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A strengthened Mulholland-type inequality with parameters

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Abstract

By means of the way of weight coefficients, technique of real analysis, and Hermite-Hadamard's inequality, a strengthened version of the Mulholland-type inequality with the best possible constant factor and multi-parameters is given. The equivalent forms, the reverses, the operator expressions and a few particular cases are considered.

MSC: 26D15; 47A07

Keywords: Mulholland-type inequality; weight coefficient; equivalent form; reverse; operator

1 Introduction

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_m, b_n \geq 0$, $a = \{a_m\}_{m=1}^{\infty} \in l^p$, $b = \{b_n\}_{n=1}^{\infty} \in l^q$, $\|a\|_p = (\sum_{m=1}^{\infty} a_m^p)^{\frac{1}{p}} > 0$, $\|b\|_q > 0$, then we have the following well-known Hardy-Hilbert's inequality with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ (*cf.* [1], Theorem 315):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\|a\|_p\|b\|_q. \tag{1}$$

Also we have the following Mulholland's inequality similar to (1) with the same best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ (*cf.* [2] or [1], Theorem 343, replacing $\frac{a_m}{n}$, $\frac{b_n}{n}$ by a_m, b_n):

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{a_m b_n}{\ln mn} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=2}^{\infty}\frac{a_m^p}{m^{1-p}}\right)^{\frac{1}{p}}\left(\sum_{n=2}^{\infty}\frac{b_n^q}{n^{1-q}}\right)^{\frac{1}{q}}. \tag{2}$$

Inequalities (1) and (2) are important in analysis and its applications (*cf.* [1, 3–8]).

In 1998, Gao and Yang [9] gave a strengthened version of (1) as follows:

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} &< \left\{\sum_{m=1}^{\infty}\left[\frac{\pi}{\sin(\pi/p)} - \frac{1-\gamma}{m^{1/p}}\right]a_m^p\right\}^{\frac{1}{p}} \\ &\quad \times \left\{\sum_{n=1}^{\infty}\left[\frac{\pi}{\sin(\pi/p)} - \frac{1-\gamma}{n^{1/q}}\right]b_n^q\right\}^{\frac{1}{q}},\end{aligned} \tag{3}$$

where $1-\gamma = 0.42278433^{+}$ (γ is Euler constant).

Suppose that $\mu_i, \upsilon_j > 0$ $(i,j \in \mathbf{N} = \{1,2,\ldots\})$,

$$U_m := \sum_{i=1}^{m} \mu_i, \qquad V_n := \sum_{j=1}^{n} \upsilon_j \quad (m,n \in \mathbf{N}), \tag{4}$$

we have the following Hardy-Hilbert-type inequality (*cf.* [1], Theorem 321):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\mu_m^{1/q}\upsilon_n^{1/p} a_m b_n}{U_m + V_n} < \frac{\pi}{\sin(\pi/p)}\|a\|_p\|b\|_q. \tag{5}$$

For $\mu_i = \upsilon_j = 1$ $(i,j \in \mathbf{N})$, inequality (5) reduces to (1). Replacing $\mu_m^{1/q} a_m$ and $\upsilon_n^{1/p} b_n$ by a_m and b_n in (5), respectively, we obtain the equivalent form of (5) as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{U_m + V_n} < \frac{\pi}{\sin(\frac{\pi}{p})}\left(\sum_{m=1}^{\infty} \frac{a_m^p}{\mu_m^{p-1}}\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} \frac{b_n^q}{\upsilon_n^{q-1}}\right)^{\frac{1}{q}}. \tag{6}$$

In 2015, Yang [10] gave an extension of (6) as follows: For $0 < \lambda_1, \lambda_2 \le 1$, $\lambda_1 + \lambda_2 = \lambda$, we have

$$\begin{aligned}&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{(U_m + V_n)^{\lambda}} \\ &\quad < B(\lambda_1,\lambda_2)\left[\sum_{m=1}^{\infty} \frac{U_m^{p(1-\lambda_1)-1} a_m^p}{\mu_m^{p-1}}\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty} \frac{V_n^{q(1-\lambda_2-1)} b_n^q}{\upsilon_n^{q-1}}\right]^{\frac{1}{q}},\end{aligned} \tag{7}$$

where $B(u,v)$ is the beta function indicated by (*cf.* [11])

$$B(u,v) := \int_0^{\infty} \frac{t^{u-1}}{(1+t)^{u+v}}\,dt \quad (u,v > 0). \tag{8}$$

In this paper, by using the way of weight coefficients, the technique of real analysis, and Hermite-Hadamard's inequality, a Mulholland-type inequality with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ is given as follows: For $\mu_1 = \upsilon_1 = 1$, $\{\mu_m\}_{m=1}^{\infty}$ and $\{\upsilon_n\}_{n=1}^{\infty}$ are decreasing, and $U_\infty = V_\infty = \infty$, we have

$$\begin{aligned}&\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} \frac{a_m b_n}{\ln U_m V_n} \\ &\quad < \frac{\pi}{\sin(\pi/p)}\left[\sum_{m=2}^{\infty}\left(\frac{U_m}{\mu_{m+1}}\right)^{p-1} a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=2}^{\infty}\left(\frac{V_n}{\upsilon_{n+1}}\right)^{q-1} b_n^q\right]^{\frac{1}{q}},\end{aligned} \tag{9}$$

which is an extension of (2) (*Note*: the series on the right-hand side of (9) are positive). Moreover, a strengthened version of (9) and some extended Mulholland-type inequalities with multi-parameters are obtained. The equivalent forms, the reverses, the operator expressions and a few particular cases are considered.

2 Some lemmas

In the following, we make appointment that $p \neq 0, 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \lambda_1, \lambda_2 \leq 1$, $\lambda_1 + \lambda_2 = \lambda$, $\mu_i, \upsilon_j > 0$ $(i, j \in \mathbf{N})$, with $\mu_1 = \upsilon_1 = 1$, U_m and V_n are defined by (4), $a_m, b_n \geq 0$, $\|a\|_{p,\Phi_\lambda} := (\sum_{m=2}^{\infty} \Phi_\lambda(m) a_m^p)^{\frac{1}{p}}$ and $\|b\|_{q,\Psi_\lambda} := (\sum_{n=2}^{\infty} \Psi_\lambda(n) b_n^q)^{\frac{1}{q}}$, where

$$\begin{aligned} \Phi_\lambda(m) &:= \frac{(\ln U_m)^{p(1-\lambda_1)-1}}{U_m^{1-p} \mu_{m+1}^{p-1}}, \\ \Psi_\lambda(n) &:= \frac{(\ln V_n)^{q(1-\lambda_2)-1}}{V_n^{1-q} \upsilon_{n+1}^{q-1}} \quad \bigl(m, n \in \mathbf{N} \backslash \{1\}\bigr). \end{aligned} \tag{10}$$

Lemma 1 *If $a \in \mathbf{R}$, $f(x)$ is continuous in $[a - \frac{1}{2}, a + \frac{1}{2}]$, $f'(x)$ is strictly increasing in $(a - \frac{1}{2}, a)$ and $(a, a + \frac{1}{2})$, respectively, and*

$$\lim_{x \to a-} f'(x) = f'(a - 0) \leq f'(a + 0) = \lim_{x \to a+} f'(x),$$

then we have the following Hermite-Hadamard's inequality (*cf.* [12]):

$$f(a) < \int_{a-\frac{1}{2}}^{a+\frac{1}{2}} f(x)\, dx. \tag{11}$$

Proof Since $f'(a - 0)$ $(\leq f'(a + 0))$ is finite, we set a function $g(x)$ as follows:

$$g(x) := f'(a - 0)(x - a) + f(a), \quad x \in \left[a - \frac{1}{2}, a + \frac{1}{2}\right].$$

