \in Tu_n$, it follows that with $x = u_n$

$$\begin{aligned} d(z,Tz) &= \lim_{n\to\infty} d(u_{n+1}, Tz) \\ &\leq H(Tu_n, Tz) \\ &\leq \lim_{n\to\infty} r\bigl[d(u_n,z) + d(u_n,Tu_n) + d(z,Tz) + d(u_n,Tz) + d(z,Tu_n)\bigr] \\ &\leq \lim_{n\to\infty}\bigl[rd(u_n,z) + rd(u_n,u_{n+1}) + rd(z,Tz) + rd(u_n,Tz) + rd(z,u_{n+1})\bigr] \\ &\leq (2r)d(z,Tz). \end{aligned} \tag{2.18}$$

Using (2.18), we have $(1-2r)d(z,Tz) \leq 0$, which implies $d(z,Tz) = 0$. Since Tz is closed, we have $z \in Tz$. This completes the proof. □

Example 2.4 Let $X = [0, \frac{1}{2}]$ with the metric $d(x,y) = \frac{|x-y|}{|x-y|+1}$ for all $x, y \in X$. Define $T: X \to \mathrm{CB}(X)$ by

$$T(x) = \bigl[0, x^2\bigr].$$

Proof We show that T satisfies (2.15). Let $x, y \in X$. Thus, if $x^2 \leq y$, then

$$\varphi\left(\frac{1}{6}\right)d(x,Tx) = \frac{|x-x^2|}{|x-x^2|+1} \geq \frac{|x-y|}{|x-y|+1} = d(x,y).$$

But if $x^2 > y$, then

$$\varphi\left(\frac{1}{6}\right)d(x,Tx) = \frac{|x-x^2|}{|x-x^2|+1} \leq \frac{|x-y|}{|x-y|+1} = d(x,y)$$

and

$$\begin{aligned} &H(Tx,Ty) \\ &\quad = \frac{|x^2-y^2|}{|x^2-y^2|+1} \\ &\quad = \frac{1}{6}\frac{6|x^2-y^2|}{|x^2-y^2|+1} \\ &\quad = \frac{1}{6}\left\{\frac{|x^2-y^2|}{|x^2-y^2|+1} + \frac{|x^2-y^2|}{|x^2-y^2|+1} + \frac{|x^2-y^2|}{|x^2-y^2|+1} + \frac{2|x^2-y^2|}{|x^2-y^2|+1} + \frac{|x^2-y^2|}{|x^2-y^2|+1}\right\} \\ &\quad < \frac{1}{6}\left\{\frac{|x-y|}{|x-y|+1} + \frac{|x-x^2|}{|x-x^2|+1} + \frac{|y-y^2|}{|y-y^2|+1} + \frac{|x-y^2|}{|x-y^2|+1} + \frac{|y-x^2|}{|y-x^2|+1}\right\} \\ &\quad = \frac{1}{6}\bigl\{d(x,y) + d(x,Tx) + d(y,Ty) + d(x,Ty) + d(y,Tx)\bigr\} \\ &\quad = \frac{1}{6}S(x,y), \end{aligned} \tag{2.19}$$

where $r = \frac{1}{6}$.

Thus we see that T satisfies condition (2.15) and satisfies Theorem 2.3. So there exists $z \in X$ such that $z \in Tz$. Moreover, $0 \in T(0)$. □

Theorem 2.5 *Define a non-increasing function* φ *from* $[0,1)$ *into* $(0,1]$ *by*

$$\varphi(r) = \begin{cases} 1, & \text{if } 0 \leq r < \frac{\sqrt{5}-1}{2}, \\ 1-r, & \text{if } \frac{\sqrt{5}-1}{2} \leq r < 1. \end{cases}$$

Let $\alpha \in [0, \frac{1}{2})$ *and* $r = \frac{\alpha}{1-\alpha}$, *and let* (X,d) *be a complete metric space and let* T *be a mapping from* X *into* $\mathrm{CB}(X)$.

