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Optimality and mixed duality in multiobjective E-convex programming

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Abstract

In this paper, we consider a class of multiobjective E-convex programming problems with inequality constraints, where the objective and constraint functions are E-convex functions which were firstly introduced by Youness (J. Optim. Theory Appl. 102:439-450, 1999). Fritz-John and Kuhn-Tucker necessary and sufficient optimality theorems for the multiobjective E-convex programming are established under the weakened assumption of the theorems in Megahed *et al.* (J. Inequal. Appl. 2013:246, 2013) and Youness (Chaos Solitons Fractals 12:1737-1745, 2001). A mixed duality for the primal problem is formulated and weak and strong duality theorems between primal and dual problems are explored. Illustrative examples are given to explain the obtained results.

MSC: 90C29; 90C30; 69K05

Keywords: E-convex function; mixed duality; multiobjective programming; optimality condition

1 Introduction

The concepts of an E-convex set and an E-convex function were introduced first by Youness [1]. Subsequently, necessary and sufficient optimality criteria for a class of E-convex programming problems were discussed by Youness [2], and E-Fritz-John and E-Kuhn-Tucker problems, which modified the Fritz-John and Kuhn-Tucker problems, were also presented. In Megahed *et al.* [3], the concept of an E-differentiable convex function which transforms a non-differentiable convex function to a differentiable function under an operator $E: \mathbb{R}^n \to \mathbb{R}^n$ was presented, then a solution of mathematical programming with a non-differentiable function could be found by applying the Fritz-John and Kuhn-Tucker conditions due to Mangasarian [4].

However, on the other hand, the results on E-convex programming in Youness [1] were not correct, and some counterexamples were given by Yang [5]. The results concerning the characterization of an E-convex function f in terms of its E-epigraph in Youness [1] were also not correct, and some characterizations of E-convex functions using a different notion of epigraph were given by Duca *et al.* [6].

Based on the correct results in Youness [1], a class of semi-E-convex functions was introduced by Chen [7], the concepts of E-quasiconvex functions and strictly E-quasiconvex functions were introduced by Syau and Stanley Lee [8], respectively.

In fact, after defining the E-convex function in 1999, Youness [1] pointed out that the E-convex function that he defined had more generalized results than a convex function. He dealt mainly with some properties of an E-convex set and an E-convex function, a programming problem without E in both objective functions or constrained functions, and the relation between solutions of objective and constrained functions with and without E. He then drew the conclusion that the E-convex set and E-convex function were more generalized than the convex set and function proposed by Hanson [9], Hanson and Mond [10], and Kaul and Kaur [11].

This paper also addresses a counterexample of Theorem 4.1 in Youness [1]. Characterization of efficient solutions based on the modification of Theorem 4.2 in Youness [1] is presented. A sufficient optimality theorem is given by using this characterization and E-convexity conditions. We obtain the scalarization method due to Chankong and Haimes [12] for multiobjective E-convex programming. By employing this scalarization method, Fritz-John and Kuhn-Tucker necessary theorems for the multiobjective case are established under the weakened assumption of the theorems in Megahed *et al.* [3] and Youness [2]. Moreover, a mixed type dual for the primal problem is given. Under the assumption of the E-convex conditions, weak and strong duality theorems between the primal and dual problems are established, and we also propose some examples to illustrate our results.

2 Preliminaries

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space. The following conventions for a vector in $\mathbb{R}^n$ will be used in this paper:

$$\begin{aligned} &x < y \quad \text{if and only if} \quad x_i < y_i \quad \text{for all } i = 1, 2, \ldots, p, \\ &x \leqq y \quad \text{if and only if} \quad x_i \leqq y_i \quad \text{for all } i = 1, 2, \ldots, p, \\ &x \leq y \quad \text{if and only if} \quad x_i \leqq y_i \quad \text{for all } i = 1, 2, \ldots, p \text{ but } x \neq y. \end{aligned}$$

We present some concepts of E-convex set and E-convex function; for convenience, we recall the definition of E-convex set first.

Definition 2.1 [1] A set $M \subset \mathbb{R}^n$ is said to be E-convex iff there is a map $E : \mathbb{R}^n \to \mathbb{R}^n$ such that $(1 - \lambda)E(x) + \lambda E(y) \in M$, for each $x, y \in M$, and $\lambda \in [0, 1]$.

It is clear that if $M \subset \mathbb{R}^n$ is convex, then M is E-convex by taking a map $E : \mathbb{R}^n \to \mathbb{R}^n$ as the identity map, but the converse may not be true; see the following example.

Example 2.1 Consider the set $S_1 = \{(x, y) \in \mathbb{R}^2 \mid y \leqq x, 0 \leqq x \leqq 1\}$. Let $E(x, y) = (\sqrt{x}, y)$, it is clear that S_1 is E-convex (since S_1 is convex). It is easy to check that $E(S_1)$ is E-convex by taking the map $E(x, y) = (\sqrt{x}, y)$, while $E(S_1)$ is not convex, where $E(S_1) = \{(x, y) \in \mathbb{R}^2 \mid y \leqq x^2, 0 \leqq x \leqq 1\}$.

However, if $E : \mathbb{R}^n \to \mathbb{R}^n$ is a surjective map, it is easy to check that the converse also holds. Note that E is said to be *surjective* if there exists $x \in M$ such that $E(x) = y$, $\forall y \in E(M)$.

Definition 2.2 [1] A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be E-convex on $M \in \mathbb{R}^n$ iff there is a map $E : \mathbb{R}^n \to \mathbb{R}^n$ such that M is an E-convex set and

$$f\bigl(\lambda E(x) + (1-\lambda)E(y)\bigr) \leqq \lambda f\bigl(E(x)\bigr) + (1-\lambda)f\bigl(E(y)\bigr)$$

for each $x, y \in M$ and $0 \leqq \lambda \leqq 1$. Moreover, if

$$f\bigl(\lambda E(x) + (1-\lambda)E(y)\bigr) \geqq \lambda f\bigl(E(x)\bigr) + (1-\lambda)f\bigl(E(y)\bigr)$$

then f is called E-concave on M. If the inequality signs in the above two inequalities are strict, then f is called strictly E-convex and strictly E-concave, respectively.