In view of $f'(x)$ being strictly increasing in $(a - \frac{1}{2}, a)$, then for $x \in (a - \frac{1}{2}, a)$, $(f(x) - g(x))' = f'(x) - f'(a - 0) < 0$. Since $f(a) - g(a) = 0$, it follows that $f(x) - g(x) > 0$, $x \in (a - \frac{1}{2}, a)$. In the same way, we can obtain $f(x) - g(x) > 0$, $x \in (a, a + \frac{1}{2})$. Hence, we find

$$\int_{a-\frac{1}{2}}^{a+\frac{1}{2}} f(x)\, dx > \int_{a-\frac{1}{2}}^{a+\frac{1}{2}} g(x)\, dx = f(a),$$

namely (11) follows. □

Example 1 If $\{\mu_m\}_{m=1}^{\infty}$ and $\{\upsilon_n\}_{n=1}^{\infty}$ are also decreasing, we set $\mu(t) := \mu_m$, $t \in (m - 1, m]$ $(m \in \mathbf{N})$; $\upsilon(t) := \upsilon_n$, $t \in (n - 1, n]$ $(n \in \mathbf{N})$,

$$U(x) := \int_0^x \mu(t)\, dt \quad (x \geq 0), \qquad V(y) := \int_0^y \upsilon(t)\, dt \quad (y \geq 0). \tag{12}$$

Then it follows that $U(m) = U_m$, $V(n) = V_n$ $(m, n \in \mathbf{N})$, $U(\infty) = U_\infty$, $V(\infty) = V_\infty$ and

$$\begin{aligned} U'(x) &= \mu(x) = \mu_m \quad \bigl(x \in (m - 1, m)\bigr), \\ V'(y) &= \upsilon(y) = \upsilon_n \quad \bigl(y \in (n - 1, n)\bigr). \end{aligned}$$

For fixed $m, n \in \mathbf{N} \backslash \{1\}$, we also set a function $f(x)$ as follows:

$$f(x) = \frac{\ln^{\lambda_2 - 1} V(x)}{V(x)(\ln U_m + \ln V(x))^\lambda}, \quad x \in \left[n - \frac{1}{2}, n + \frac{1}{2}\right].$$

Then $f(x)$ in continuous in $[n-\frac{1}{2}, n+\frac{1}{2}]$. For $x \in (n-\frac{1}{2}, n)$ ($n \in \mathbf{N}\backslash\{1\}$), we find

$$\begin{aligned} f'(x) = &-\left[\frac{\ln^{\lambda_2-1} V(x)}{V(x)} + \frac{\lambda \ln^{\lambda_2-1} V(x)}{\ln U_m + \ln V(x)} + \frac{1-\lambda_2}{V^{2-\lambda_2}(x)}\right] \\ &\times \frac{\upsilon_n}{V(x)(\ln U_m + \ln V(x))^{\lambda}}. \end{aligned}$$

Since $1-\lambda_2 \geq 0$, it follows that $f'(x)$ (< 0) is strictly increasing in $(n-\frac{1}{2}, n)$ and

$$\begin{aligned} \lim_{x\to n-} f'(x) &= f'(n-0) \\ &= -\left[\frac{\ln^{\lambda_2-1} V_n}{V_n} + \frac{\lambda \ln^{\lambda_2-1} V_n}{\ln U_m + \ln V_n} + \frac{1-\lambda_2}{V_n^{2-\lambda_2}}\right] \\ &\quad \times \frac{\upsilon_n}{V_n(\ln U_m + \ln V_n)^{\lambda}}. \end{aligned}$$

In the same way, for $x \in (n, n+\frac{1}{2})$, we find

$$\begin{aligned} f'(x) = &-\left[\frac{\ln^{\lambda_2-1} V(x)}{V(x)} + \frac{\lambda \ln^{\lambda_2-1} V(x)}{\ln U_m + \ln V(x)} + \frac{1-\lambda_2}{V^{2-\lambda_2}(x)}\right] \\ &\times \frac{\upsilon_{n+1}}{V(x)(\ln U_m + \ln V(x))^{\lambda}}, \end{aligned}$$

$f'(x)$ (< 0) is strictly increasing in $(n, n+\frac{1}{2})$. In view of $\upsilon_{n+1} \leq \upsilon_n$, it follows that $\lim_{x\to n+} f'(x) = f'(n+0) \geq f'(n-0)$. Then by (11) we have

$$f(n) < \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} f(x)\,dx = \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{\ln^{\lambda_2-1} V(x)}{V(x)(\ln U_m + \ln V(x))^{\lambda}}\,dx. \tag{13}$$

Definition 1 Define the following weight coefficients:

$$\omega(\lambda_2, m) := \sum_{n=2}^{\infty} \frac{1}{\ln^{\lambda}(U_m V_n)} \frac{\upsilon_{n+1} \ln^{\lambda_1} U_m}{V_n \ln^{1-\lambda_2} V_n}, \quad m \in \mathbf{N}\backslash\{1\}, \tag{14}$$

$$\varpi(\lambda_1, n) := \sum_{m=2}^{\infty} \frac{1}{\ln^{\lambda}(U_m V_n)} \frac{\mu_{m+1} \ln^{\lambda_2} V_n}{U_m \ln^{1-\lambda_1} U_m}, \quad n \in \mathbf{N}\backslash\{1\}. \tag{15}$$

Lemma 2 *If $\{\mu_m\}_{m=1}^{\infty}$ and $\{\upsilon_n\}_{n=1}^{\infty}$ are decreasing, and $U_\infty = V_\infty = \infty$, then we have the following inequalities*:

$$\begin{aligned} &\omega(\lambda_2, m) < B(\lambda_1, \lambda_2)\left(1 - \frac{\theta_1}{\ln^{\lambda_2} U_m}\right) \\ &\quad \bigl(0 < \lambda_2 \leq 1, \lambda_1 > 0; m \in \mathbf{N}\backslash\{1\}\bigr), \end{aligned} \tag{16}$$

$$\begin{aligned} &\varpi(\lambda_1, n) < B(\lambda_1, \lambda_2)\left(1 - \frac{\theta_2}{\ln^{\lambda_1} V_n}\right) \\ &\quad \bigl(0 < \lambda_1 \leq 1, \lambda_2 > 0; n \in \mathbf{N}\backslash\{1\}\bigr), \end{aligned} \tag{17}$$

where

$$\theta_1 := \frac{1}{B(\lambda_1,\lambda_2)} \frac{\ln^{\lambda_2}(1+\upsilon_2/2)}{\lambda_2[1+\frac{\ln(1+\upsilon_2/2)}{\ln(1+\mu_2/2)}]^\lambda}, \tag{18}$$

$$\theta_2 := \frac{1}{B(\lambda_1,\lambda_2)} \frac{\ln^{\lambda_1}(1+\mu_2/2)}{\lambda_1[1+\frac{\ln(1+\mu_2/2)}{\ln(1+\upsilon_2/2)}]^\lambda}. \tag{19}$$

Proof Since for $x \in (n-\frac{1}{2}, n+\frac{1}{2})\backslash\{n\}$, $\upsilon_{n+1} \le V'(x)$, by (13) we find

$$\begin{aligned}
\omega(\lambda_2,m) &< \sum_{n=2}^{\infty} \upsilon_{n+1} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{\ln^{\lambda_1} U_m \ln^{\lambda_2-1} V(x)}{V(x)(\ln U_m + \ln V(x))^\lambda}\,dx \\
&\le \sum_{n=2}^{\infty} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{\ln^{\lambda_1} U_m \ln^{\lambda_2-1} V(x)}{V(x)(\ln U_m + \ln V(x))^\lambda} V'(x)\,dx \\
&= \int_{\frac{3}{2}}^{\infty} \frac{\ln^{\lambda_1} U_m \ln^{\lambda_2-1} V(x)}{V(x)(\ln U_m + \ln V(x))^\lambda} V'(x)\,dx \\
&= \int_{1}^{\infty} \frac{\ln^{\lambda_1} U_m \ln^{\lambda_2-1} V(x)}{V(x)(\ln U_m + \ln V(x))^\lambda} V'(x)\,dx \\
&\quad - \int_{1}^{\frac{3}{2}} \frac{\ln^{\lambda_1} U_m \ln^{\lambda_2-1} V(x)}{V(x)(\ln U_m + \ln V(x))^\lambda} V'(x)\,dx.
\end{aligned}$$

Setting $t = \frac{\ln V(x)}{\ln U_m}$, we obtain $\frac{V'(x)}{V(x)}\,dx = \ln U_m\,dt$ and

$$\begin{aligned}
\omega(\lambda_2,m) &< \int_0^\infty \frac{1}{(1+t)^\lambda} t^{\lambda_2-1}\,dt - \int_1^{\frac{3}{2}} \frac{\ln^{\lambda_1} U_m}{(\ln U_m + \ln V(x))^\lambda} \frac{\ln^{\lambda_2-1} V(x)}{V(x)} V'(x)\,dx \\
&= B(\lambda_1,\lambda_2)\bigl(1-\theta(m)\bigr),
\end{aligned} \tag{20}$$

where

$$\theta(m) := \frac{1}{B(\lambda_1,\lambda_2)} \int_1^{\frac{3}{2}} \frac{\ln^{\lambda_1} U_m}{(\ln U_m + \ln V(x))^\lambda} \frac{\ln^{\lambda_2-1} V(x)}{V(x)} V'(x)\,dx. \tag{21}$$