Assume that

$$\varphi(r)d(x,Tx) \leq d(x,y) \quad \text{implies} \quad H(Tx,Ty) \leq \alpha M(x,y) \tag{2.20}$$

where $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$, *for all* $x, y \in X$. *Then there exists* $z \in X$ *such that* $z \in Tz$.

Proof Let r_1 be a real number such that $0 \leq r < r_1 < \frac{1}{2}$. Let $u_1 \in X$ and $u_2 \in Tu_1$ be arbitrary. Since $u_2 \in Tu_1$, we have $d(u_2, Tu_2) \leq H(Tu_1, Tu_2)$ and

$$\varphi(r)d(u_1,Tu_1) \leq d(u_1,Tu_1) \leq d(u_1,u_2).$$

Thus, from the assumption (2.20),

$$d(u_2,Tu_2) \leq H(Tu_1,Tu_2) \leq \alpha M(u_1,u_2)$$

where $M(u_1,u_2) = \max\{d(u_1,u_2), d(u_1,Tu_1), d(u_2,Tu_2), d(u_1,Tu_2), d(u_2,Tu_1)\}$.

Consider

$$\begin{aligned} d(u_2,Tu_2) &\leq \alpha \max\bigl\{d(u_1,u_2), d(u_1,Tu_1), d(u_2,Tu_2), d(u_1,Tu_2), d(u_2,Tu_1)\bigr\} \\ &= \alpha \max\bigl\{d(u_1,u_2), d(u_1,Tu_2)\bigr\}. \end{aligned}$$

If $\max\{d(u_1,u_2), d(u_1,Tu_2)\} = d(u_1,Tu_2)$, then

$$\begin{aligned} d(u_2,Tu_2) &\leq \alpha d(u_1,Tu_2) \\ &\leq \alpha d(u_1,u_2) + \alpha d(u_2,Tu_2) \end{aligned}$$

and then

$$d(u_2,Tu_2) \leq \left(\frac{\alpha}{1-\alpha}\right)d(u_1,u_2) = rd(u_1,u_2),$$

where $r = \frac{\alpha}{1-\alpha}$.

So there exists $u_3 \in Tu_2$ such that $d(u_2, u_3) \leq r_1 d(u_1, u_2)$. Thus, we can construct a sequence $\{x_n\}$ in X such that $u_{n+1} \in Tu_n$ and

$$d(u_{n+1}, u_{n+2}) \leq r_1 d(u_n, u_{n+1}).$$

Hence, by induction

$$d(u_n, u_{n+1}) \leq (r_1)^{n-1} d(u_1, u_2).$$

Then by the triangle inequality, we have

$$\sum_{n=1}^{\infty} d(u_n, u_{n+1}) \leq \sum_{n=1}^{\infty} (r_1)^{n-1} d(u_1, u_2) < \infty.$$

Hence we conclude that $\{u_n\}$ is a Cauchy sequence. Since X is complete, there is some point $z \in X$ such that

$$\lim_{n\to\infty} u_n = z.$$

Now, we will show that $d(z, Tx) \leq rd(x, Tx)$ for all $x \in X \setminus \{z\}$.

Let $x \in X \setminus \{z\}$. Since $u_n \to z$, there exists $n_0 \in N$ such that $d(z, u_n) \leq (\frac{1}{3})d(z, x)$ for all $n \geq n_0$. By using (2.2), we get

$$\varphi(r) d(u_n, Tu_n) \leq d(x, u_n).$$

Then from (2.20), we have

$$H(Tu_n, T_x) \leq \alpha \max\bigl\{d(u_n, x), d(u_n, Tu_n), d(x, Tx), d(u_n, Tx), d(x, Tu_n)\bigr\}.$$

