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Remarks on some inequalities for s-convex functions and applications

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Abstract

Some new inequalities for functions whose second derivatives in absolute value at certain powers are s-convex in the second sense are established. Two mistakes in a recently published paper are pointed out and corrected.

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1 Introduction

It is well known that a function $f : I \to \mathbf{R}$, $\emptyset \neq I \subset \mathbf{R}$ is called convex if

$$f\bigl(\lambda x + (1-\lambda)y\bigr) \le \lambda f(x) + (1-\lambda)f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$. If $-f : I \to \mathbf{R}$ is convex, then we say that $f : I \to \mathbf{R}$ is concave.

In [1], the class of s-convex function in the second sense is defined in the following way: a function $f : [0,\infty) \to \mathbf{R}$ is said to be s-convex in the second sense if

$$f\bigl(\lambda x + (1-\lambda)y\bigr) \le \lambda^s f(x) + (1-\lambda)^s f(y)$$

holds for all $x, y \in [0,\infty)$, $\lambda \in [0,1]$ and for some fixed $s \in (0,1]$. The class of s-convex functions in the second sense is usually denoted by K_s^2. It can be easily seen that for $s = 1$ s-convexity reduces to ordinary convexity of functions defined on $[0,\infty)$. It is proved in [1] that all functions from $K_s^2, s \in (0,1)$ are nonnegative. Similarly, a function $f : [0,\infty) \to \mathbf{R}$ is said to be s-concave in the second sense for some fixed $s \in (0,1]$ if $-f \in K_s^2$. Thus we can conclude that an s-concave function is always nonpositive for any $s \in (0,1)$.

Example 1 [1] Let $s \in (0,1)$ and $a, b, c, \in \mathbf{R}$. Define the function $f : [0,\infty) \to \mathbf{R}$ as

$$f(t) = \begin{cases} a, & t = 0, \\ bt^s + c, & t > 0. \end{cases}$$

It can be easily checked that

(i) if $b \ge 0$ and $0 \le c \le a$, then $f \in K_s^2$,

(ii) if $b > 0$ and $c < 0$, then $f \notin K_s^2$.

Along this paper, we consider a real interval $I \subset \mathbf{R}$ and denote that I° is the interior of I.

In a recent paper [2], Özdemir *et al.* proved the following inequalities for functions whose second derivatives in absolute value at certain powers are s-convex in the second sense.

Theorem 1 ([2], Theorem 2) *Let $I \subset [0,\infty)$, $f: I \to \mathbf{R}$ be a twice differentiable function on I° such that $f'' \in L^1[a,b]$, where $a, b \in I$ with $a < b$. If $|f''|$ is s-convex in the second sense on $[a,b]$ for some fixed $s \in (0,1]$, then the following inequality holds*:

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)\right| \\
&\quad \le \frac{(b-a)^2}{8(s+1)(s+2)(s+3)}\left[\left|f''(a)\right| + (s+1)(s+2)\left|f''\left(\frac{a+b}{2}\right)\right| + \left|f''(b)\right|\right] \\
&\quad \le \frac{[1+(s+2)2^{1-s}](b-a)^2}{8(s+1)(s+2)(s+3)}\left[\left|f''(a)\right| + \left|f''(b)\right|\right]. \qquad (1)
\end{aligned}$$

Theorem 2 ([2], Theorem 4) *Let $I \subset [0,\infty)$, $f: I \to \mathbf{R}$ be a twice differentiable function on I° such that $f'' \in L^1[a,b]$, where $a, b \in I$ with $a < b$. If $|f''|^q$ is s-convex in the second sense on $[a,b]$ for some fixed $s \in (0,1]$ and $q \ge 1$, then the following inequality holds*:

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)\right| \\
&\quad \le \frac{(b-a)^2}{16}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left\{\left(\frac{2}{(s+1)(s+2)(s+3)}\left|f''(a)\right|^q + \frac{1}{s+3}\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right. \\
&\qquad \left. + \left(\frac{1}{s+3}\left|f''\left(\frac{a+b}{2}\right)\right|^q + \frac{2}{(s+1)(s+2)(s+3)}\left|f''(b)\right|^q\right)^{\frac{1}{q}}\right\}. \qquad (2)
\end{aligned}$$

However, it is a pity that Theorem 5 in [2] is not valid since nonnegative $|f''|^q$ could not be an s-concave function for any fixed $s \in (0,1)$ which has been mentioned in [3], and it is the Hölder inequality but not the power mean inequality that has been used in proving Theorem 4 of [2].

In this work, we will first derive a new general inequality for functions whose second derivatives in absolute value at certain powers are s-convex in the second sense, which not only provides generalization of Theorem 1 and Theorem 2 but also gives some other interesting special results. Then we establish another new general inequality for functions whose second derivatives in absolute value at certain powers are s-convex in the second sense, which also gives some interesting special results. Finally, applications to some special means of real numbers are considered.

2 Main results

We first provide a new general inequality for functions whose second derivatives in absolute value at certain powers are s-convex in the second sense, and so we need the following lemma.