We find

$$\begin{aligned}
\theta(m) &\ge \frac{1}{B(\lambda_1,\lambda_2)} \frac{\ln^{\lambda_1} U_m}{(\ln U_m + \ln V(\frac{3}{2}))^\lambda} \int_1^{\frac{3}{2}} \frac{\ln^{\lambda_2-1} V(x)}{V(x)} V'(x)\,dx \\
&= \frac{1}{B(\lambda_1,\lambda_2)} \frac{\ln^{\lambda_1} U_m}{\lambda_2(\ln U_m + \ln V(\frac{3}{2}))^\lambda} \ln^{\lambda_2} V\left(\frac{3}{2}\right) \\
&= \frac{1}{B(\lambda_1,\lambda_2)} \frac{\ln^{\lambda_2}(1+\upsilon_2/2)}{\lambda_2(1+\frac{\ln(1+\upsilon_2/2)}{\ln U_m})^\lambda} \frac{1}{\ln^{\lambda_2} U_m} \\
&\ge \frac{1}{B(\lambda_1,\lambda_2)} \frac{\ln^{\lambda_2}(1+\upsilon_2/2)}{\lambda_2(1+\frac{\ln(1+\upsilon_2/2)}{\ln U_2})^\lambda} \frac{1}{\ln^{\lambda_2} U_m} = \frac{\theta_1}{\ln^{\lambda_2} U_m}.
\end{aligned}$$

Hence, by (20), we have (16) and (18). In the same way, we obtain (17) and (19). □

Note For example, $\mu_n, \upsilon_n = \frac{1}{n^\sigma}$ $(0 \le \sigma \le 1)$ are satisfied the assumptions of Lemma 2.

Lemma 3 *With the assumptions of Lemma* 2, (i) *for* $m, n \in \mathbf{N}\backslash\{1\}$, *we have*

$$B(\lambda_1,\lambda_2)\bigl(1-\theta(\lambda_2,m)\bigr) < \omega(\lambda_2,m) \quad (0<\lambda_2\le 1, \lambda_1>0), \tag{22}$$

$$B(\lambda_1,\lambda_2)\bigl(1-\vartheta(\lambda_1,n)\bigr) < \varpi(\lambda_1,n) \quad (0<\lambda_1\le 1, \lambda_2>0), \tag{23}$$

where

$$\begin{aligned}\theta(\lambda_2,m) &= \frac{1}{B(\lambda_1,\lambda_2)} \frac{\ln^{\lambda_2}(1+\upsilon_2)}{\lambda_2[1+\frac{\ln(1+\theta(m)\upsilon_2)}{\ln U_m}]^{\lambda}} \frac{1}{\ln^{\lambda_2} U_m} \\ &= O\biggl(\frac{1}{\ln^{\lambda_2} U_m}\biggr) \in (0,1) \quad \bigl(\theta(m)\in(0,1)\bigr),\end{aligned} \tag{24}$$

$$\begin{aligned}\vartheta(\lambda_1,n) &= \frac{1}{B(\lambda_1,\lambda_2)} \frac{\ln^{\lambda_1}(1+\mu_2)}{\lambda_1[1+\frac{\ln(1+\vartheta(n)\mu_2)}{\ln V_n}]^{\lambda}} \frac{1}{\ln^{\lambda_1} V_n} \\ &= O\biggl(\frac{1}{\ln^{\lambda_1} V_n}\biggr) \in (0,1) \quad \bigl(\vartheta(n)\in(0,1)\bigr);\end{aligned} \tag{25}$$

(ii) *for any* $a>0$, *we have*

$$\sum_{m=2}^{\infty} \frac{\mu_{m+1}}{U_m \ln^{1+a} U_m} = \frac{1}{a}\biggl[\frac{1}{\ln^a(1+\mu_2)} + aO(1)\biggr], \tag{26}$$

$$\sum_{n=2}^{\infty} \frac{\upsilon_{n+1}}{V_n \ln^{1+a} V_n} = \frac{1}{a}\biggl[\frac{1}{\ln^a(1+\upsilon_2)} + a\widetilde{O}(1)\biggr]. \tag{27}$$

Proof Since by Example 1, $f(x)$ is strictly decreasing in $[n,n+1]$, then we find

$$\begin{aligned}\omega(\lambda_2,m) &> \sum_{n=2}^{\infty}\int_n^{n+1} \upsilon_{n+1} \frac{\ln^{\lambda_1} U_m \ln^{\lambda_2-1} V(x)}{V(x)(\ln U_m + \ln V(x))^{\lambda}}\,dx \\ &= \int_2^{\infty} \frac{\ln^{\lambda_1} U_m \ln^{\lambda_2-1} V(x)}{V(x)(\ln U_m + \ln V(x))^{\lambda}} V'(x)\,dx \\ &= \int_1^{\infty} \frac{\ln^{\lambda_1} U_m \ln^{\lambda_2-1} V(x)}{V(x)(\ln U_m + \ln V(x))^{\lambda}} V'(x)\,dx \\ &\quad - \int_1^{2} \frac{\ln^{\lambda_1} U_m \ln^{\lambda_2-1} V(x)}{V(x)(\ln U_m + \ln V(x))^{\lambda}} V'(x)\,dx \\ &= B(\lambda_1,\lambda_2)\bigl(1-\theta(\lambda_2,m)\bigr),\end{aligned}$$

where

$$\theta(\lambda_2,m) := \frac{1}{B(\lambda_1,\lambda_2)} \int_1^2 \frac{V'(x)\ln^{\lambda_1} U_m \ln^{\lambda_2-1} V(x)}{V(x)(\ln U_m + \ln V(x))^{\lambda}}\,dx \in (0,1).$$

There exists $\theta(m)\in(0,1)$ such that

$$\begin{aligned}\theta(\lambda_2,m) &= \frac{1}{B(\lambda_1,\lambda_2)} \frac{\ln^{\lambda_1} U_m}{[\ln U_m + \ln V(1+\theta(m))]^{\lambda}} \\ &\quad \times \int_1^2 \frac{\ln^{\lambda_2-1} V(x)}{V(x)} V'(x)\,dx\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{B(\lambda_1,\lambda_2)} \frac{\ln^{\lambda_1} U_m \ln^{\lambda_2}(1+\upsilon_2)}{\lambda_2[\ln U_m + \ln V(1+\theta(m))]^{\lambda}} \\
&= \frac{1}{B(\lambda_1,\lambda_2)} \frac{\ln^{\lambda_2}(1+\upsilon_2)}{\lambda_2[1+\frac{\ln(1+\theta(m)\upsilon_2)}{\ln U_m}]^{\lambda}} \frac{1}{\ln^{\lambda_2} U_m}.
\end{aligned}$$

Since we obtain

$$0 < \theta(\lambda_2, m) \le \frac{\ln^{\lambda_2}(1+\upsilon_2)}{\lambda_2 B(\lambda_1,\lambda_2)} \frac{1}{\ln^{\lambda_2} U_m},$$

namely $\theta(\lambda_2, m) = O(\frac{1}{\ln^{\lambda_2} U_m})$, we have (22). In the same way, we obtain (23).