Since $u_{n+1} \in Tu_n$, we have $d(u_{n+1}, T_x) \leq H(Tu_n, T_x)$, so that

$$d(u_{n+1}, Tx) \leq \alpha \max\bigl\{d(u_n, x), d(u_n, u_{n+1}), d(x, Tx), d(u_n, Tx), d(x, u_{n+1})\bigr\}$$

for all $n \geq n_0$. Letting $n \to \infty$, we obtain

$$d(z, Tx) \leq \alpha \max\bigl\{d(z, x), d(x, Tx), d(z, Tx)\bigr\}.$$

We obtain

$$d(z, Tx) \leq \left(\frac{\alpha}{1-\alpha}\right) d(x, Tx) = rd(x, Tx) \quad \text{for all } x \in X \setminus \{z\}. \tag{2.21}$$

Next, we show that $z \in Tz$. Suppose that z is not an element in Tz.

Case (i): $0 \leq r < \frac{\sqrt{5}-1}{2}$. Let $a \in Tz$. Then $a \neq z$ and so by (2.21), we have

$$d(z, Ta) \leq rd(a, Ta).$$

On the other hand, since $\varphi(r)d(z,Tz) = d(z,Tz) \leq d(z,a)$, from (2.20) we have

$$H(Tz,Ta) \leq \alpha \max\bigl\{d(z,a), d(z,Tz), d(a,Ta), d(z,Ta), d(a,Tz)\bigr\}.$$

So

$$d(a,Ta) \leq H(Tz,Ta) \leq \alpha \max\bigl\{d(z,a), d(z,Tz), d(z,Ta)\bigr\}. \tag{2.22}$$

It implies that

$$d(a,Ta) \leq \alpha \max\bigl\{d(z,a), d(z,Tz), d(z,Ta)\bigr\}.$$

Since $d(z,a) \leq d(z,Tz) + d(Tz,a) = d(z,Tz)$, we have

$$d(a,Ta) \leq rd(z,Tz). \tag{2.23}$$

Using (2.20), (2.21), (2.22), and (2.23), we have

$$\begin{aligned}
d(z,Tz) &\leq d(z,Ta) + H(Ta,Tz) \\
&\leq rd(a,Ta) + \alpha \max\bigl\{d(z,a), d(z,Tz), d(z,Ta)\bigr\} \\
&\leq rd(a,Ta) + \alpha \max\bigl\{d(z,a), d(z,Tz), rd(a,Ta)\bigr\} \\
&\leq rd(a,Ta) + \alpha \max\bigl\{d(z,a), d(z,Tz)\bigr\} \\
&\leq rd(a,Ta) + \alpha d(z,Tz) \\
&\leq (r)^2 d(z,Tz) + rd(z,Tz) \\
&\leq \bigl(r^2 + r\bigr) d(z,Tz),
\end{aligned}$$

where $r = \frac{\alpha}{1-\alpha}$.

Since $r < \frac{\sqrt{5}-1}{2}$, we have $r^2 + r < 1$ and so $d(z,Tz) < d(z,Tz)$, a contradiction. Thus $z \in Tz$.

Case (ii) $\frac{\sqrt{5}-1}{2} \leq r < 1$. Let $x \in X$. If $x = z$, then $H(Tx,Tz) \leq \alpha \max\{d(x,z), d(x,Tx), d(z,Tz), d(x,Tz), d(z,Tx)\}$ holds. If $x \neq z$, then for all $n \in \mathbb{N}$, there exists $y_n \in Tx$ such that

$$d(z,y_n) \leq d(z,Tx) + \left(\frac{1}{n}\right) d(x,z).$$

We consider

$$\begin{aligned}
d(x,Tx) &\leq d(x,y_n) \\
&\leq d(x,z) + d(z,y_n) \\
&\leq d(x,z) + d(z,Tx) + \left(\frac{1}{n}\right) d(x,z) \\
&\leq d(x,z) + rd(x,Tx) + \left(\frac{1}{n}\right) d(x,z).
\end{aligned}$$