Lemma 1 *Let $I \subset \mathbf{R}$, $f: I \to \mathbf{R}$ be a twice differentiable function on I° such that $f'' \in L^1[a,b]$, where $a, b \in I$ with $a < b$. Then, for any $\theta \in [0,1]$, the following equality holds*:

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(t)\,dt - (1-\theta)f\left(\frac{a+b}{2}\right) - \theta\frac{f(a)+f(b)}{2} \\ &\quad = \frac{(b-a)^2}{16}\left[\int_0^1 (t^2-2\theta t)f''\left(t\frac{a+b}{2}+(1-t)a\right)dt \right. \\ &\qquad \left. + \int_0^1 (t^2-2\theta t)f''\left(t\frac{a+b}{2}+(1-t)b\right)dt\right]. \end{aligned} \tag{3}$$

Proof Integrating by parts, we have the following identity:

$$\begin{aligned} I_1 &= \int_0^1 (t^2-2\theta t)f''\left(t\frac{a+b}{2}+(1-t)a\right)dt \\ &= (t^2-2\theta t)\frac{2}{b-a}f'\left(t\frac{a+b}{2}+(1-t)a\right)\bigg|_0^1 - \frac{4}{b-a}\int_0^1 (t-\theta)f'\left(t\frac{a+b}{2}+(1-t)a\right)dt \\ &= \frac{2(1-2\theta)}{b-a}f'\left(\frac{a+b}{2}\right) - \frac{4}{b-a}\left[\frac{2(t-\theta)}{b-a}f\left(t\frac{a+b}{2}+(1-t)a\right)\bigg|_0^1 \right. \\ &\quad \left. - \frac{2}{b-a}\int_0^1 f\left(t\frac{a+b}{2}+(1-t)a\right)dt\right] \\ &= \frac{2(1-2\theta)}{b-a}f'\left(\frac{a+b}{2}\right) - \frac{8(1-\theta)}{(b-a)^2}f\left(\frac{a+b}{2}\right) - \frac{8\theta}{(b-a)^2}f(a) \\ &\quad + \frac{8}{(b-a)^2}\int_0^1 f\left(t\frac{a+b}{2}+(1-t)a\right)dt. \end{aligned} \tag{4}$$

Using the change of variable $x = t\frac{a+b}{2}+(1-t)a$ for $t \in [0,1]$ and multiplying both sides of (4) by $\frac{(b-a)^2}{16}$, we obtain

$$\begin{aligned} &\frac{(b-a)^2}{16}\int_0^1 (t^2-2\theta t)f''\left(t\frac{a+b}{2}+(1-t)a\right)dt \\ &\quad = \frac{b-a}{8}(1-2\theta)f'\left(\frac{a+b}{2}\right) - \frac{1-\theta}{2}f\left(\frac{a+b}{2}\right) - \frac{\theta}{2}f(a) + \frac{1}{b-a}\int_a^{\frac{a+b}{2}} f(x)\,dx. \end{aligned} \tag{5}$$

Similarly, we observe that

$$\begin{aligned} &\frac{(b-a)^2}{16}\int_0^1 (t^2-2\theta t)f''\left(t\frac{a+b}{2}+(1-t)b\right)dt \\ &\quad = -\frac{b-a}{8}(1-2\theta)f'\left(\frac{a+b}{2}\right) - \frac{1-\theta}{2}f\left(\frac{a+b}{2}\right) - \frac{\theta}{2}f(b) + \frac{1}{b-a}\int_{\frac{a+b}{2}}^b f(x)\,dx. \end{aligned} \tag{6}$$

Thus, adding (5) and (6), we get the required identity (3). □

Theorem 3 *Let $I \subset [0,\infty)$, $f: I \to \mathbf{R}$ be a twice differentiable function on I° such that $f'' \in L^1[a,b]$, where $a, b \in I$ with $a < b$. If $|f''|^q$ is s-convex in the second sense on $[a,b]$ for* *some fixed* $s \in (0,1]$ *and* $q \geq 1$, *then the following inequalities hold*:

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,dx - (1-\theta)f\left(\frac{a+b}{2}\right) - \theta\frac{f(a)+f(b)}{2}\right| \\
&\quad\leq \frac{(b-a)^2}{16}\left(\frac{8\theta^3-3\theta+1}{3}\right)^{1-\frac{1}{q}}\left\{\left[\frac{2(2\theta)^{s+3}-2(s+3)\theta+s+2}{(s+2)(s+3)}\left|f''\left(\frac{a+b}{2}\right)\right|^q\right.\right. \\
&\qquad\left.+\frac{4(1-2\theta)^{s+2}[(s+1)\theta+1]+2(s+3)\theta-2}{(s+1)(s+2)(s+3)}\left|f''(a)\right|^q\right]^{\frac{1}{q}} \\
&\qquad+\left[\frac{2(2\theta)^{s+3}-2(s+3)\theta+s+2}{(s+2)(s+3)}\left|f''\left(\frac{a+b}{2}\right)\right|^q\right. \\
&\qquad\left.\left.+\frac{4(1-2\theta)^{s+2}[(s+1)\theta+1]+2(s+3)\theta-2}{(s+1)(s+2)(s+3)}\left|f''(b)\right|^q\right]^{\frac{1}{q}}\right\}
\end{aligned}\tag{7}$$