For $a > 0$, we find

$$\begin{aligned}
\sum_{m=2}^{\infty} \frac{\mu_{m+1}}{U_m \ln^{1+a} U_m} &\le \sum_{m=2}^{\infty} \frac{\mu_m}{U_m \ln^{1+a} U_m} \\
&= \frac{\mu_2}{U_2 \ln^{1+a} U_2} + \sum_{m=3}^{\infty} \frac{\mu_m}{U_m \ln^{1+a} U_m} \\
&= \frac{\mu_2}{U_2 \ln^{1+a} U_2} + \sum_{m=3}^{\infty} \int_{m-1}^{m} \frac{U'(x)}{U_m \ln^{1+a} U_m}\,dx \\
&< \frac{\mu_2}{U_2 \ln^{1+a} U_2} + \sum_{m=3}^{\infty} \int_{m-1}^{m} \frac{U'(x)}{U(x) \ln^{1+a} U(x)}\,dx \\
&= \frac{\mu_2}{U_2 \ln^{1+a} U_2} + \int_{2}^{\infty} \frac{U'(x)}{U(x) \ln^{1+a} U(x)}\,dx \\
&= \frac{\mu_2}{(1+\mu_2) \ln^{1+a}(1+\mu_2)} + \frac{1}{a \ln^{a}(1+\mu_2)} \\
&= \frac{1}{a}\left(\frac{1}{\ln^{a}(1+\mu_2)} + \frac{a\mu_2}{(1+\mu_2)\ln^{1+a}(1+\mu_2)}\right), \\
\sum_{m=2}^{\infty} \frac{\mu_{m+1}}{U_m \ln^{1+a} U_m} &= \sum_{m=2}^{\infty} \int_{m}^{m+1} \frac{U'(x)\,dx}{U_m \ln^{1+a} U_m} \\
&> \sum_{m=2}^{\infty} \int_{m}^{m+1} \frac{U'(x)}{U(x) \ln^{1+a} U(x)}\,dx \\
&= \int_{2}^{\infty} \frac{U'(x)\,dx}{U(x) \ln^{1+a} U(x)} = \frac{1}{a \ln^{a}(1+\mu_2)}.
\end{aligned}$$

Hence we have (26). In the same way, we have (27). □

3 Main results and operator expressions

We also set

$$\begin{aligned}
&\widetilde{\Phi}_{\lambda}(m) := \omega(\lambda_2, m) \frac{(\ln U_m)^{p(1-\lambda_1)-1}}{U_m^{1-p} \mu_{m+1}^{p-1}}, \\
&\widetilde{\Psi}_{\lambda}(n) := \varpi(\lambda_1, n) \frac{(\ln V_n)^{q(1-\lambda_2)-1}}{V_n^{1-q} \upsilon_{n+1}^{q-1}} \quad \bigl(m,n \in \mathbf{N}\backslash\{1\}\bigr).
\end{aligned} \tag{28}$$

Theorem 1 (i) *For* $p > 1$, *we have the following equivalent inequalities*:

$$I := \sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{\ln^{\lambda}(U_m V_n)} \le \|a\|_{p,\widetilde{\Phi}_\lambda}\|b\|_{q,\widetilde{\Psi}_\lambda}, \tag{29}$$

$$J := \left\{\sum_{n=2}^{\infty}\frac{\upsilon_{n+1}\ln^{p\lambda_2-1}V_n}{(\varpi(\lambda_1,n))^{p-1}V_n}\left[\sum_{m=2}^{\infty}\frac{a_m}{\ln^{\lambda}(U_m V_n)}\right]^p\right\}^{\frac{1}{p}} \le \|a\|_{p,\widetilde{\Phi}_\lambda}. \tag{30}$$

(ii) *For* $0 < p < 1$ (*or* $p < 0$), *we have the equivalent reverses of* (29) *and* (30).

Proof (i) By Hölder's inequality with weight (*cf.* [12]) and (15), we have

$$\begin{aligned}
\left[\sum_{m=2}^{\infty}\frac{a_m}{\ln^{\lambda}(U_m V_n)}\right]^p &= \left[\sum_{m=2}^{\infty}\frac{1}{\ln^{\lambda}(U_m V_n)}\left(\frac{U_m^{1/q}(\ln U_m)^{(1-\lambda_1)/q}\upsilon_{n+1}^{1/p}}{(\ln V_n)^{(1-\lambda_2)/p}\mu_{m+1}^{1/q}}a_m\right)\right.\\
&\quad\left.\times\left(\frac{(\ln V_n)^{(1-\lambda_2)/p}\mu_{m+1}^{1/q}}{U_m^{1/q}(\ln U_m)^{(1-\lambda_1)/q}\upsilon_{n+1}^{1/p}}\right)\right]^p\\
&\le \sum_{m=2}^{\infty}\frac{1}{\ln^{\lambda}(U_m V_n)}\frac{U_m^{p-1}(\ln U_m)^{(1-\lambda_1)p/q}\upsilon_{n+1}}{(\ln V_n)^{1-\lambda_2}\mu_{m+1}^{p/q}}a_m^p\\
&\quad\times\left[\sum_{m=2}^{\infty}\frac{1}{\ln^{\lambda}(U_m V_n)}\frac{(\ln V_n)^{(1-\lambda_2)(q-1)}\mu_{m+1}}{U_m(\ln U_m)^{1-\lambda_1}\upsilon_{n+1}^{q-1}}\right]^{p-1}\\
&= \frac{(\varpi(\lambda_1,n))^{p-1}V_n}{(\ln V_n)^{p\lambda_2-1}\upsilon_{n+1}}\sum_{m=2}^{\infty}\frac{\upsilon_{n+1}U_m^{p-1}(\ln U_m)^{(1-\lambda_1)(p-1)}a_m^p}{\ln^{\lambda}(U_m V_n)V_n(\ln V_n)^{1-\lambda_2}\mu_{m+1}^{p-1}}.
\end{aligned} \tag{31}$$

Then by (14) we find

$$\begin{aligned}
J &\le \left[\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{\upsilon_{n+1}}{\ln^{\lambda}(U_m V_n)}\frac{U_m^{p-1}(\ln U_m)^{(1-\lambda_1)(p-1)}}{V_n(\ln V_n)^{1-\lambda_2}\mu_{m+1}^{p-1}}a_m^p\right]^{\frac{1}{p}}\\
&= \left[\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{\upsilon_{n+1}(\ln U_m)^{\lambda_1}}{\ln^{\lambda}(U_m V_n)}\frac{U_m^{p-1}(\ln U_m)^{p(1-\lambda_1)-1}}{V_n(\ln V_n)^{1-\lambda_2}\mu_{m+1}^{p-1}}a_m^p\right]^{\frac{1}{p}}\\
&= \left[\sum_{m=2}^{\infty}\omega(\lambda_2,m)\frac{(\ln U_m)^{p(1-\lambda_1)-1}}{U_m^{1-p}\mu_{m+1}^{p-1}}a_m^p\right]^{\frac{1}{p}},
\end{aligned} \tag{32}$$

and then (30) follows.

By Hölder's inequality (*cf.* [12]), we have

$$\begin{aligned}
I &= \sum_{n=2}^{\infty}\left[\frac{(\ln V_n)^{\lambda_2-\frac{1}{p}}\upsilon_{n+1}^{1/p}}{(\varpi(\lambda_1,n))^{\frac{1}{q}}V_n^{\frac{1}{p}}}\sum_{m=1}^{\infty}\frac{a_m}{\ln^{\lambda}(U_m V_n)}\right]\\
&\quad\times\left[(\varpi(\lambda_1,n))^{\frac{1}{q}}\frac{(\ln V_n)^{\frac{1}{p}-\lambda_2}}{V_n^{\frac{-1}{p}}\upsilon_{n+1}^{\frac{1}{p}}}b_n\right] \le J\|b\|_{q,\widetilde{\Psi}_\lambda}.
\end{aligned} \tag{33}$$

Then by (30) we have (29).

On the other hand, assuming that (29) is valid, we set

$$b_n := \frac{(\ln V_n)^{p\lambda_2-1}\upsilon_{n+1}}{(\varpi(\lambda_1,n))^{p-1}V_n}\left[\sum_{m=2}^{\infty}\frac{a_m}{\ln^{\lambda}(U_m V_n)}\right]^{p-1}, \quad n\in\mathbf{N}\backslash\{1\}. \tag{34}$$

Then we find $J^p = \|b\|_{q,\widetilde{\Psi}_\lambda}^q$. If $J = 0$, then (30) is trivially valid; if $J = \infty$, then, by (32), (30) takes the form of equality. Suppose that $0 < J < \infty$. By (29), it follows that

$$\|b\|_{q,\widetilde{\Psi}_\lambda}^q = J^p = I \le \|a\|_{p,\widetilde{\Phi}_\lambda}\|b\|_{q,\widetilde{\Psi}_\lambda}, \tag{35}$$

$$\|b\|_{q,\widetilde{\Psi}_\lambda}^{q-1} = J \le \|a\|_{p,\widetilde{\Phi}_\lambda}, \tag{36}$$

and then (30) follows, which is equivalent to (29).