Remark 2.1 Let f, g be E-convex on M. Then $f + g$, αf ($\alpha \geqq 0$) are E-convex on the set M.

It is easy to check that every convex function f on a convex set M is an E-convex function, where E is the identity map. But the converse may not hold, we recall the example from [1].

Example 2.2 Define the function $f : \mathbb{R} \to \mathbb{R}$ as

$$f(x) = \begin{cases} 1, & \text{if } x > 0, \\ -x, & \text{if } x \leqq 0, \end{cases}$$

and let $E : \mathbb{R} \to \mathbb{R}$ be defined as $E(x) = -x^2$. Then $\mathbb{R}$ is an E-convex set and f is E-convex but not convex.

Obviously, if f is a real-valued differentiable function on an E-convex set $M \subset \mathbb{R}^n$, we can define a differentiable E-convex function in the following.

Definition 2.3 f is E-convex on M if and only if for each $x, y \in M$

$$f\bigl(E(x)\bigr) - f\bigl(E(y)\bigr) \geqq \nabla f\bigl(E(y)\bigr)\bigl(E(x) - E(y)\bigr).$$

3 Optimality criteria

In this section, we suppose that $E : M \to M$ ($M \subset \mathbb{R}^n$) is a surjective map. In addition, as we know if a set $M \subset \mathbb{R}^n$ is E-convex with respect to a mapping $E : \mathbb{R}^n \to \mathbb{R}^n$, then $E(M) \subset M$ (see [1], Proposition 2.2). For an E-convex function f, we say that the function $(f \circ E) : M \to \mathbb{R}$ defined by $(f \circ E)(x) = f(E(x))$ for all $x \in M$ is well defined (see [8]).

Consider the following multiobjective nonlinear program:

$$\begin{aligned} \text{(MP)} \quad & \text{Maximize} \quad f(x) = \bigl(f_1(x), f_2(x), \ldots, f_p(x)\bigr), \\ & \text{subject to} \quad x \in M = \bigl\{x \in \mathbb{R}^n \mid g_j(x) \leqq 0, j = 1, 2, \ldots, m\bigr\}, \end{aligned}$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$, $i \in P = \{1, 2, \ldots, p\}$ *and* $g_j : \mathbb{R}^n \to \mathbb{R}$, $j \in Q = \{1, 2, \ldots, m\}$ *are E-convex functions.*

Then we give the following E-convex program related to (MP):

$$(\mathbf{MP_E}) \quad \text{Maximize} \quad (f \circ E)(x) = \bigl((f_1 \circ E)(x), f_2 \circ E\bigr)(x), \ldots, f_p \circ E)(x)),$$
$$\text{subject to} \quad x \in E(M) = \bigl\{x \in \mathbb{R}^n \mid (g_j \circ E)(x) \leqq 0, j = 1, 2, \ldots, m\bigr\},$$

where $f_i \circ E$, $i \in P$ and $g_j \circ E$, $j \in Q$ are differentiable on M.

It states that, for a surjective map E, if f is E-convex, then $f \circ E$ is obviously convex.

Definition 3.1 A point $\bar{x} \in E(M)$ is said to be an efficient solution of ($\mathrm{MP_E}$) if and only if there is no other $x \in E(M)$ such that

$$(f_{i_0} \circ E)(x) < (f_{i_0} \circ E)(\bar{x}) \quad \text{for some } i_0 \in P$$

and

$$(f_i \circ E)(x) \leqq (f_i \circ E)(\bar{x}) \quad \text{for all } i \in P,$$

where $P = \{1, 2, \ldots, p\}$, that is

$$(f \circ E)(x) \leq (f \circ E)(\bar{x}).$$

Now we give a counterexample which is easier to understand than the one in [5], to show that Theorem 4.1 (In (MP), the set M is an E-convex set.) in Youness [1] is incorrect.

Example 3.1 In (MP), g_j, $j \in Q$ are E-convex, but M does not always need to be E-convex set.

Let $g(x) = x \in \mathbb{R}$ and define the map E as $E(x) = |x|$. Then $g(x)$ is E-convex. Take $x = -1$, $y = -1/2$. Then $g(-1) = -1$, $g(-1/2) = -1/2$.

So, $-1, -1/2 \in M = \{x \in \mathbb{R} \mid g(x) \leqq 0\}$. But, for all $\lambda \in [0, 1]$,

$$g\bigl(\lambda E(x) + (1 - \lambda)E(y)\bigr) = g\bigl(\lambda|x| + (1 - \lambda)|y|\bigr) = \frac{1 + \lambda}{2} > 0.$$

Hence, M is not E-convex set.

Also, Theorem 4.2 in Youness [1] is incorrect. The counterexample was given by Yang [5].

Now we would like to present the characterization of efficient solutions modifying Theorem 4.2 in Youness [1] by using only surjective assumption of the mapping E as follows.

Theorem 3.1 *Let $E : M \to M$ be a surjective map. Then $\bar{x}$ is an efficient solution of* ($\mathrm{MP_E}$) *if and only if $E(\bar{x})$ is an efficient solution of* (MP).

Proof Suppose that $E(\bar{x})$ is not an efficient solution of (MP). Then there exists $\bar{z} \in M$ such that $f(\bar{z}) \leq f(E(\bar{x}))$. Since E is surjective, we have $E(M) = M$, then there exists $\bar{y} \in M$ such that $\bar{z} = E(\bar{y})$, that is, $(f \circ E)(\bar{y}) \leq (f \circ E)(\bar{x})$, which contradicts that $\bar{x}$ is an efficient solution of ($\mathrm{MP_E}$).