for $0 \leq \theta \leq \frac{1}{2}$ *and*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,dx - (1-\theta)f\left(\frac{a+b}{2}\right) - \theta\frac{f(a)+f(b)}{2}\right| \\
&\quad\leq \frac{(b-a)^2}{16}\left(\theta-\frac{1}{3}\right)^{1-\frac{1}{q}}\left\{\left[\frac{2(s+3)\theta-s-2}{(s+2)(s+3)}\left|f''\left(\frac{a+b}{2}\right)\right|^q\right.\right. \\
&\qquad\left.+\frac{2(s+3)\theta-2}{(s+1)(s+2)(s+3)}\left|f''(a)\right|^q\right]^{\frac{1}{q}}+\left[\frac{2(s+3)\theta-s-2}{(s+2)(s+3)}\left|f''\left(\frac{a+b}{2}\right)\right|^q\right. \\
&\qquad\left.\left.+\frac{2(s+3)\theta-2}{(s+1)(s+2)(s+3)}\left|f''(b)\right|^q\right]^{\frac{1}{q}}\right\}
\end{aligned}\tag{8}$$

for $\frac{1}{2} \leq \theta \leq 1$.

Proof In case $0 \leq \theta \leq \frac{1}{2}$, by Lemma 1 and using the Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,dt - (1-\theta)f\left(\frac{a+b}{2}\right) - \theta\frac{f(a)+f(b)}{2}\right| \\
&\quad\leq \frac{(b-a)^2}{16}\left[\int_0^1\left|t^2-2\theta t\right|\left|f''\left(t\frac{a+b}{2}+(1-t)a\right)\right|dt\right. \\
&\qquad\left.+\int_0^1\left|t^2-2\theta t\right|\left|f''\left(t\frac{a+b}{2}+(1-t)b\right)\right|dt\right] \\
&\quad\leq \frac{(b-a)^2}{16}\left[\left(\int_0^1\left|t^2-2\theta t\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|t^2-2\theta t\right|\left|f''\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\qquad\left.+\left(\int_0^1\left|t^2-2\theta t\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|t^2-2\theta t\right|\left|f''\left(t\frac{a+b}{2}+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}}\right] \\
&\quad\leq \frac{(b-a)^2}{16}\left[\left(\frac{8\theta^3-3\theta+1}{3}\right)^{1-\frac{1}{q}}\left(\int_0^1\left|t^2-2\theta t\right|\left(t^s\left|f''\left(\frac{a+b}{2}\right)\right|^q\right.\right.\right. \\
&\qquad\left.\left.+(1-t)^s\left|f''(a)\right|^q\right)dt\right)^{\frac{1}{q}}
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{8\theta^3-3\theta+1}{3}\right)^{1-\frac{1}{q}}\left(\int_0^1\left|t^2-2\theta t\right|\left(t^s\left|f''\left(\frac{a+b}{2}\right)\right|^q+(1-t)^s\left|f''(b)\right|^q\right)dt\right)^{\frac{1}{q}}\Bigg]\\
&=\frac{(b-a)^2}{16}\Bigg[\left(\frac{8\theta^3-3\theta+1}{3}\right)^{1-\frac{1}{q}}\left(\int_0^{2\theta}t(2\theta-t)\left(t^s\left|f''\left(\frac{a+b}{2}\right)\right|^q\right.\right.\\
&\quad\left.+(1-t)^s\left|f''(a)\right|^q\right)dt+\int_{2\theta}^1 t(t-2\theta)\left(t^s\left|f''\left(\frac{a+b}{2}\right)\right|^q+(1-t)^s\left|f''(a)\right|^q\right)dt\Bigg)^{\frac{1}{q}}\\
&\quad+\left(\frac{8\theta^3-3\theta+1}{3}\right)^{1-\frac{1}{q}}\left(\int_0^{2\theta}t(2\theta-t)\left(t^s\left|f''\left(\frac{a+b}{2}\right)\right|^q+(1-t)^s\left|f''(b)\right|^q\right)dt\right.\\
&\quad\left.+\int_{2\theta}^1 t(t-2\theta)\left(t^s\left|f''\left(\frac{a+b}{2}\right)\right|^q+(1-t)^s\left|f''(b)\right|^q\right)dt\right)^{\frac{1}{q}}\Bigg]\\
&=\frac{(b-a)^2}{16}\left(\frac{8\theta^3-3\theta+1}{3}\right)^{1-\frac{1}{q}}\Bigg\{\Bigg[\frac{2(2\theta)^{s+3}-2(s+3)\theta+s+2}{(s+2)(s+3)}\left|f''\left(\frac{a+b}{2}\right)\right|^q\\
&\quad+\frac{4(1-2\theta)^{s+2}[(s+1)\theta+1]+2(s+3)\theta-2}{(s+1)(s+2)(s+3)}\left|f''(a)\right|^q\Bigg]^{\frac{1}{q}}\\
&\quad+\Bigg[\frac{2(2\theta)^{s+3}-2(s+3)\theta+s+2}{(s+2)(s+3)}\left|f''\left(\frac{a+b}{2}\right)\right|^q\\
&\quad+\frac{4(1-2\theta)^{s+2}[(s+1)\theta+1]+2(s+3)\theta-2}{(s+1)(s+2)(s+3)}\left|f''(b)\right|^q\Bigg]^{\frac{1}{q}}\Bigg\},
\end{aligned}
$$