Thus, $(1-r)d(x,Tx) \leq (1+\frac{1}{n})d(x,z)$. Take $n \to \infty$, we obtain

$$(1-r)d(x,Tx) \leq d(x,z),$$

by using (2.20), this implies $H(Tx,Tz) \leq \alpha \max\{d(x,z), d(x,Tx), d(z,Tz), d(x,Tz), d(z,Tx)\}$. Hence, as $u_{n+1} \in Tu_n$, it follows that with $x = u_n$

$$\begin{aligned} d(z,Tz) &= \lim_{n\to\infty} d(u_{n+1},Tz) \\ &\leq H(Tu_n,Tz) \\ &\leq \lim_{n\to\infty} \alpha \max\bigl\{d(u_n,z), d(u_n,Tu_n), d(z,Tz), d(u_n,Tz), d(z,Tu_n)\bigr\} \\ &\leq \lim_{n\to\infty} \alpha \max\bigl\{d(u_n,z), d(u_n,u_{n+1}), d(z,Tz), d(u_n,Tz), d(z,u_{n+1})\bigr\} \\ &\leq \alpha d(z,Tz). \end{aligned}$$

Therefore, $(1-\alpha)d(z,Tz) \leq 0$, which implies $d(z,Tz) = 0$. Since Tz is closed, we have $z \in Tz$. This completes the proof. □

Corollary 2.6 *Let be (X,d) a complete metric space and let T be a mapping from X into* $\mathrm{CB}(X)$. *Let $\alpha \in [0,\frac{1}{5})$ and $r = 5\alpha$. Assume that*

$$\varphi(r)d(x,Tx) \leq d(x,y) \quad \textit{implies} \quad H(Tx,Ty) \leq S(x,y)$$

where $S(x,y) = \alpha d(x,y) + \alpha d(x,Tx) + \alpha d(y,Ty) + \alpha d(x,Ty) + \alpha d(y,Tx)$ for all $x,y \in X$, where the function φ is defined as Theorem 2.5. *Then there exists $z \in X$ such that $z \in Tz$.*

Remark 2.7 We see that Theorem 2.5 is a multi-valued mapping generalization of Theorem 2.3 of Kikkawa and Suzuki [7] and therefore the Kannan fixed point theorem [6] for generalized Kannan mappings.

Theorem 2.8 *Define a non-increasing function φ from* $[0,1)$ *into* $(0,1]$ *by*

$$\varphi(r) = \begin{cases} 1, & \text{if } 0 \leq r < \frac{1}{2}, \\ 1-r, & \text{if } \frac{1}{2} \leq r < 1. \end{cases}$$

Let $\alpha \in [0,\frac{1}{5})$ and $r = \frac{3\alpha}{1-2\alpha}$, and let be (X,d) a complete metric space and let T be a mapping from X into $\mathrm{CB}(X)$.

Assume that

$$\varphi(r)d(x,Tx) \leq d(x,y) \quad \textit{implies} \quad H(Tx,Ty) \leq S(x,y) \tag{2.24}$$

where $S(x,y) = \alpha d(x,y) + \alpha d(x,Tx) + \alpha d(y,Ty) + \alpha d(x,Ty) + \alpha d(y,Tx)$ for all $x,y \in X$. Then there exists $z \in X$ such that $z \in Tz$.

Proof Let r_1 be a real number such that $0 \leq r < r_1 < 1$. Let $u_1 \in X$ and $u_2 \in Tu_1$ be arbitrary. Since $u_2 \in Tu_1$, we have $d(u_2,Tu_2) \leq H(Tu_1,Tu_2)$ and

$$\varphi(r)d(u_1,Tu_1) \leq d(u_1,Tu_1) \leq d(u_1,u_2).$$

Thus, from the assumption (2.24),

$$d(u_2, Tu_2) \leq H(Tu_1, Tu_2) \leq S(u_1, u_2)$$

where $S(u_1, u_2) = \alpha d(u_1, u_2) + \alpha d(u_1, Tu_1) + \alpha d(u_2, Tu_2) + \alpha d(u_1, Tu_2) + \alpha d(u_2, Tu_1)$. Consider

$$\begin{aligned} d(u_2, Tu_2) &\leq \alpha d(u_1, u_2) + \alpha d(u_1, Tu_1) + \alpha d(u_2, Tu_2) + \alpha d(u_1, Tu_2) + \alpha d(u_2, Tu_1) \\ &\leq 3\alpha d(u_1, u_2) + 2\alpha d(u_2, Tu_2). \end{aligned}$$