Conversely, suppose that $\bar{x}$ is not an efficient solution of $(\mathrm{MP_E})$, then there exists $y^* \in E(M)$ such that $(f \circ E)(y^*) \leq (f \circ E)(\bar{x})$. Since E is surjective, there exists $z^* \in M$ such that $E(y^*) = z^*$. Hence $f(z^*) \leq f(E(\bar{x}))$, which contradicts that $E(\bar{x})$ is an efficient solution of (MP). □

With the help of Theorem 3.1 and the E-convexity assumption, we now give the sufficient optimality condition.

Theorem 3.2 (Sufficient optimality condition) *Assume that* $(\bar{x}, \bar{\lambda}, \bar{\mu})$ *satisfies the following conditions*:

$$\bar{\lambda}\nabla(f \circ E)(\bar{x}) + \bar{\mu}\nabla(g \circ E)(\bar{x}) = 0,$$
$$\bar{\mu}(g \circ E)(\bar{x}) = 0,$$
$$(g \circ E)(\bar{x}) \leqq 0,$$
$$\bar{\lambda} > 0, \qquad \bar{\mu} \geqq 0,$$

where $\bar{\lambda} \in \mathbb{R}^p$, $\bar{\mu} \in \mathbb{R}^m$.

Then $\bar{x}$ *is an efficient solution of* $(\mathrm{MP_E})$.

Proof Suppose that $\bar{x}$ is not an efficient solution of $(\mathrm{MP_E})$, then there exists $x^* \in E(M)$ such that

$$(f \circ E)\bigl(x^*\bigr) \leq (f \circ E)(\bar{x}). \tag{3.1}$$

Since f_i and g_j are E-convex and $f_i \circ E$ and $g_j \circ E$ are differentiable on M, for any $x \in E(M)$, we have

$$(f_i \circ E)(x) - (f_i \circ E)(\bar{x}) \geqq (x - \bar{x})\nabla(f_i \circ E)(\bar{x}), \tag{3.2}$$
$$(g_j \circ E)(x) - (g_j \circ E)(\bar{x}) \geqq (x - \bar{x})\nabla(g_j \circ E)(\bar{x}). \tag{3.3}$$

Since $\bar{\lambda} > 0$, $\bar{\mu} \geqq 0$, from (3.2) and (3.3), for each $i \in P$ and $j \in Q$, we have

$$\begin{aligned}&\bar{\lambda}(f \circ E)(x) - \bar{\lambda}(f \circ E)(\bar{x}) + \bar{\mu}(g \circ E)(x) - \bar{\mu}(g \circ E)(\bar{x})\\ &\quad \geqq (x - \bar{x})\bigl[\bar{\lambda}\nabla(f \circ E)(\bar{x}) + \bar{\mu}\nabla(g \circ E)(\bar{x})\bigr].\end{aligned}$$

Since $\bar{\lambda}\nabla(f \circ E)(\bar{x}) + \bar{\mu}\nabla(g \circ E)(\bar{x}) = 0$, $\bar{\mu}(g \circ E)(\bar{x}) = 0$ and $(g \circ E)(\bar{x}) \leqq 0$, we get

$$(f \circ E)(x) \geqq (f \circ E)(\bar{x}),$$

which contradicts (3.1). □

Remark 3.1 If we replace the E-convexity of f_i and $\bar{\lambda} > 0$ by the strictly E-convexity of f_i and $\bar{\lambda} \geq 0$, respectively, then Theorem 3.2 also holds.

Now we present the following lemma due to Chankong and Haimes [12] to deal with the relationship between the scalar and multiobjective programming problems.

Lemma 3.1 *$\bar{x}$ is an efficient solution for* $(\mathrm{MP_E})$ *if and only if $\bar{x}$ solves*

$$\begin{aligned} (\mathbf{MP_E})_{\boldsymbol{k}} \quad & \textit{Minimize} \quad (f_k \circ E)(x) \\ & \textit{subject to} \quad (f_i \circ E)(x) \leqq (f_i \circ E)(\bar{x}), \quad i \in P^k := P \setminus \{k\}, \\ & \qquad (g \circ E)(x) \leqq 0, \end{aligned}$$

for each $k = 1, 2, \ldots, p$.

Proof Suppose that $\bar{x}$ is not a solution of $(\mathrm{MP_E})_k$. Then there exists $x \in E(M)$ such that

$$(f_k \circ E)(x) < (f_k \circ E)(\bar{x}), \quad k \in P, \tag{3.4}$$

$$(f_i \circ E)(x) \leqq (f_i \circ E)(\bar{x}), \quad i \neq k. \tag{3.5}$$

From (3.4) and (3.5), we conclude that $\bar{x}$ is not efficient for $(\mathrm{MP_E})$.

Conversely, assume that $\bar{x}$ is a solution of $(\mathrm{MP_E})_k$ for every $k \in P$, then for all $x \in E(M)$ with $(f_i \circ E)(x) \leqq (f_i \circ E)(\bar{x})$, $i \neq k$, we have $(f_k \circ E)(\bar{x}) \leqq (f_k \circ E)(x)$. Then there exists no other $x \in E(M)$ such that $(f_i \circ E)(x) \leqq (f_i \circ E)(\bar{x})$, $i \in P$, with strict inequality holding for at least one i. This implies that $\bar{x}$ is efficient for $(\mathrm{MP_E})$. □

Remark 3.2 Without loss of generality, we assume that $P \cap Q = \emptyset$. Set

$$(G_t \circ E)(x) = \begin{cases} (f_t \circ E)(x) - (f_t \circ E)(\bar{x}), & t \in P^k, \\ (g_t \circ E)(x), & t \in Q, \end{cases}$$

and $T = P^k \cup Q$. Then $(\mathrm{MP_E})_k$ is equivalent to the following problem:

$$\min (f_k \circ E)(x) \quad \text{subject to} \quad (G_t \circ E)(x) \leqq 0, \quad t \in T, \text{for each } k \in P.$$

In order to obtain the necessary optimality condition, we employ the following generalized linearization lemma due to Mangasarian [4].