where

$$
\begin{aligned}
&\int_0^1\left|t^2-2\theta t\right|dt=\int_0^{2\theta}t(2\theta-t)\,dt+\int_{2\theta}^1 t(t-2\theta)\,dt=\frac{8\theta^3-3\theta+1}{3},\\
&\int_0^{2\theta}t^{s+1}(2\theta-t)\,dt=\frac{(2\theta)^{s+3}}{(s+2)(s+3)},\\
&\int_{2\theta}^1 t^{s+1}(t-2\theta)\,dt=\frac{(2\theta)^{s+3}-2(s+3)\theta+s+2}{(s+2)(s+3)},\\
&\int_0^{2\theta}t(1-t)^s(2\theta-t)\,dt=\frac{(1-2\theta)^{s+2}[2(s+1)\theta+2]+2(s+3)\theta-2}{(s+1)(s+2)(s+3)},
\end{aligned}
$$

and

$$
\int_{2\theta}^1 t(1-t)^s(t-2\theta)\,dt=\frac{(1-2\theta)^{s+2}[2(s+1)\theta+2]}{(s+1)(s+2)(s+3)}.
$$

In case $\frac{1}{2}\le\theta\le 1$, by Lemma 1 and using the Hölder inequality, we have

$$
\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,dt-(1-\theta)f\left(\frac{a+b}{2}\right)-\theta\frac{f(a)+f(b)}{2}\right|\\
&\quad\le\frac{(b-a)^2}{16}\Bigg[\int_0^1\left|t^2-2\theta t\right|\left|f''\left(t\frac{a+b}{2}+(1-t)a\right)\right|dt\\
&\quad+\int_0^1\left|t^2-2\theta t\right|\left|f''\left(t\frac{a+b}{2}+(1-t)b\right)\right|dt\Bigg]
\end{aligned}
$$

$$\begin{aligned}
&\leq \frac{(b-a)^2}{16}\biggl[\biggl(\int_0^1 \bigl|t^2-2\theta t\bigr|\,dt\biggr)^{1-\frac{1}{q}}\biggl(\int_0^1 \bigl|t^2-2\theta t\bigr|\biggl|f''\biggl(t\frac{a+b}{2}+(1-t)a\biggr)\biggr|^q\,dt\biggr)^{\frac{1}{q}} \\
&\quad + \biggl(\int_0^1 \bigl|t^2-2\theta t\bigr|\,dt\biggr)^{1-\frac{1}{q}}\biggl(\int_0^1 \bigl|t^2-2\theta t\bigr|\biggl|f''\biggl(t\frac{a+b}{2}+(1-t)b\biggr)\biggr|^q\,dt\biggr)^{\frac{1}{q}}\biggr] \\
&\leq \frac{(b-a)^2}{16}\biggl\{\biggl(\theta-\frac{1}{3}\biggr)^{1-\frac{1}{q}}\biggl(\int_0^1 \bigl|t^2-2\theta t\bigr|\biggl[t^s\biggl|f''\biggl(\frac{a+b}{2}\biggr)\biggr|^q+(1-t)^s\bigl|f''(a)\bigr|^q\biggr]\,dt\biggr)^{\frac{1}{q}} \\
&\quad + \biggl(\theta-\frac{1}{3}\biggr)^{1-\frac{1}{q}}\biggl(\int_0^1 \bigl|t^2-2\theta t\bigr|\biggl[t^s\biggl|f''\biggl(\frac{a+b}{2}\biggr)\biggr|^q+(1-t)^s\bigl|f''(b)\bigr|^q\biggr]\,dt\biggr)^{\frac{1}{q}}\biggr\} \\
&= \frac{(b-a)^2}{16}\biggl(\theta-\frac{1}{3}\biggr)^{1-\frac{1}{q}}\biggl\{\biggl[\biggl(\int_0^1 t^{s+1}(2\theta-t)\,dt\biggr)\biggl|f''\biggl(\frac{a+b}{2}\biggr)\biggr|^q \\
&\quad + \biggl(\int_0^1 t(1-t)^s(2\theta-t)\,dt\biggr)\bigl|f''(a)\bigr|^q\biggr]^{\frac{1}{q}} \\
&\quad + \biggl[\biggl(\int_0^1 t^{s+1}(2\theta-t)\,dt\biggr)\biggl|f''\biggl(\frac{a+b}{2}\biggr)\biggr|^q + \biggl(\int_0^1 t(1-t)^s(2\theta-t)\,dt\biggr)\bigl|f''(b)\bigr|^q\biggr]^{\frac{1}{q}}\biggr\} \\
&= \frac{(b-a)^2}{16}\biggl(\theta-\frac{1}{3}\biggr)^{1-\frac{1}{q}}\biggl\{\biggl[\frac{2(s+3)\theta-s-2}{(s+2)(s+3)}\biggl|f''\biggl(\frac{a+b}{2}\biggr)\biggr|^q \\
&\quad + \frac{2(s+3)\theta-2}{(s+1)(s+2)(s+3)}\bigl|f''(a)\bigr|^q\biggr]^{\frac{1}{q}} + \biggl[\frac{2(s+3)\theta-s-2}{(s+2)(s+3)}\biggl|f''\biggl(\frac{a+b}{2}\biggr)\biggr|^q \\
&\quad + \frac{2(s+3)\theta-2}{(s+1)(s+2)(s+3)}\bigl|f''(b)\bigr|^q\biggr]^{\frac{1}{q}}\biggr\},
\end{aligned}$$