Then

$$d(u_2, Tu_2) \leq \left(\frac{3\alpha}{1-2\alpha}\right) d(u_1, u_2) = rd(u_1, u_2),$$

where $r = \frac{3\alpha}{1-2\alpha}$.

So there exists $u_3 \in Tu_2$ such that $d(u_2, u_3) \leq r_1 d(u_1, u_2)$. Thus, we can construct a sequence $\{x_n\}$ in X such that $u_{n+1} \in Tu_n$ and

$$d(u_{n+1}, u_{n+2}) \leq r_1 d(u_n, u_{n+1}).$$

Hence, by induction

$$d(u_n, u_{n+1}) \leq (r_1)^{n-1} d(u_1, u_2).$$

Then by the triangle inequality, we have

$$\sum_{n=1}^{\infty} d(u_n, u_{n+1}) \leq \sum_{n=1}^{\infty} (r_1)^{n-1} d(u_1, u_2) < \infty.$$

Hence we conclude that $\{u_n\}$ is a Cauchy sequence. Since X is complete, there is some point $z \in X$ such that

$$\lim_{n\to\infty} u_n = z.$$

Now, we will show that $d(z, Tx) \leq rd(x, Tx)$ for all $x \in X \setminus \{z\}$.

Let $x \in X \setminus \{z\}$. Since $u_n \to z$, there exists $n_0 \in N$ such that $d(z, u_n) \leq (\frac{1}{3}) d(z, x)$ for all $n \geq n_0$. By using (2.2), we get

$$\varphi(r) d(u_n, Tu_n) \leq d(x, u_n).$$

Then from (2.1), we have

$$H(Tu_n, T_x) \leq \alpha\big[d(u_n, x) + d(u_n, Tu_n) + d(x, Tx) + d(u_n, Tx) + d(x, Tu_n)\big].$$

Since $u_{n+1} \in Tu_n$, $d(u_{n+1}, T_x) \leq H(Tu_n, T_x)$, so that

$$d(u_{n+1}, Tx) \leq \alpha\big[d(u_n, x) + d(u_n, u_{n+1}) + d(x, Tx) + d(u_n, Tx) + d(x, u_{n+1})\big]$$

for all $n \geq n_0$. Letting $n \to \infty$, we obtain

$$\begin{aligned} d(z, Tx) &\leq \alpha\bigl[2d(z,x) + d(x, Tx) + d(z, Tx)\bigr] \\ &\leq \alpha 3d(z,x) + \alpha 2d(z, Tx). \end{aligned}$$

It follows that

$$d(z, Tx) \leq \left(\frac{3\alpha}{1-2\alpha}\right)d(x, Tx) = rd(x, Tx) \quad \text{for all } x \in X \setminus \{z\}. \tag{2.25}$$

Next, we show that $z \in Tz$. Suppose that z is not element in Tz.