Lemma 3.2 *Let $\bar{x}$ be a local solution of* $(\mathrm{MP_E})_k$, *let $f_k \circ E$, for each $k \in P$ and $G_t \circ E$, $t \in T$ be differentiable at $\bar{x}$. Then the system*

$$\nabla (f_k \circ E)(\bar{x}) z < 0,$$

$$\nabla (G_W \circ E)(\bar{x}) z < 0,$$

$$\nabla (G_V \circ E)(\bar{x}) z \leqq 0,$$

has no solution $z \in \mathbb{R}^n$, for each $k \in P$, where we denote

$$I = \{ t \mid (G_t \circ E)(\bar{x}) = 0 \}, \qquad J = \{ t \mid (G_t \circ E)(\bar{x}) < 0 \}, \qquad I \cup J = T,$$

$$V = \{ t \mid (G_t \circ E)(\bar{x}) = 0 \textit{ and } (G_t \circ E) \textit{ is E-concave at } \bar{x} \},$$

$$W = \{ t \mid (G_t \circ E)(\bar{x}) = 0 \textit{ and } (G_t \circ E) \textit{ is not E-concave at } \bar{x} \},$$

$$I = V \cup W.$$

We establish the following Fritz-John necessary optimality criteria by using Lemma 3.2.

Theorem 3.3 (Fritz-John necessary condition) *Assume that* $\bar{x} \in E(M)$ *is an efficient solution of* $(\mathrm{MP_E})$, *then there exist* $\bar{\lambda} \in \mathbb{R}^p$, $\bar{\mu} \in \mathbb{R}^m$ *such that*

$$\begin{aligned} &\bar{\lambda}\nabla(f \circ E)(\bar{x}) + \bar{\mu}\nabla(g \circ E)(\bar{x}) = 0, \\ &\bar{\mu}(g \circ E)(\bar{x}) = 0, \\ &(g \circ E)(\bar{x}) \leqq 0, \\ &(\bar{\lambda}, \bar{\mu}) \geq 0. \end{aligned}$$

Proof Since $\bar{x}$ is an efficient solution of $(\mathrm{MP_E})$, then from Lemma 3.1, $\bar{x}$ solves $(\mathrm{MP_E})_k$ for each $k \in P$. By Lemma 3.2 and Remark 3.2, we see that the system

$$\begin{aligned} &\nabla(f_k \circ E)(\bar{x})z < 0, \\ &\nabla(G_W \circ E)(\bar{x})z < 0, \\ &\nabla(G_V \circ E)(\bar{x})z \leqq 0, \end{aligned}$$

has no solution $z \in \mathbb{R}^n$. Hence by Motzin's theorem [4], there exist $\bar{\lambda}_k$, $\bar{\mu}_W$, $\bar{\mu}_V$ such that

$$\begin{aligned} &\bar{\lambda}_k\nabla(f_k \circ E)(\bar{x}) + \bar{\mu}_W\nabla(G_W \circ E)(\bar{x}) + \bar{\mu}_V\nabla(G_V \circ E)(\bar{x}) = 0, \\ &(\bar{\lambda}_k, \bar{\mu}_W) \geq 0, \\ &\bar{\mu}_V \geqq 0. \end{aligned}$$

Since $(G_W \circ E)(\bar{x}) = 0$ and $(G_V \circ E)(\bar{x}) = 0$, it follows that if we define $\bar{\mu}_J = 0$ and $\bar{\mu} = (\bar{\mu}_W, \bar{\mu}_V, \bar{\mu}_J)$, then

$$\bar{\mu}(G \circ E)(\bar{x}) = \bar{\mu}_W(G_W \circ E)(\bar{x}) + \bar{\mu}_V(G_V \circ E)(\bar{x}) + \bar{\mu}_J(G_J \circ E)(\bar{x}) = 0,$$

here, we can reduce $\bar{\mu}(g \circ E)(\bar{x}) = 0$. Thus $\bar{\lambda}_k\nabla(f_k \circ E)(\bar{x}) + \bar{\mu}(g \circ E)(\bar{x}) = 0$ and $(\bar{\lambda}_k, \bar{\mu}) \geq 0$. Then, for each $k \in P$, we have

$$\begin{aligned} &\bar{\lambda}\nabla(f \circ E)(\bar{x}) + \bar{\mu}\nabla(g \circ E)(\bar{x}) = 0, \\ &(\bar{\lambda}, \bar{\mu}) \geq 0. \end{aligned}$$

Since $x^* \in E(M)$, $(g \circ E)(x^*) \leqq 0$.

The proof is complete. □

Theorem 3.4 (Kuhn-Tucker necessary condition) *If* $\bar{x} \in E(M)$ *is an efficient solution of* $(\mathrm{MP_E})$ *and* $G_t \circ E$, $t \in T$ *satisfies a constraint qualification* [4] *for* $(\mathrm{MP_E})_k$ *for at least one* $k \in P$. *Then there exist* $\bar{\lambda} \in \mathbb{R}^p$ *and* $\bar{\mu} \in \mathbb{R}^m$ *such that*

$$\begin{aligned} &\bar{\lambda}\nabla(f \circ E)(\bar{x}) + \bar{\mu}\nabla(g \circ E)(\bar{x}) = 0, \\ &\bar{\mu}(g \circ E)(\bar{x}) = 0, \end{aligned}$$

$$(g \circ E)(\bar{x}) \leqq 0,$$
$$\bar{\lambda} \geq 0, \qquad \bar{\mu} \geqq 0.$$

Proof Since $\bar{x}$ is an efficient solution of $(\mathrm{MP_E})$, then by Theorem 3.3 there exist $\bar{\lambda} \in \mathbb{R}^p$, $\bar{\mu} \in \mathbb{R}^m$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ satisfies

$$\bar{\lambda}\nabla(f \circ E)(\bar{x}) + \bar{\mu}\nabla(g \circ E)(\bar{x}) = 0,$$
$$\bar{\mu}(g \circ E)(\bar{x}) = 0,$$
$$(g \circ E)(\bar{x}) \leqq 0,$$
$$(\bar{\lambda}, \bar{\mu}) \geq 0.$$

We only have to show that $\bar{\lambda} \geq 0$, that is, $\bar{\lambda}_k > 0$ for at least one $k \in P$.