where

$$\begin{aligned}
&\int_0^1 \bigl|t^2-2\theta t\bigr|\,dt = \int_0^1 t(2\theta-t)\,dt = \theta-\frac{1}{3}, \\
&\int_0^1 t^{s+1}(2\theta-t)\,dt = \frac{2(s+3)\theta-s-2}{(s+2)(s+3)},
\end{aligned}$$

and

$$\int_0^1 t(1-t)^s(2\theta-t)\,dt = \frac{2(s+3)\theta-2}{(s+1)(s+2)(s+3)}.$$

The proof is thus completed. □

Remark 1 If we take $\theta = 0$ in (7), then we get a midpoint type inequality

$$\begin{aligned}
&\biggl|\frac{1}{b-a}\int_a^b f(t)\,dt - f\biggl(\frac{a+b}{2}\biggr)\biggr| \\
&\quad\leq \frac{(b-a)^2}{16}\biggl(\frac{1}{3}\biggr)^{1-\frac{1}{q}}\biggl\{\biggl[\frac{1}{s+3}\biggl|f''\biggl(\frac{a+b}{2}\biggr)\biggr|^q + \frac{2}{(s+1)(s+2)(s+3)}\bigl|f''(a)\bigr|^q\biggr]^{\frac{1}{q}} \\
&\qquad + \biggl[\frac{1}{s+3}\biggl|f''\biggl(\frac{a+b}{2}\biggr)\biggr|^q + \frac{2}{(s+1)(s+2)(s+3)}\bigl|f''(b)\bigr|^q\biggr]^{\frac{1}{q}}\biggr\}. \qquad (9)
\end{aligned}$$

If we take $\theta = 1$ in (8), then we get a trapezoid type inequality

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}\right| \\
&\quad\le \frac{(b-a)^2}{16}\left(\frac{2}{3}\right)^{1-\frac{1}{q}}\left\{\left[\frac{s+4}{(s+2)(s+3)}\left|f''\left(\frac{a+b}{2}\right)\right|^q + \frac{2}{(s+1)(s+3)}\left|f''(a)\right|^q\right]^{\frac{1}{q}}\right. \\
&\qquad\left. + \left[\frac{s+4}{(s+2)(s+3)}\left|f''\left(\frac{a+b}{2}\right)\right|^q + \frac{2}{(s+1)(s+3)}\left|f''(b)\right|^q\right]^{\frac{1}{q}}\right\}. \qquad (10)
\end{aligned}$$

If we take $\theta = \frac{1}{3}$ in (7), then we get a Simpson type inequality

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \\
&\quad\le \frac{(b-a)^2}{16}\left(\frac{8}{81}\right)^{1-\frac{1}{q}}\left\{\left[\left(\frac{s}{3(s+2)(s+3)} + \frac{2}{(s+2)(s+3)}\left(\frac{2}{s+3}\right)^{s+3}\right)\left|f''\left(\frac{a+b}{2}\right)\right|^q\right.\right. \\
&\qquad\left. + \left(\frac{4s+16}{(s+1)(s+2)(s+3)}\left(\frac{1}{3}\right)^{s+3} + \frac{2s}{3(s+1)(s+2)(s+3)}\right)\left|f''(a)\right|^q\right]^{\frac{1}{q}} \\
&\qquad\times\left[\left(\frac{s}{3(s+2)(s+3)} + \frac{2}{(s+2)(s+3)}\left(\frac{2}{s+3}\right)^{s+3}\right)\left|f''\left(\frac{a+b}{2}\right)\right|^q\right. \\
&\qquad\left.\left. + \left(\frac{4s+16}{(s+1)(s+2)(s+3)}\left(\frac{1}{3}\right)^{s+3} + \frac{2s}{3(s+1)(s+2)(s+3)}\right)\left|f''(b)\right|^q\right]^{\frac{1}{q}}\right\}. \qquad (11)
\end{aligned}$$

If we take $\theta = \frac{1}{2}$ in (7) or (8), then we get an averaged midpoint-trapezoid type inequality

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \\
&\quad\le \frac{(b-a)^2}{16}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left\{\left[\frac{1}{(s+2)(s+3)}\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q + \left|f''(a)\right|^q\right)\right]^{\frac{1}{q}}\right. \\
&\qquad\left. + \left[\frac{1}{(s+2)(s+3)}\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q + \left|f''(b)\right|^q\right)\right]^{\frac{1}{q}}\right\}. \qquad (12)
\end{aligned}$$