(ii) For $0 < p < 1$ (or $p < 0$), by the reverse Hölder's inequality with weight (*cf.* [12]) and (15), we obtain the reverse of (31) (or (31)), then we have the reverse of (32), and then the reverse of (30) follows. By Hölder's inequality (*cf.* [12]), we have the reverse of (33) and then by the reverse of (30), the reverse of (29) follows.

On the other hand, assuming that the reverse of (29) is valid, we set b_n as (34). Then we find $J^p = \|b\|_{q,\widetilde{\Psi}_\lambda}^q$. If $J = \infty$, then the reverse of (30) is trivially valid; if $J = 0$, then, by the reverse of (32), (30) takes the form of equality (= 0). Suppose that $0 < J < \infty$. By the reverse of (29), it follows that the reverses of (35) and (36) are valid, and then the reverse of (30) follows, which is equivalent to the reverse of (29). □

Setting

$$\begin{aligned}\Omega_\lambda(m) &:= \left(1-\frac{\theta_1}{\ln^{\lambda_2} U_m}\right)\frac{(\ln U_m)^{p(1-\lambda_1)-1}}{U_m^{1-p}\mu_{m+1}^{p-1}},\\ \digamma_\lambda(n) &:= \left(1-\frac{\theta_2}{\ln^{\lambda_1} V_n}\right)\frac{(\ln V_n)^{q(1-\lambda_2)-1}}{V_n^{1-q}\upsilon_{n+1}^{q-1}} \quad \bigl(m,n\in\mathbf{N}\backslash\{1\}\bigr),\end{aligned} \tag{37}$$

we have the following.

Theorem 2 *If $p > 1$, $\{\mu_m\}_{m=1}^\infty$ and $\{\upsilon_n\}_{n=1}^\infty$ are decreasing, $U_\infty = V_\infty = \infty$, $\|a\|_{p,\Phi_\lambda} \in \mathbf{R}_+$ and $\|b\|_{q,\Psi_\lambda} \in \mathbf{R}_+$, then we have the following equivalent inequalities*:

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{\ln^{\lambda}(U_m V_n)} < B(\lambda_1,\lambda_2)\|a\|_{p,\Omega_\lambda}\|b\|_{q,\digamma_\lambda}, \tag{38}$$

$$J_1 := \left\{\sum_{n=2}^{\infty}\frac{\upsilon_{n+1}\ln^{p\lambda_2-1}V_n}{(1-\frac{\theta_2}{\ln^{\lambda_1}V_n})^{p-1}V_n}\left[\sum_{m=2}^{\infty}\frac{a_m}{\ln^{\lambda}(U_m V_n)}\right]^p\right\}^{\frac{1}{p}} < B(\lambda_1,\lambda_2)\|a\|_{p,\Omega_\lambda}, \tag{39}$$

where the constant factor $B(\lambda_1,\lambda_2)$ is the best possible.

Proof Using (16) and (17) in (29) and (30), since

$$\bigl(\omega(\lambda_2,m)\bigr)^{\frac{1}{p}} < \bigl(B(\lambda_1,\lambda_2)\bigr)^{\frac{1}{p}}\left(1-\frac{\theta_1}{\ln^{\lambda_2}U_m}\right)^{\frac{1}{p}} \quad (p>1),$$

$$\bigl(\varpi(\lambda_1,n)\bigr)^{\frac{1}{q}} < \bigl(B(\lambda_1,\lambda_2)\bigr)^{\frac{1}{q}}\left(1-\frac{\theta_2}{\ln^{\lambda_1}V_n}\right)^{\frac{1}{q}} \quad (q>1)$$

and

$$\frac{1}{(B(\lambda_1,\lambda_2))^{p-1}(1-\frac{\theta_2}{\ln^{\lambda_1} V_n})^{p-1}} < \frac{1}{(\varpi(\lambda_1,n))^{p-1}} \quad (p>1),$$

we obtain equivalent inequalities (38) and (39).

For $\varepsilon \in (0, p\lambda_1)$, we set $\widetilde{\lambda}_1 = \lambda_1 - \frac{\varepsilon}{p}$ ($\in (0,1)$), $\widetilde{\lambda}_2 = \lambda_2 + \frac{\varepsilon}{p}$ (>0), and

$$\begin{aligned} \widetilde{a}_m &:= \frac{\mu_{m+1}}{U_m}\ln^{\widetilde{\lambda}_1-1} U_m = \frac{\mu_{m+1}}{U_m}\ln^{\lambda_1-\frac{\varepsilon}{p}-1} U_m, \\ \widetilde{b}_n &= \frac{\upsilon_{n+1}}{V_n}\ln^{\widetilde{\lambda}_2-\varepsilon-1} V_n = \frac{\upsilon_{n+1}}{V_n}\ln^{\lambda_2-\frac{\varepsilon}{q}-1} V_n. \end{aligned} \tag{40}$$

Then, by (26), (27) and (23), we have

$$\begin{aligned} &\|\widetilde{a}\|_{p,\Omega_\lambda}\|\widetilde{b}\|_{q,\digamma_\lambda} \\ &\quad\le \|\widetilde{a}\|_{p,\Phi_\lambda}\|\widetilde{b}\|_{q,\Psi_\lambda} \\ &\quad= \left(\sum_{m=2}^\infty \frac{\mu_{m+1}}{U_m\ln^{1+\varepsilon} U_m}\right)^{\frac{1}{p}}\left(\sum_{n=2}^\infty \frac{\upsilon_{n+1}}{V_n\ln^{1+\varepsilon} V_n}\right)^{\frac{1}{q}} \\ &\quad= \frac{1}{\varepsilon}\left[\frac{1}{\ln^\varepsilon(1+\mu_2)}+\varepsilon O(1)\right]^{\frac{1}{p}}\left[\frac{1}{\ln^\varepsilon(1+\upsilon_2)}+\varepsilon\widetilde{O}(1)\right]^{\frac{1}{q}}, \\ &\widetilde{I} := \sum_{n=2}^\infty\sum_{m=2}^\infty \frac{\widetilde{a}_m\widetilde{b}_n}{\ln^\lambda(U_mV_n)} \\ &\quad= \sum_{n=2}^\infty\left[\sum_{m=2}^\infty \frac{1}{\ln^\lambda(U_mV_n)}\frac{\mu_{m+1}\ln^{\widetilde{\lambda}_2} V_n}{U_m\ln^{1-\widetilde{\lambda}_1} U_m}\right]\frac{\upsilon_{n+1}}{V_n\ln^{\varepsilon+1} V_n} \\ &\quad= \sum_{n=2}^\infty \varpi(\widetilde{\lambda}_1,n)\frac{\upsilon_{n+1}}{V_n\ln^{\varepsilon+1} V_n} \\ &\quad\ge B(\widetilde{\lambda}_1,\widetilde{\lambda}_2)\sum_{n=2}^\infty\left(1-O\left(\frac{1}{\ln^{\widetilde{\lambda}_1} V_n}\right)\right)\frac{\upsilon_{n+1}}{V_n\ln^{\varepsilon+1} V_n} \\ &\quad= B(\widetilde{\lambda}_1,\widetilde{\lambda}_2)\left[\sum_{n=2}^\infty \frac{\upsilon_{n+1}}{V_n\ln^{\varepsilon+1} V_n} - \sum_{n=2}^\infty O\left(\frac{\upsilon_{n+1}}{V_n(\ln V_n)^{(\frac{\varepsilon}{q}+\lambda_1)+1}}\right)\right] \\ &\quad= \frac{1}{\varepsilon}B(\widetilde{\lambda}_1,\widetilde{\lambda}_2)\left[\frac{1}{\ln^\varepsilon(1+\upsilon_2)}+\varepsilon\big(\widetilde{O}(1)-O(1)\big)\right]. \end{aligned}$$

If there exists a positive constant $K \le B(\lambda_1,\lambda_2)$ such that (38) is valid when replacing $B(\lambda_1,\lambda_2)$ by K, then, in particular, we have $\varepsilon\widetilde{I} < \varepsilon K\|\widetilde{a}\|_{p,\Omega_\lambda}\|\widetilde{b}\|_{q,\digamma_\lambda}$, namely

$$\begin{aligned} &B(\widetilde{\lambda}_1,\widetilde{\lambda}_2)\left[\frac{1}{\ln^\varepsilon(1+\upsilon_2)}+\varepsilon\big(\widetilde{O}(1)-O(1)\big)\right] \\ &\quad< K\left[\frac{1}{\ln^\varepsilon(1+\mu_2)}+\varepsilon O(1)\right]^{\frac{1}{p}}\left[\frac{1}{\ln^\varepsilon(1+\upsilon_2)}+\varepsilon\widetilde{O}(1)\right]^{\frac{1}{q}}. \end{aligned}$$

It follows that $B(\lambda_1,\lambda_2) \le K$ ($\varepsilon \to 0^+$). Hence, $K = B(\lambda_1,\lambda_2)$ is the best possible constant factor of (38).