Since $(\bar{\lambda}, \bar{\mu}) \geq 0$, $(\bar{\lambda}, \bar{\mu}_W) \geq 0$, we have $\bar{\lambda}_k > 0$ for at least one $k \in P$ if W is empty. Now, we show that $\bar{\lambda}_k > 0$ for at least one $k \in P$ if W is nonempty by contradiction.

Suppose that $\bar{\lambda}_k = 0$ for all $k \in P$. Since $\bar{\mu}_J = 0$ as we define in the proof of Theorem 3.3, we have $\bar{\mu}_W \nabla(G_W \circ E)(\bar{x}) + \bar{\mu}_V \nabla(G_V \circ E)(\bar{x}) = 0$, $\bar{\mu}_W \geq 0$, $\bar{\mu}_V \geqq 0$. Since $G_t \circ E$ satisfies the Arrow-Hurwicz-Uzawa constraint qualification [4] at $\bar{x}$ for $(\mathrm{MP_E})_k$ for at least one $k \in P$, there exists $\bar{z} \in \mathbb{R}^n$ such that

$$\nabla(G_W \circ E)(\bar{x})\bar{z} > 0, \tag{3.6}$$
$$\nabla(G_V \circ E)(\bar{x})\bar{z} \geqq 0. \tag{3.7}$$

Multiplying (3.6) and (3.7) by $\bar{\mu}_W$ and $\bar{\mu}_V$, respectively, then yields

$$\bar{\mu}_W \nabla(G_W \circ E)(\bar{x})\bar{z} + \bar{\mu}_V \nabla(G_W \circ E)(\bar{x})\bar{z} > 0,$$

which contradicts the fact that

$$\bar{\mu}_W \nabla(G_W \circ E)(\bar{x})\bar{z} + \bar{\mu}_V \nabla(G_W \circ E)(\bar{x})\bar{z} = 0.$$

Hence $\bar{\lambda}_k > 0$ for at least one $k \in P$. Then we obtain $\bar{\lambda} \geq 0$. □

Remark 3.3 If we replace our surjective assumption of E by bijection (or linearity) of E, then our Fritz-John and Kuhn-Tucker necessary optimality results reduce to the ones in Megahed *et al.* [3] (or Youness [2]).

Example 3.2 Consider the following problem:

$$\begin{aligned} (\widehat{\mathbf{MP}}) \quad & \text{Minimize} \quad \big(f_1(x), f_2(x)\big), \\ & \text{subject to} \quad x \in M = \big\{x \in \mathbb{R} \mid g_1(x) \leqq 0, g_2(x) \leqq 0\big\}, \end{aligned}$$

where $f_1(x) = x$, $f_2(x) = x^2$, $g_1(x) = x - 1$, and $g_2(x) = -x$.

Let $E : M \to E(M)$ defined by $E(x) = x + 1$ be the surjective map, then we get the following E-convex programming problem related to $(\widehat{\mathrm{MP}})$:

$$\begin{aligned}
&(\widehat{\mathbf{MP_E}}) \quad \text{Minimize} \quad \big((f_1 \circ E)(x), (f_2 \circ E)(x)\big), \\
&\qquad\qquad \text{subject to} \quad x \in E(M) = \big\{x \in \mathbb{R} \mid (g_1 \circ E)(x) \leqq 0, (g_2 \circ E)(x) \leqq 0\big\},
\end{aligned}$$

where $(f_1 \circ E)(x) = x - 1$, $(f_2 \circ E)(x) = x^2 - 2x + 1$, $(g_1 \circ E)(x) = x - 2$, and $(g_2 \circ E)(x) = -x + 1$.

(a) It is easy to check that the feasible sets of $(\widehat{\mathrm{MP}})$ and $(\widehat{\mathrm{MP_E}})$ are $M = [0, 1]$ and $E(M) = [1, 2]$, respectively.

(b) By the definition of an efficient solution, we see that $x^* = 0 \in M$ is the efficient solution of $(\widehat{\mathrm{MP}})$ and $\bar{x} = E(x^*) = 1 \in E(M)$ is the efficient solution of $(\widehat{\mathrm{MP_E}})$, hence Theorem 3.1 holds.

(c) We can easily check that $(\bar{x}, (\bar{\lambda}_1, \bar{\lambda}_2), (\bar{\mu}_1, \bar{\mu}_2)) = (1, (\frac{1}{2}, 1), (0, \frac{1}{2}))$ satisfy the conditions in Theorem 3.2, and $\bar{x} = 1$ is the efficient solution of $(\widehat{\mathrm{MP_E}})$, hence Theorem 3.2 holds.

(d) Since the efficient solution $\bar{x} = 1$ for $(\widehat{\mathrm{MP_E}})$, also solves both $(\widehat{\mathrm{MP_E}})_1$ and $(\widehat{\mathrm{MP_E}})_2$, Lemma 3.1 holds, where

$$\begin{aligned}
(\widehat{\mathbf{MP_E}})_\mathbf{1} \quad &\text{Mimimize} \quad (f_1 \circ E)(x), \\
&\text{subject to} \quad (f_2 \circ E)(x) \leqq (f_2 \circ E)(\bar{x}), \\
&\qquad\qquad\quad x \in E(M),
\end{aligned}$$

and

$$\begin{aligned}
(\widehat{\mathbf{MP_E}})_\mathbf{2} \quad &\text{Mimimize} \quad (f_2 \circ E)(x), \\
&\text{subject to} \quad (f_1 \circ E)(x) \leqq (f_1 \circ E)(\bar{x}), \\
&\qquad\qquad\quad x \in E(M).
\end{aligned}$$

(e) As $\bar{x} = 1$ is the efficient solution of $(\widehat{\mathrm{MP_E}})$, then there exist $\bar{\lambda} = (\frac{1}{2}, 1)$ and $\bar{\mu} = (0, \frac{1}{2})$ satisfy the conditions in Theorem 3.3, hence Theorem 3.3 holds.