Corollary 1 *Let $I \subset [0,\infty)$, $f : I \to \mathbf{R}$ be a twice differentiable function on I° such that $f'' \in L^1[a,b]$, where $a, b \in I$ with $a < b$. If $|f''|$ is s-convex in the second sense on $[a,b]$ for some fixed $s \in (0,1]$, then the following inequalities hold*:

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,dx - (1-\theta)f\left(\frac{a+b}{2}\right) - \theta\frac{f(a)+f(b)}{2}\right| \\
&\quad\le \frac{(b-a)^2}{8}\left[\frac{2(2\theta)^{s+3} - 2(s+3)\theta + s + 2}{(s+2)(s+3)}\left|f''\left(\frac{a+b}{2}\right)\right|\right. \\
&\qquad\left. + \frac{2(1-2\theta)^{s+2}[(s+1)\theta + 1] + (s+3)\theta - 1}{(s+1)(s+2)(s+3)}\left(\left|f''(a)\right| + \left|f''(b)\right|\right)\right] \qquad (13)
\end{aligned}$$

for $0 \le \theta \le \frac{1}{2}$ *and*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)\,dx-(1-\theta)f\left(\frac{a+b}{2}\right)-\theta\frac{f(a)+f(b)}{2}\right|\\&\quad\le\frac{(b-a)^2}{8}\left[\frac{2(s+3)\theta-s-2}{(s+2)(s+3)}\left|f''\left(\frac{a+b}{2}\right)\right|\right.\\&\qquad\left.+\frac{(s+3)\theta-1}{(s+1)(s+2)(s+3)}\left(\left|f''(a)\right|+\left|f''(b)\right|\right)\right]\end{aligned}\tag{14}$$

for $\frac{1}{2} \le \theta \le 1$.

Proof Inequalities (13) and (14) are immediate by setting $q = 1$ in (7) and (8) of Theorem 3. □

Remark 2 If we take $\theta = 0$ in (13), then we get a midpoint type inequality

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)\,dt-f\left(\frac{a+b}{2}\right)\right|\\&\quad\le\frac{(b-a)^2}{8}\left[\frac{1}{s+3}\left|f''\left(\frac{a+b}{2}\right)\right|+\frac{1}{(s+1)(s+2)(s+3)}\left(\left|f''(a)\right|+\left|f''(b)\right|\right)\right].\end{aligned}\tag{15}$$

If we take $\theta = 1$ in (14), then we get a trapezoid type inequality

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{f(a)+f(b)}{2}\right|\\&\quad\le\frac{(b-a)^2}{8}\left[\frac{s+4}{(s+2)(s+3)}\left|f''\left(\frac{a+b}{2}\right)\right|+\frac{1}{(s+1)(s+3)}\left(\left|f''(a)\right|+\left|f''(b)\right|\right)\right].\end{aligned}\tag{16}$$

If we take $\theta = \frac{1}{3}$ in (13), then we get a Simpson type inequality

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|\\&\quad\le\frac{(b-a)^2}{8}\left\{\left[\frac{s}{3(s+2)(s+3)}+\frac{2}{(s+2)(s+3)}\left(\frac{2}{3}\right)^{s+3}\right]\left|f''\left(\frac{a+b}{2}\right)\right|\right.\\&\qquad\left.+\left[\frac{2s+8}{(s+1)(s+2)(s+3)}\left(\frac{1}{3}\right)^{s+3}+\frac{s}{3(s+1)(s+2)(s+3)}\right]\left(\left|f''(a)\right|+\left|f''(b)\right|\right)\right\}.\end{aligned}\tag{17}$$

If we take $\theta = \frac{1}{2}$ in (13) or (14), then we get an averaged midpoint-trapezoid type inequality

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{1}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]\right|\\&\quad\le\frac{(b-a)^2}{8}\left[\frac{1}{(s+2)(s+3)}\left|f''\left(\frac{a+b}{2}\right)\right|+\frac{1}{2(s+2)(s+3)}\right]\left(\left|f''(a)\right|+\left|f''(b)\right|\right).\end{aligned}\tag{18}$$

Remark 3 If we put $M = \sup_{x\in[a,b]} |f''|$ in (15)-(18), then we have

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)\right| \le \frac{M(s^2+3s+4)(b-a)^2}{8(s+1)(s+2)(s+3)}, \tag{19}$$

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}\right| \le \frac{M(s^2+7s+8)(b-a)^2}{8(s+1)(s+2)(s+3)}, \tag{20}$$

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \\ &\quad\le \frac{M(b-a)^2}{24(s+1)(s+2)(s+3)}\left[s^2+3s+(4s+16)\left(\frac{1}{3}\right)^{s+2}+6(s+1)\left(\frac{2}{3}\right)^{s+3}\right],\end{aligned} \tag{21}$$

and

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \le \frac{M(b-a)^2}{4(s+2)(s+3)}. \tag{22}$$

Remark 4 If we further take $s = 1$ in (19)-(22), *i.e.*, for functions f with convex $|f''|$, we have

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)\right| \le \frac{M(b-a)^2}{24}, \tag{23}$$

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}\right| \le \frac{M(b-a)^2}{12}, \tag{24}$$

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \le \frac{M(b-a)^2}{81}, \tag{25}$$

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \le \frac{M(b-a)^2}{48}. \tag{26}$$

Obviously, (23)-(26) indicate that the Simpson type inequality has the best error estimation for functions f with convex $|f''|$.