Similarly to (33), we still can find that

$$I \le J_1 \|b\|_{q,\tilde{F}_\lambda}. \tag{41}$$

Hence, we can prove that the constant factor $B(\lambda_1,\lambda_2)$ in (39) is the best possible. Otherwise, we would reach a contradiction by (41) that the constant factor in (38) is not the best possible. □

Remark 1 (i) It is evident that (38) and (39) are strengthened versions of the following equivalent Mulholland-type inequalities:

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{\ln^{\lambda}(U_m V_n)} < B(\lambda_1,\lambda_2)\|a\|_{p,\Phi_\lambda}\|b\|_{q,\Psi_\lambda}, \tag{42}$$

$$\left\{\sum_{n=2}^{\infty}\frac{\upsilon_{n+1}}{V_n}\ln^{p\lambda_2-1}V_n\left[\sum_{m=2}^{\infty}\frac{a_m}{\ln^{\lambda}(U_m V_n)}\right]^p\right\}^{\frac{1}{p}} < B(\lambda_1,\lambda_2)\|a\|_{p,\Phi_\lambda}, \tag{43}$$

where the constant factor $B(\lambda_1,\lambda_2)$ is still the best possible.

(ii) For $\lambda = 1$, $\lambda_1 = \frac{1}{q}$, $\lambda_2 = \frac{1}{p}$,

$$\theta_1 = \vartheta_1 := \frac{p\sin(\pi/p)\ln^{1/p}(1+\upsilon_2/2)}{\pi[1+\frac{\ln(1+\upsilon_2/2)}{\ln(1+\mu_2/2)}]},$$

$$\theta_2 = \vartheta_2 := \frac{q\sin(\pi/p)\ln^{1/q}(1+\mu_2/2)}{\pi[1+\frac{\ln(1+\mu_2/2)}{\ln(1+\upsilon_2/2)}]},$$

(38) reduces to the strengthened version of (9) as follows:

$$\begin{aligned}\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{a_m b_n}{\ln U_m V_n} &< \frac{\pi}{\sin(\pi/p)}\left[\sum_{m=2}^{\infty}\left(1-\frac{\vartheta_1}{\ln^{1/p}U_m}\right)\left(\frac{U_m}{\mu_{m+1}}\right)^{p-1}a_m^p\right]^{\frac{1}{p}} \\ &\quad\times\left[\sum_{n=2}^{\infty}\left(1-\frac{\vartheta_2}{\ln^{1/q}V_n}\right)\left(\frac{V_n}{\upsilon_{n+1}}\right)^{q-1}b_n^q\right]^{\frac{1}{q}}.\end{aligned} \tag{44}$$

For $\mu_i = \upsilon_j = 1$ ($i,j \in \mathbf{N}$), (44) reduces to the following strengthened Mulholland's inequality:

$$\begin{aligned}\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{a_m b_n}{\ln mn} &< \left\{\sum_{m=2}^{\infty}\left[\frac{\pi}{\sin(\pi/p)}-\frac{\ln\sqrt{3/2}}{\ln^{1/p}m}\right]\frac{a_m^p}{m^{1-p}}\right\}^{\frac{1}{p}} \\ &\quad\times\left\{\sum_{n=2}^{\infty}\left[\frac{\pi}{\sin(\pi/p)}-\frac{\ln\sqrt{3/2}}{\ln^{1/q}m}\right]\frac{b_n^q}{n^{1-q}}\right\}^{\frac{1}{q}},\end{aligned} \tag{45}$$

where $\ln\sqrt{3/2} = 0.20275^+$.

For $p > 1$, $\Psi_\lambda^{1-p}(n) = \frac{\upsilon_{n+1}}{V_n}(\ln V_n)^{p\lambda_2 - 1}$, we define the following normed spaces:

$$l_{p,\Phi_\lambda} := \{a = \{a_m\}_{m=2}^\infty; \|a\|_{p,\Phi_\lambda} < \infty\},$$
$$l_{q,\Psi_\lambda} := \{b = \{b_n\}_{n=2}^\infty; \|b\|_{q,\Psi_\lambda} < \infty\},$$
$$l_{p,\Psi_\lambda^{1-p}} := \{c = \{c_n\}_{n=2}^\infty; \|c\|_{p,\Psi_\lambda^{1-p}} < \infty\}.$$

Assuming that $a = \{a_m\}_{m=2}^\infty \in l_{p,\Phi_\lambda}$, setting

$$c = \{c_n\}_{n=2}^\infty, c_n := \sum_{m=2}^\infty \frac{a_m}{\ln^\lambda(U_m V_n)}, \quad n \in \mathbf{N},$$

we can rewrite (43) as follows:

$$\|c\|_{p,\Psi_\lambda^{1-p}} < B(\lambda_1, \lambda_2)\|a\|_{p,\Phi_\lambda} < \infty,$$

namely $c \in l_{p,\Psi_\lambda^{1-p}}$.

Definition 2 Define a Mulholland-type operator $T : l_{p,\Phi_\lambda} \to l_{p,\Psi_\lambda^{1-p}}$ as follows: For any $a = \{a_m\}_{m=2}^\infty \in l_{p,\Phi_\lambda}$, there exists a unique representation $Ta = c \in l_{p,\Psi_\lambda^{1-p}}$. Define the formal inner product of Ta and $b = \{b_n\}_{n=2}^\infty \in l_{q,\Psi_\lambda}$ as follows:

$$(Ta, b) := \sum_{n=2}^\infty \left[\sum_{m=2}^\infty \frac{a_m}{\ln^\lambda(U_m V_n)}\right] b_n. \tag{46}$$

Then we can rewrite (42) and (43) as follows:

$$(Ta, b) < B(\lambda_1, \lambda_2)\|a\|_{p,\Phi_\lambda}\|b\|_{q,\Psi_\lambda}, \tag{47}$$
$$\|Ta\|_{p,\Psi_\lambda^{1-p}} < B(\lambda_1, \lambda_2)\|a\|_{p,\Phi_\lambda}. \tag{48}$$

Define the norm of operator T as follows:

$$\|T\| := \sup_{a(\neq\theta)\in l_{p,\Phi_\lambda}} \frac{\|Ta\|_{p,\Psi_\lambda^{1-p}}}{\|a\|_{p,\Phi_\lambda}}.$$

Then by (43) we find $\|T\| \leq B(\lambda_1, \lambda_2)$. Since the constant factor in (48) is the best possible, we have

$$\|T\| = B(\lambda_1, \lambda_2). \tag{49}$$

4 Some strengthened versions of the reverses

In the following, we also set

$$\begin{aligned} \widetilde{\Omega}_\lambda(m) &:= \bigl(1 - \theta(\lambda_2, m)\bigr)\frac{(\ln U_m)^{p(1-\lambda_1)-1}}{U_m^{1-p}\mu_{m+1}^{p-1}}, \\ \widetilde{F}_\lambda(n) &:= \bigl(1 - \vartheta(\lambda_1, n)\bigr)\frac{(\ln V_n)^{q(1-\lambda_2)-1}}{V_n^{1-q}\upsilon_{n+1}^{q-1}} \quad \bigl(m, n \in \mathbf{N}\backslash\{1\}\bigr). \end{aligned} \tag{50}$$

For $0 < p < 1$ or $p < 0$, we still use the formal symbols $\|a\|_{p,\Phi_\lambda}$, $\|b\|_{q,\Psi_\lambda}$, $\|a\|_{p,\Omega_\lambda}$, $\|b\|_{q,F_\lambda}$, $\|a\|_{p,\widetilde{\Omega}_\lambda}$ and $\|b\|_{q,\widetilde{F}_\lambda}$.