Case (i): $0 \leq r < \frac{1}{2}$. Let $a \in Tz$. Then $a \neq z$ and so by (2.25), we have

$$d(z, Ta) \leq rd(a, Ta).$$

On the other hand, since $\varphi(r)d(z, Tz) = d(z, Tz) \leq d(z, a)$, from (2.24) we have

$$H(Tz, Ta) \leq \alpha\bigl[d(z,a) + d(z, Tz) + d(a, Ta) + d(z, Ta) + d(a, Tz)\bigr].$$

So

$$\begin{aligned} d(a, Ta) \leq H(Tz, Ta) &\leq \alpha\bigl[2d(z,a) + d(a, Ta) + d(z, Ta)\bigr] \\ &\leq \alpha\bigl[3d(z,a) + 2d(a, Ta)\bigr]. \end{aligned} \tag{2.26}$$

Since $d(z,a) \leq d(z, Tz) + d(Tz, a) = d(z, Tz)$, we have

$$d(a, Ta) \leq \left(\frac{3\alpha}{1-2\alpha}\right)d(z, Tz) = rd(z, Tz). \tag{2.27}$$

Using (2.24), (2.26), and (2.27), we have

$$\begin{aligned} d(z, Tz) &\leq d(z, Ta) + H(Ta, Tz) \\ &\leq rd(a, Ta) + S(a,z) \\ &\leq rd(a, Ta) + \alpha\bigl[d(z,a) + d(z, Tz) + d(a, Ta) + d(z, Ta) + d(a, Tz)\bigr] \\ &\leq (r + 2\alpha)d(a, Ta) + 3\alpha d(z,a) \\ &\leq (r + 2\alpha)rd(z, Tz) + 3\alpha d(z, Tz) \\ &\leq (r + r)rd(z, Tz) + rd(z, Tz) \\ &\leq \bigl(2r^2 + r\bigr)d(z, Tz). \end{aligned}$$

Since $0 \leq r < \frac{1}{2}$, we have $0 \leq 2r^2 + r < 1$ and so, $d(z, Tz) < d(z, Tz)$, a contradiction. Thus $z \in Tz$.

Case (ii): $\frac{1}{2} \leq r < 1$. Let $x \in X$. If $x = z$, then $H(Tx, Tz) \leq \alpha[d(x,z) + d(x, Tx) + d(z, Tz) + d(x, Tz) + d(z, Tx)]$ holds. If $x \neq z$, then for all $n \in \mathbb{N}$, there exists $y_n \in Tx$ such that

$$d(z, y_n) \leq d(z, Tx) + \left(\frac{1}{n}\right)d(x,z).$$

We consider

$$\begin{aligned}
d(x,Tx) &\leq d(x,y_n)\\
&\leq d(x,z)+d(z,y_n)\\
&\leq d(x,z)+d(z,Tx)+\left(\frac{1}{n}\right)d(x,z)\\
&\leq d(x,z)+rd(x,Tx)+\left(\frac{1}{n}\right)d(x,z).
\end{aligned}$$

Thus, $(1-r)d(x,Tx)\leq(1+\frac{1}{n})d(x,z)$. Take $n\to\infty$, we obtain

$$(1-r)d(x,Tx)\leq d(x,z),$$

by using (2.24), this implies $H(Tx,Tz)\leq S(x,z)$, where $S(x,z)=\alpha[d(x,z)+d(x,Tx)+d(z,Tz)+d(x,Tz)+d(z,Tx)]$. Hence, as $u_{n+1}\in Tu_n$, it follows that with $x=u_n$

$$\begin{aligned}
d(z,Tz) &= \lim_{n\to\infty} d(u_{n+1},Tz)\\
&\leq H(Tu_n,Tz)\\
&\leq \lim_{n\to\infty}\alpha\big[d(u_n,z)+d(u_n,Tu_n)+d(z,Tz)+d(u_n,Tz)+d(z,Tu_n)\big]\\
&\leq \lim_{n\to\infty}\big[\alpha d(u_n,z)+\alpha d(u_n,u_{n+1})+\alpha d(z,Tz)+\alpha d(u_n,Tz)+\alpha d(z,u_{n+1})\big]\\
&\leq (2\alpha)d(z,Tz). \qquad (2.28)
\end{aligned}$$

Therefore, $(1-2\alpha)d(z,Tz)\leq 0$, which implies $d(z,Tz)=0$. Since Tz is closed, we have $z\in Tz$. This completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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