(f) $\bar{x} = 1$ is the efficient solution of $(\widehat{\mathrm{MP_E}})$ and it is easy to check the problem $(\widehat{\mathrm{MP_E}})_1$ satisfies the Kuhn-Tucker constraint qualification [4], and there exist $\bar{\lambda} = (\frac{1}{2}, 1)$ and $\bar{\mu} = (0, \frac{1}{2})$ satisfying the conditions in Theorem 3.4, hence Theorem 3.4 holds.

4 Duality

Recently, several researchers found some results on mixed dual model under some generalized convexity; see [13–15], for example. In this section, first we establish the following mixed dual problem (MD) to (MP):

$$\begin{aligned}
(\mathbf{MD}) \quad \text{Maximize} \quad &\Big(f_1(u) + \sum_{j \in J_0} \mu_j^T g_j(u), \ldots, f_p(u) + \sum_{j \in J_0} \mu_j^T g_j(u)\Big) \\
\text{subject to} \quad &\sum_{i=1}^{p} \lambda_i^T \nabla f_i^T(u) + \sum_{j=1}^{q} \mu_j^T \nabla g_j(u) = 0, \\
&\sum_{j \in J_\alpha} \mu_j^T g_j(u) \geqq 0, \quad \alpha = 1, 2, \ldots, r, \\
&\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p) \in \Lambda^+, \\
&\mu_j \geqq 0, \quad j \in Q = \{1, 2, \ldots, q\},
\end{aligned}$$

where $J_\alpha \subset Q = \{1,2,\ldots,q\}$, $\alpha = 0,1,\ldots,r$ with $\bigcup_{\alpha=0}^{r} J_\alpha = Q$ and $J_\alpha \cap J_\beta = \emptyset$ if $\alpha \neq \beta$. $\Lambda^+ = \{\lambda \in \mathbb{R}^p \mid \lambda \geqq 0, \lambda^T e = 1, e = (1,\ldots,1)^T \in \mathbb{R}^p\}$.

Then we formulate the following mixed dual problem ($\mathrm{MD_E}$) to ($\mathrm{MP_E}$):

$$(\mathbf{MD_E}) \quad \text{Maximize} \quad \Big((f_1 \circ E)(u) + \sum_{j\in J_0} \mu_j^T (g_j \circ E)(u), \ldots, (f_p \circ E)(u) + \sum_{j\in J_0} \mu_j^T (g_j \circ E)(u)\Big)$$

$$\text{subject to} \quad \sum_{i=1}^{p} \lambda_i^T \nabla (f_i \circ E)(u) + \sum_{j=1}^{q} \mu_j^T \nabla (g_j \circ E)(u) = 0,$$

$$\sum_{j\in J_\alpha} \mu_j^T (g_j \circ E)(u) \geqq 0, \quad \alpha = 1,2,\ldots,r,$$

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p) \in \Lambda^+,$$

$$\mu_j \geqq 0, \quad j \in Q = \{1,2,\ldots,q\},$$

where $J_\alpha \subset Q = \{1,2,\ldots,q\}$, $\alpha = 0,1,\ldots,r$ with $\bigcup_{\alpha=0}^{r} J_\alpha = Q$ and $J_\alpha \cap J_\beta = \emptyset$ if $\alpha \neq \beta$; $\Lambda^+ = \{\lambda \in \mathbb{R}^p \mid \lambda \geqq 0, \lambda^T e = 1, e = (1,\ldots,1)^T \in \mathbb{R}^p\}$.

(1) If $J_0 = Q$, then our mixed dual type ($\mathrm{MD_E}$) (or (MD)) reduces to the Wolfe dual type.
(2) If $J_0 = \emptyset$, then our mixed dual type ($\mathrm{MD_E}$) (or (MD)) reduces to the Mond-Weir dual type.

Theorem 4.1 *Let $E : M \to M$ be a surjective map. Then $\bar{u}$ is an efficient solution of* ($\mathrm{MD_E}$) *if and only if $E(\bar{u})$ is an efficient solution of* (MD).

Proof By Lemma 3.1, we can obtain this theorem. □

Assume that f is an E-convex function and $E : M \to M$ ($M \subset \mathbb{R}^n$) is a surjective map, by Lemma 3.1, we can study dual problem between (MP) and (MD). Here, we would like to study the dual problem between ($\mathrm{MP_E}$) and ($\mathrm{MD_E}$).

Theorem 4.2 (Weak duality) *Assume that for all feasible x of* ($\mathrm{MP_E}$) *and all feasible (u, λ, μ) of* ($\mathrm{MD_E}$), *f_i, g_j are E-convex functions. If also either*

(a) $\lambda_i > 0$ *for all $i = 1,2,\ldots,p$, or*
(b) $\sum_{i=1}^{p} \lambda_i f_i(\cdot) + \sum_{j=1}^{q} \mu_j g_j(\cdot)$ *is strictly E-convex at u,*

then the following cannot hold:

$$(f_{i_0} \circ E)(x) \leqq (f_{i_0} \circ E)(u) + \sum_{j\in J_0} \mu_j^T (g_j \circ E)(u) \quad \text{for all } i_0 \in P, \tag{4.1}$$

$$(f_i \circ E)(x) < (f_i \circ E)(u) + \sum_{j\in J_0} \mu_j^T (g_j \circ E)(u) \quad \text{for some } i \in P. \tag{4.2}$$