Theorem 3 *If $0 < p < 1$, $\{\mu_m\}_{m=1}^\infty$ and $\{\upsilon_n\}_{n=1}^\infty$ are decreasing, $U_\infty = V_\infty = \infty$, $\|a\|_{p,\Phi_\lambda} \in \mathbf{R}_+$ and $\|b\|_{q,\Psi_\lambda} \in \mathbf{R}_+$, then we have the following equivalent inequalities with the best possible constant factor $B(\lambda_1,\lambda_2)$:*

$$\sum_{n=2}^\infty \sum_{m=2}^\infty \frac{a_m b_n}{\ln^\lambda(U_m V_n)} > B(\lambda_1,\lambda_2)\|a\|_{p,\widetilde{\Omega}_\lambda}\|b\|_{q,F_\lambda}, \tag{51}$$

$$\left\{\sum_{n=2}^\infty \frac{\upsilon_{n+1}\ln^{p\lambda_2-1} V_n}{(1-\frac{\theta_2}{\ln^{\lambda_1} V_n})^{p-1} V_n}\left[\sum_{m=2}^\infty \frac{a_m}{\ln^\lambda(U_m V_n)}\right]^p\right\}^{\frac{1}{p}} > B(\lambda_1,\lambda_2)\|a\|_{p,\widetilde{\Omega}_\lambda}. \tag{52}$$

Proof Using (22) and (17) in the reverses of (29) and (30), since

$$\bigl(\omega(\lambda_2,m)\bigr)^{\frac{1}{p}} > \bigl(B(\lambda_1,\lambda_2)\bigr)^{\frac{1}{p}}\bigl(1-\theta(\lambda_2,m)\bigr)^{\frac{1}{p}} \quad (0 < p < 1),$$

$$\bigl(\varpi(\lambda_1,n)\bigr)^{\frac{1}{q}} > \bigl(B(\lambda_1,\lambda_2)\bigr)^{\frac{1}{q}}\left(1-\frac{\theta_2}{\ln^{\lambda_1} V_n}\right)^{\frac{1}{q}} \quad (q < 0)$$

and

$$\frac{1}{(B(\lambda_1,\lambda_2))^{p-1}(1-\frac{\theta_2}{V_n^{\lambda_1}})^{p-1}} > \frac{1}{(\varpi(\lambda_1,n))^{p-1}} \quad (0 < p < 1),$$

we obtain equivalent inequalities (51) and (52).

For $\varepsilon \in (0,p\lambda_1)$, we set $\widetilde{\lambda}_1$,$\widetilde{\lambda}_2$, $\widetilde{a}_m$ and $\widetilde{b}_n$ as (40). Then, by (26), (27) and (17), we find

$$\begin{aligned}
&\|a\|_{p,\widetilde{\Omega}_\lambda}\|b\|_{q,F_\lambda} \\
&\quad \geq \|a\|_{p,\widetilde{\Omega}_\lambda}\|b\|_{q,\Psi_\lambda} \\
&\quad = \left[\sum_{m=2}^\infty \bigl(1-\theta(\lambda_2,m)\bigr)\frac{\mu_{m+1}}{U_m \ln^{1+\varepsilon} U_m}\right]^{\frac{1}{p}}\left(\sum_{n=2}^\infty \frac{\upsilon_{n+1}}{V_n \ln^{1+\varepsilon} V_n}\right)^{\frac{1}{q}} \\
&\quad = \left(\sum_{m=2}^\infty \frac{\mu_{m+1}}{U_m \ln^{1+\varepsilon} U_m} - \sum_{m=2}^\infty O\left(\frac{\mu_{m+1}}{U_m \ln^{1+\lambda_2+\varepsilon} U_m}\right)\right)^{\frac{1}{p}} \\
&\qquad \times \left(\sum_{n=2}^\infty \frac{\upsilon_{n+1}}{V_n \ln^{1+\varepsilon} V_n}\right)^{\frac{1}{q}} \\
&\quad = \frac{1}{\varepsilon}\left[\frac{1}{\ln^\varepsilon(1+\mu_2)} + \varepsilon\bigl(O(1)-O_1(1)\bigr)\right]^{\frac{1}{p}}\left[\frac{1}{\ln^\varepsilon(1+\upsilon_2)} + \varepsilon\widetilde{O}(1)\right]^{\frac{1}{q}},
\end{aligned}$$

$$\begin{aligned}
\widetilde{I} &:= \sum_{n=2}^\infty \sum_{m=2}^\infty \frac{\widetilde{a}_m \widetilde{b}_n}{\ln^\lambda(U_m V_n)} \\
&= \sum_{n=2}^\infty \left[\sum_{m=2}^\infty \frac{1}{\ln^\lambda(U_m V_n)} \frac{\mu_{m+1}\ln^{\widetilde{\lambda}_2} V_n}{U_m \ln^{1-\widetilde{\lambda}_1} U_m}\right]\frac{\upsilon_{n+1}}{V_n \ln^{\varepsilon+1} V_n}
\end{aligned}$$

$$= \sum_{n=2}^{\infty} \varpi(\widetilde{\lambda}_1, n) \frac{\upsilon_{n+1}}{V_n \ln^{\varepsilon+1} V_n} \le B(\widetilde{\lambda}_1, \widetilde{\lambda}_2) \sum_{n=2}^{\infty} \frac{\upsilon_{n+1}}{V_n \ln^{\varepsilon+1} V_n}$$
$$= \frac{1}{\varepsilon} B(\widetilde{\lambda}_1, \widetilde{\lambda}_2) \left[\frac{1}{\ln^{\varepsilon}(1+\upsilon_2)} + \varepsilon \widetilde{O}(1) \right].$$

If there exists a positive constant $K \ge B(\lambda_1, \lambda_2)$ such that (51) is valid when replacing $B(\lambda_1, \lambda_2)$ by K, then, in particular, we have $\varepsilon \widetilde{I} > \varepsilon K \|\widetilde{a}\|_{p,\widetilde{\Omega}_\lambda} \|\widetilde{b}\|_{q,\widetilde{F}_\lambda}$, namely

$$B(\widetilde{\lambda}_1, \widetilde{\lambda}_2) \left[\frac{1}{\ln^{\varepsilon}(1+\upsilon_2)} + \varepsilon \widetilde{O}(1) \right]$$
$$> K \left[\frac{1}{\ln^{\varepsilon}(1+\mu_2)} + \varepsilon \bigl(O(1) - O_1(1) \bigr) \right]^{\frac{1}{p}} \left[\frac{1}{\ln^{\varepsilon}(1+\upsilon_2)} + \varepsilon \widetilde{O}(1) \right]^{\frac{1}{q}}.$$

It follows that $B(\lambda_1, \lambda_2) \ge K$ ($\varepsilon \to 0^+$). Hence, $K = B(\lambda_1, \lambda_2)$ is the best possible constant factor of (51).

The constant factor $B(\lambda_1, \lambda_2)$ in (52) is still the best possible. Otherwise, we would reach a contradiction by the reverse of (41) that the constant factor in (51) is not the best possible. □

Remark 2 It is evident that (51) and (52) are strengthened versions of the following equivalent inequalities:

$$\sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \frac{a_m b_n}{\ln^{\lambda}(U_m V_n)} > B(\lambda_1, \lambda_2) \|a\|_{p,\widetilde{\Omega}_\lambda} \|b\|_{q,\Psi_\lambda}, \tag{53}$$

$$\left\{ \sum_{n=2}^{\infty} \frac{\upsilon_{n+1}}{V_n} \ln^{p\lambda_2 - 1} V_n \left[\sum_{m=2}^{\infty} \frac{a_m}{\ln^{\lambda}(U_m V_n)} \right]^p \right\}^{\frac{1}{p}} > B(\lambda_1, \lambda_2) \|a\|_{p,\widetilde{\Omega}_\lambda}, \tag{54}$$

where the constant factor $B(\lambda_1, \lambda_2)$ is still the best possible.