Now we turn to establish another new general inequality for functions whose second derivatives in absolute value at certain powers are s-convex in the second sense, we need the following lemma.

Lemma 2 *Let $I \subset \mathbf{R}$, $f : I \to \mathbf{R}$ be a twice differentiable function on I° such that $f'' \in L^1[a,b]$, where $a, b \in I$ with $a < b$. Then, for any $\theta \in [0,1]$, the following equality holds*:

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(t)\,dt - (1-\theta)f\left(\frac{a+b}{2}\right) - \theta\frac{f(a)+f(b)}{2} \\ &\quad= (b-a)^2\int_0^1 k(t)f''\bigl(ta+(1-t)b\bigr)\,dt,\end{aligned} \tag{27}$$

where

$$k(t) = \begin{cases}\frac{1}{2}t(t-\theta), & 0 \le t \le \frac{1}{2}, \\ \frac{1}{2}(1-t)(1-\theta-t), & \frac{1}{2} \le t \le 1.\end{cases}$$

Proof See, *e.g.*, Lemma 2 in [4]. □

Theorem 4 *Let $I \subset [0,\infty)$, $f : I \to \mathbf{R}$ be a twice differentiable function on I° such that $f'' \in L^1[a,b]$, where $a, b \in I$ with $a < b$. If $|f''|^q$ is s-convex in the second sense on $[a,b]$ for some fixed $s \in (0,1]$ and $q \geq 1$, then the following inequalities hold*:

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,dx - (1-\theta)f\left(\frac{a+b}{2}\right) - \theta\frac{f(a)+f(b)}{2}\right| \\
&\quad\leq \frac{(b-a)^2}{2}\left(\frac{8\theta^3-3\theta+1}{24}\right)^{1-\frac{1}{q}}\Biggl\{\biggl[\frac{2\theta^{s+3}}{(s+2)(s+3)} \\
&\qquad + \frac{2(1-\theta)^{s+2}[(s+1)\theta+2]+(s+3)\theta-2}{(s+1)(s+2)(s+3)} - \frac{1-\theta}{2^{s+1}(s+1)(s+2)}\biggr] \\
&\qquad \times \left(\left|f''(a)\right|^q + \left|f''(b)\right|^q\right)\Biggr\}^{\frac{1}{q}} \qquad (28)
\end{aligned}$$

for $0 \leq \theta \leq \frac{1}{2}$ *and*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,dx - (1-\theta)f\left(\frac{a+b}{2}\right) - \theta\frac{f(a)+f(b)}{2}\right| \\
&\quad\leq \frac{(b-a)^2}{2}\left(\frac{3\theta-1}{24}\right)^{1-\frac{1}{q}}\Biggl\{\biggl[\frac{(s+3)\theta-2}{(s+1)(s+2)(s+3)} + \frac{1-\theta}{2^{s+1}(s+1)(s+2)}\biggr] \\
&\qquad \times \left(\left|f''(a)\right|^q + \left|f''(b)\right|^q\right)\Biggr\}^{\frac{1}{q}} \qquad (29)
\end{aligned}$$

for $\frac{1}{2} \leq \theta \leq 1$.