Proof Suppose to the contrary that (4.1) and (4.2) hold. Since x is feasible for ($\mathrm{MP_E}$) and $\mu \geqq 0$, from (4.1) and (4.2), we imply

$$(f_{i_0} \circ E)(x) + \sum_{j\in J_0} \mu_j^T (g_j \circ E)(x) \leqq (f_{i_0} \circ E)(u) + \sum_{j\in J_0} \mu_j^T (g_j \circ E)(u) \quad \text{for all } i_0 \in P, \tag{4.3}$$

$$(f_i \circ E)(x) + \sum_{j \in J_0} \mu_j^T (g_j \circ E)(x) < (f_i \circ E)(u) + \sum_{j \in J_0} \mu_j^T (g_j \circ E)(u) \quad \text{for some } i \in P. \tag{4.4}$$

If hypothesis (a) holds, then with $\sum_{i=1}^p \lambda_i = 1$, one has

$$\sum_{i=1}^p \lambda_i (f_i \circ E)(x) + \sum_{j \in J_0} \mu_j^T (g_j \circ E)(x) < \sum_{i=1}^p \lambda_i (f_i \circ E)(u) + \sum_{j \in J_0} \mu_j^T (g_j \circ E)(u) \tag{4.5}$$

and since f_i, g_j are E-convex and $\lambda_i > 0$, $i = 1, 2, \dots, p$, $\mu \geqq 0$, it now follows from (4.5) that

$$\bigl(E(x) - E(u)\bigr)^T \left(\sum_{i=1}^p \lambda_i \nabla (f_i \circ E)(u) + \sum_{j \in J_0} \mu_j^T (g_j \circ E)(u) \right) < 0,$$

which contradicts the fact that

$$\sum_{i=1}^p \lambda_i \nabla (f_i \circ E)(u) + \sum_{j=1}^q \mu_j^T \nabla (g_j \circ E)(u) = 0.$$

On the other hand, since $\lambda_i \geqq 0$, $i = 1, 2, \dots, p$ and $\sum_{i=1}^p \lambda_i = 1$, (4.3) and (4.4) imply

$$\sum_{i=1}^p \lambda_i (f_i \circ E)(x) + \sum_{j \in J_0} \mu_j^T (g_j \circ E)(x) \leqq \sum_{i=1}^p \lambda_i (f_i \circ E)(u) + \sum_{j \in J_0} \mu_j^T (g_j \circ E)(u). \tag{4.6}$$

Now (4.6) and hypothesis (b) imply (4.5), which also contradicts the fact that

$$\sum_{i=1}^p \lambda_i \nabla (f_i \circ E)(u) + \sum_{j=1}^q \mu_j^T \nabla (g_j \circ E)(u) = 0.$$

□

Corollary 4.1 *Assume that weak duality* (*Theorem* 4.2) *holds between* ($\mathrm{MP_E}$) *and* ($\mathrm{MD_E}$). *If* $(\bar{u}, \bar{\lambda}, \bar{\mu})$ *is feasible for* ($\mathrm{MD_E}$) *with* $\bar{\mu}^T (g \circ E)(\bar{u}) = 0$ *and if* $\bar{u}$ *is feasible for* ($\mathrm{MP_E}$), *then* $\bar{u}$ *is efficient for* ($\mathrm{MP_E}$) *and* $(\bar{u}, \bar{\lambda}, \bar{\mu})$ *is efficient for* ($\mathrm{MD_E}$).

Proof Suppose that $\bar{u}$ is not efficient for ($\mathrm{MP_E}$). Then there exists a feasible x for ($\mathrm{MP_E}$) such that

$$(f_{i_0} \circ E)(x) \leqq (f_{i_0} \circ E)(\bar{u}) \quad \text{for all } i_0 \in P, \tag{4.7}$$

$$(f_i \circ E)(x) < (f_i \circ E)(\bar{u}) \quad \text{for some } i \in P. \tag{4.8}$$

By hypothesis $\bar{\mu}^T (g \circ E)(\bar{u}) = 0$, so (4.7) and (4.8) can be written as

$$(f_{i_0} \circ E)(x) \leqq (f_{i_0} \circ E)(\bar{u}) + \bar{\mu}^T (g \circ E)(\bar{u}) \quad \text{for all } i_0 \in P,$$

$$(f_i \circ E)(x) < (f_i \circ E)(\bar{u}) + \bar{\mu}^T (g \circ E)(\bar{u}) \quad \text{for some } i \in P.$$

Since $(\bar{u}, \bar{\lambda}, \bar{\mu})$ is feasible for ($\mathrm{MD_E}$) and x is feasible for ($\mathrm{MP_E}$), these inequalities contradict weak duality (Theorem 4.2).

Also, suppose that $(\bar{u}, \bar{\lambda}, \bar{\mu})$ is not efficient for $(\mathrm{MD_E})$, then there exists a feasible solution (u, λ, μ) for $(\mathrm{MD_E})$ such that

$$(f_j \circ E)(u) + \mu^T (g \circ E)(u) \geqq (f_j \circ E)(\bar{u}) + \bar{\mu}^T (g \circ E)(\bar{u}) \quad \text{for all } j \in P, \tag{4.9}$$

$$(f_i \circ E)(u) + \mu^T (g \circ E)(u) > (f_i \circ E)(\bar{u}) + \bar{\mu}^T (g \circ E)(\bar{u}) \quad \text{for some } i \in P. \tag{4.10}$$

Since $\bar{\mu}^T (g \circ E)(\bar{u}) = 0$, (4.9) and (4.10) reduce to

$$(f_j \circ E)(u) + \mu^T (g \circ E)(u) \geq (f_j \circ E)(\bar{u}) \quad \text{for all } j \in P,$$

$$(f_i \circ E)(u) + \mu^T (g \circ E)(u) > (f_i \circ E)(\bar{u}) \quad \text{for some } i \in P.$$