Theorem 4 *If $p < 0$, $\{\mu_m\}_{m=1}^{\infty}$ and $\{\upsilon_n\}_{n=1}^{\infty}$ are decreasing, $U_\infty = V_\infty = \infty$, $\|a\|_{p,\Phi_\lambda} \in \mathbf{R}_+$ and $\|b\|_{q,\Psi_\lambda} \in \mathbf{R}_+$, then we have the following equivalent inequalities with the best possible constant factor $B(\lambda_1, \lambda_2)$:*

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\ln^{\lambda}(U_m V_n)} > B(\lambda_1, \lambda_2) \|a\|_{p,\Omega_\lambda} \|b\|_{q,\widetilde{F}_\lambda}, \tag{55}$$

$$J_2 := \left\{ \sum_{n=1}^{\infty} \frac{\upsilon_{n+1} \ln^{p\lambda_2 - 1} V_n}{(1 - \vartheta(\lambda_1, n))^{p-1} V_n} \left[\sum_{m=1}^{\infty} \frac{a_m}{\ln^{\lambda}(U_m V_n)} \right]^p \right\}^{\frac{1}{p}} > B(\lambda_1, \lambda_2) \|a\|_{p,\Omega_\lambda}. \tag{56}$$

Proof Using (16) and (23) in the reverses of (29) and (30), since

$$\bigl(\omega(\lambda_2, m)\bigr)^{\frac{1}{p}} > \bigl(B(\lambda_1, \lambda_2)\bigr)^{\frac{1}{p}} \left(1 - \frac{\theta_1}{U_m^{\lambda_2}} \right)^{\frac{1}{p}} \quad (p < 0),$$

$$\bigl(\varpi(\lambda_1, n)\bigr)^{\frac{1}{q}} > \bigl(B(\lambda_1, \lambda_2)\bigr)^{\frac{1}{q}} \bigl(1 - \vartheta(\lambda_1, n)\bigr)^{\frac{1}{q}} \quad (0 < q < 1)$$

and

$$\left[\frac{1}{(B(\lambda_1,\lambda_2))^{p-1}(1-\vartheta(\lambda_1,n))^{p-1}}\right]^{\frac{1}{p}} > \left[\frac{1}{(\varpi(\lambda_1,n))^{p-1}}\right]^{\frac{1}{p}} \quad (p<0),$$

we obtain equivalent inequalities (55) and (56).

For $\varepsilon \in (0, q\lambda_2)$, we set $\widetilde{\lambda}_1 = \lambda_1 + \frac{\varepsilon}{q}$ (>0), $\widetilde{\lambda}_2 = \lambda_2 - \frac{\varepsilon}{q}$ ($\in (0,1)$), and

$$\begin{aligned}
\widetilde{a}_m &:= \frac{\mu_{m+1}}{U_m}\ln^{\widetilde{\lambda}_1-\varepsilon-1} U_m = \frac{\mu_{m+1}}{U_m}\ln^{\lambda_1-\frac{\varepsilon}{p}-1} U_m,\\
\widetilde{b}_n &= \frac{\upsilon_{n+1}}{V_n}\ln^{\widetilde{\lambda}_2-1} V_n = \frac{\upsilon_{n+1}}{V_n}\ln^{\lambda_2-\frac{\varepsilon}{q}-1} V_n.
\end{aligned}$$

Then, by (26), (27) and (16), we have

$$\begin{aligned}
&\|\widetilde{a}\|_{p,\Omega_\lambda}\|\widetilde{b}\|_{q,\widetilde{F}_\lambda}\\
&\quad\geq \|\widetilde{a}\|_{p,\Phi_\lambda}\|\widetilde{b}\|_{q,\widetilde{F}_\lambda}\\
&\quad= \left(\sum_{m=2}^{\infty}\frac{\mu_{m+1}}{U_m\ln^{\varepsilon+1} U_m}\right)^{\frac{1}{p}}\left[\sum_{n=2}^{\infty}\bigl(1-\vartheta(\lambda_1,n)\bigr)\frac{\upsilon_{n+1}}{V_n\ln^{\varepsilon+1} V_n}\right]^{\frac{1}{q}}\\
&\quad= \left(\sum_{m=2}^{\infty}\frac{\mu_{m+1}}{U_m\ln^{\varepsilon+1} U_m}\right)^{\frac{1}{p}}\left[\sum_{n=2}^{\infty}\frac{\upsilon_{n+1}}{V_n\ln^{\varepsilon+1} V_n} - \sum_{n=2}^{\infty} O\left(\frac{\upsilon_{n+1}}{V_n\ln^{1+(\lambda_1+\varepsilon)} V_n}\right)\right]^{\frac{1}{q}}\\
&\quad= \frac{1}{\varepsilon}\left[\frac{1}{\ln^{\varepsilon}(1+\mu_2)}+\varepsilon O(1)\right]^{\frac{1}{p}}\left[\frac{1}{\ln^{\varepsilon}(1+\upsilon_2)}+\varepsilon\bigl(\widetilde{O}(1)-O_1(1)\bigr)\right]^{\frac{1}{q}},\\
&\widetilde{I} = \sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{\widetilde{a}_m\widetilde{b}_n}{\ln^{\lambda}(U_m V_n)}\\
&\quad= \sum_{m=2}^{\infty}\left[\sum_{n=2}^{\infty}\frac{\ln^{\widetilde{\lambda}_1} U_m}{\ln^{\lambda}(U_m V_n)}\frac{\upsilon_{n+1}}{V_n}\ln^{\widetilde{\lambda}_2-1} V_n\right]\frac{\mu_{m+1}}{U_m\ln^{\varepsilon+1} U_m}\\
&\quad= \sum_{m=2}^{\infty}\omega(\widetilde{\lambda}_2,m)\frac{\mu_{m+1}}{U_m\ln^{\varepsilon+1} U_m} \leq B(\widetilde{\lambda}_1,\widetilde{\lambda}_2)\sum_{n=2}^{\infty}\frac{\mu_{m+1}}{U_m\ln^{\varepsilon+1} U_m}\\
&\quad= \frac{1}{\varepsilon}B(\widetilde{\lambda}_1,\widetilde{\lambda}_2)\left[\frac{1}{\ln^{\varepsilon}(1+\mu_2)}+\varepsilon O(1)\right].
\end{aligned}$$

If there exists a positive constant $K \geq B(\lambda_1,\lambda_2)$ such that (55) is valid when replacing $B(\lambda_1,\lambda_2)$ by K, then, in particular, we have $\varepsilon\widetilde{I} > \varepsilon K\|\widetilde{a}\|_{p,\Omega_\lambda}\|\widetilde{b}\|_{q,\widetilde{F}_\lambda}$, namely

$$\begin{aligned}
&B(\widetilde{\lambda}_1,\widetilde{\lambda}_2)\left[\frac{1}{\ln^{\varepsilon}(1+\mu_2)}+\varepsilon O(1)\right]\\
&\quad> K\left[\frac{1}{\ln^{\varepsilon}(1+\mu_2)}+\varepsilon O(1)\right]^{\frac{1}{p}}\left[\frac{1}{\ln^{\varepsilon}(1+\upsilon_2)}+\varepsilon\bigl(\widetilde{O}(1)-O_1(1)\bigr)\right]^{\frac{1}{q}}.
\end{aligned}$$

It follows that $B(\lambda_1,\lambda_2) \geq K$ ($\varepsilon \to 0^+$). Hence, $K = B(\lambda_1,\lambda_2)$ is the best possible constant factor of (55).

Similarly to the reverse of (33), we still find that

$$I \geq J_2 \|b\|_{q,\widetilde{F}_\lambda}. \tag{57}$$

Hence the constant factor $B(\lambda_1, \lambda_2)$ in (56) is still the best possible. Otherwise, we would reach a contradiction by (57) that the constant factor in (55) is not the best possible. □

Remark 3 It is evident that (55) and (56) are strengthened versions of the following equivalent inequalities:

$$\sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \frac{a_m b_n}{\ln^{\lambda}(U_m V_n)} > B(\lambda_1, \lambda_2) \|a\|_{p,\Phi_\lambda} \|b\|_{q,\widetilde{F}_\lambda}, \tag{58}$$

$$\left\{ \sum_{n=2}^{\infty} \frac{\upsilon_{n+1} \ln^{p\lambda_2 - 1} V_n}{(1 - \vartheta(\lambda_1, n))^{p-1} V_n} \left[\sum_{m=2}^{\infty} \frac{a_m}{\ln^{\lambda}(U_m V_n)} \right]^p \right\}^{\frac{1}{p}} > B(\lambda_1, \lambda_2) \|a\|_{p,\Phi_\lambda}, \tag{59}$$

where the constant factor $B(\lambda_1, \lambda_2)$ is still the best possible.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
BY carried out the mathematical studies, participated in the sequence alignment and drafted the manuscript. AW and QH participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

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