Since $\bar{u}$ is feasible for $(\mathrm{MP_E})$, these inequalities contradict weak duality (Theorem 4.2). Therefore $\bar{u}$ and $(\bar{u}, \bar{\lambda}, \bar{\mu})$ are efficient for their respective problems. □

Theorem 4.3 (Strong duality) *Let $\bar{x}$ be an efficient solution for* $(\mathrm{MP_E})$ *and assume that $\bar{x}$ satisfies a constraint qualification* [4] *for* $(\mathrm{MP_E})_k$ *for at least one $k = 1, 2, \ldots, p$. Then there exist $\bar{\lambda} \in \mathbb{R}^p$ and $\bar{\mu} \in \mathbb{R}^q$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is feasible for* $(\mathrm{MD_E})$. *Moreover, if weak duality* (*Theorem* 4.2) *holds between* $(\mathrm{MP_E})$ *and* $(\mathrm{MD_E})$, *then $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is efficient for* $(\mathrm{MD_E})$.

Proof Since $\bar{x}$ is efficient for $(\mathrm{MP_E})$, by Lemma 3.1, $\bar{x}$ solves $(\mathrm{MP_E})_k$ for all $k = 1, 2, \ldots, p$. By hypothesis, there exists a $k \in P = \{1, 2, \ldots, p\}$ for which $\bar{x}$ satisfies a constraint qualification of $(\mathrm{MP_E})_k$.

From the Kuhn-Tucker necessary conditions [4], there exist $\lambda_i \geqq 0$ such that, for all $i \neq k$ and $\mu \geqq 0$, $\mu \in \mathbb{R}^m$,

$$(f_k \circ E)(\bar{x}) + \sum_{i \neq k} \lambda_i \nabla (f_i \circ E)(\bar{x}) + \nabla \mu^T (g \circ E)(\bar{x}) = 0, \tag{4.11}$$

$$\mu^T (g \circ E)(\bar{x}) = 0. \tag{4.12}$$

Now we divide all terms in (4.11) and (4.12) by $1 + \sum_{i \neq k} \lambda_i$ and set $\bar{\lambda}_k = \frac{1}{1 + \sum_{i \neq k} \lambda_i} > 0$, $\bar{\lambda}_j = \frac{\lambda_i}{1 + \sum_{i \neq k} \lambda_i} \geqq 0$, $\bar{\mu} = \frac{\mu}{1 + \sum_{i \neq k} \lambda_i} \geqq 0$. Since weak duality (Theorem 4.2) holds, from Corollary 4.1, we conclude that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is feasible as well as efficient for $(\mathrm{MD_E})$. □

Example 4.1 Recall the problem in Example 3.2, and we now give the mixed dual problem to $\widehat{(\mathrm{MP_E})}$.

$$\begin{array}{lll} \widehat{(\mathbf{MD_E})} & \text{Maximize} & \big((f_1 \circ E)(u) + \mu_1 (g_1 \circ E)(u), (f_2 \circ E)(u) + \mu_1 (g_1 \circ E)(u)\big) \\ & \text{subject to} & \lambda^T \nabla (f \circ E)(u) + \mu^T \nabla (g \circ E)(u) = 0, \\ & & \mu_2 (g_2 \circ E)(u) \geqq 0, \\ & & \lambda_1 + \lambda_2 = 1, \qquad \lambda \geqq 0, \qquad \mu \geqq 0, \end{array}$$

where $\lambda = (\lambda_1, \lambda_2)$ and $\mu = (\mu_1, \mu_2)$.

As we know the feasible set of $\widehat{(\mathrm{MP_E})}$ is $E(M) = [1,2]$ and it is easy to check that the feasible set of $\widehat{(\mathrm{MD_E})}$ denoted by G is $G = \{(u,\lambda,\mu) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \mid \lambda_2(2u-3)+1+\mu_1-\mu_2 = 0, \mu_2(-u+1) \geqq 0, 0 \leqq \lambda_2 \leqq 1, \mu_1 \geqq 0, \mu_2 \geqq 0\}$.

Now we check the validity of weak duality, say Theorem 4.2, that is, for any feasible point $x \in E(M)$ and $(u,\lambda,\mu) \in G$ with positive λ_1 and λ_2,

$$\begin{pmatrix} x-1 \\ x^2-2x+1 \end{pmatrix} \leq \begin{pmatrix} u-1+\mu_1(u-2) \\ u^2-2u+1+\mu_1(u-2) \end{pmatrix} \tag{4.13}$$

cannot hold. In fact, by the positivity of λ_2, we have $G = \{(u,\lambda,\mu) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \mid 1 \leqq u \leqq \frac{3}{2} - \frac{1+\mu_1}{2\lambda_2}, 0 < \lambda_2 < 1, \mu_1 \geqq 0\}$, and

$$\min(x-1) = 0 > \max\bigl(u-1+\mu_1(u-2)\bigr) = \frac{1}{2} - \frac{1}{2\lambda_2},$$

which implies (4.13) cannot hold, and we conclude that weak duality (Theorem 4.2) holds.

Finally we turn to strong duality (Theorem 4.3), as we know $\bar{x} = 1$ is an efficient solution of $\widehat{(\mathrm{MP_E})}$, and with the satisfy of Kuhn-Tucker constraint qualification [4], it is easy to check that there exist $\bar{\lambda} = (1,0)$ and $\bar{\mu} = (0,1)$ such that $(\bar{x},\bar{\lambda},\bar{\mu}) = (1,(1,0),(0,1))$ is a feasible solution of $\widehat{(\mathrm{MD_E})}$. Moreover, if weak duality (Theorem 4.2) holds, $(\bar{x},\bar{\lambda},\bar{\mu}) = (1,(1,0),(0,1))$ is efficient for $\widehat{(\mathrm{MD_E})}$, hence strong duality (Theorem 4.3) holds.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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