Proof In case $0 \leq \theta \leq \frac{1}{2}$, by Lemma 2 and using the Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,dt - (1-\theta)f\left(\frac{a+b}{2}\right) - \theta\frac{f(a)+f(b)}{2}\right| \\
&\quad\leq (b-a)^2\int_0^1 \left|k(t)\right|\left|f''\bigl(ta+(1-t)b\bigr)\right|dt \\
&\quad\leq (b-a)^2\left[\int_0^1 \left|k(t)\right|dt\right]^{1-\frac{1}{q}}\left[\int_0^1 \left|k(t)\right|\left|f''\bigl(ta+(1-t)b\bigr)\right|^q dt\right]^{\frac{1}{q}} \\
&\quad\leq (b-a)^2\left(\frac{8\theta^3-3\theta+1}{24}\right)^{1-\frac{1}{q}}\left(\int_0^1 \left|k(t)\right|\left[t^s\left|f''(a)\right|^q + (1-t)^s\left|f''(b)\right|^q\right]dt\right)^{\frac{1}{q}} \\
&\quad= \frac{(b-a)}{2}\left(\frac{8\theta^3-3\theta+1}{24}\right)^{1-\frac{1}{q}} \\
&\qquad\times\biggl[\left(\int_0^{\frac{1}{2}} t^{s+1}|t-\theta|\,dt + \int_{\frac{1}{2}}^1 t^s(1-t)|1-\theta-t|\,dt\right)\left|f''(a)\right|^q \\
&\qquad + \left(\int_0^{\frac{1}{2}} t(1-t)^s|t-\theta|\,dt + \int_{\frac{1}{2}}^1 (1-t)^{s+1}|1-\theta-t|\,dt\right)\left|f''(b)\right|^q\biggr]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(b-a)}{2}\left(\frac{8\theta^3-3\theta+1}{24}\right)^{1-\frac{1}{q}}\biggl[\biggl(\int_0^{\theta} t^{s+1}(\theta-t)\,dt+\int_{\theta}^{\frac{1}{2}} t^{s+1}(t-\theta)\,dt \\
&\quad+\int_{\frac{1}{2}}^{1-\theta} t^s(1-t)(1-\theta-t)\,dt+\int_{1-\theta}^{1} t^s(1-t)(\theta+t-1)\,dt\biggr)\bigl|f''(a)\bigr|^q \\
&\quad+\biggl(\int_0^{\theta} t(1-t)^s(\theta-t)\,dt+\int_{\theta}^{\frac{1}{2}} t(1-t)^s(t-\theta)\,dt \\
&\quad+\int_{\frac{1}{2}}^{1-\theta} (1-t)^{s+1}(1-\theta-t)\,dt+\int_{1-\theta}^{1} (1-t)^{s+1}(\theta+t-1)\,dt\biggr)\bigl|f''(b)\bigr|^q\biggr] \\
&= \frac{(b-a)^2}{2}\left(\frac{8\theta^3-3\theta+1}{24}\right)^{1-\frac{1}{q}}\biggl\{\biggl[\frac{2\theta^{s+3}}{(s+2)(s+3)} \\
&\quad+\frac{2(1-\theta)^{s+2}[(s+1)\theta+2]+(s+3)\theta-2}{(s+1)(s+2)(s+3)}-\frac{1-\theta}{2^{s+1}(s+1)(s+2)}\biggr] \\
&\quad\times\bigl(\bigl|f''(a)\bigr|^q+\bigl|f''(b)\bigr|^q\bigr)\biggr\}^{\frac{1}{q}},
\end{aligned}
$$

where

$$
\begin{aligned}
\int_0^1\bigl|k(t)\bigr|\,dt &= \int_0^{\frac{1}{2}}\left|\frac{1}{2}t(t-\theta)\right|dt+\int_{\frac{1}{2}}^{1}\left|\frac{1}{2}(1-t)(1-\theta-t)\right|dt \\
&= \frac{1}{2}\biggl[\int_0^{\theta} t(\theta-t)\,dt+\int_{\theta}^{\frac{1}{2}} t(t-\theta)\,dt \\
&\quad+\int_{\frac{1}{2}}^{1-\theta}(1-t)(1-\theta-t)\,dt+\int_{1-\theta}^{1}(1-t)(t-1+\theta)\,dt\biggr] \\
&= \frac{8\theta^3-3\theta+1}{24}, \\
\int_0^{\theta} t^{s+1}(\theta-t)\,dt &= \int_{1-\theta}^{1}(1-t)^{s+1}(\theta+t-1)\,dt=\frac{\theta^{s+3}}{(s+2)(s+3)}, \\
\int_0^{\theta} t(1-t)^s(\theta-t)\,dt &= \int_{1-\theta}^{1}(1-t)(\theta+t-1)t^s\,dt \\
&= \frac{(1-\theta)^{s+2}[(s+1)\theta+2]+(s+3)\theta-2}{(s+1)(s+2)(s+3)}, \\
\int_{\theta}^{\frac{1}{2}} t^{s+1}(t-\theta)\,dt &= \int_{\frac{1}{2}}^{1-\theta}(1-t)^{s+1}(1-\theta-t)\,dt \\
&= \frac{(2\theta)^{s+3}-2(s+3)\theta+s+2}{2^{s+3}(s+2)(s+3)},
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{\theta}^{\frac{1}{2}} t(1-t)^s(t-\theta)\,dt &= \int_{\frac{1}{2}}^{1-\theta}(1-t)(1-\theta-t)t^s\,dt \\
&= \frac{(1-\theta)^{s+2}[(s+1)\theta+2]}{(s+1)(s+2)(s+3)}+\frac{2(s+3)^2\theta-s^2-7s-14}{2^{s+3}(s+1)(s+2)(s+3)}.
\end{aligned}
$$

In case $0 \le \theta \le \frac{1}{2}$, by Lemma 2 and using the Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,dt-(1-\theta)f\left(\frac{a+b}{2}\right)-\theta\frac{f(a)+f(b)}{2}\right| \\
&\quad\le (b-a)^2\int_0^1 \left|k(t)\right|\left|f''\big(ta+(1-t)b\big)\right|dt \\
&\quad\le (b-a)^2\left[\int_0^1\left|k(t)\right|dt\right]^{1-\frac{1}{q}}\left[\int_0^1\left|k(t)\right|\left|f''\big(ta+(1-t)b\big)\right|^q dt\right]^{\frac{1}{q}} \\
&\quad\le (b-a)^2\left(\frac{3\theta-1}{24}\right)^{1-\frac{1}{q}}\left(\int_0^1\left|k(t)\right|\left[t^s\left|f''(a)\right|^q++(1-t)^s\left|f''(b)\right|^q\right]dt\right)^{\frac{1}{q}} \\
&\quad=\frac{(b-a)}{2}\left(\frac{3\theta-1}{24}\right)^{1-\frac{1}{q}}\left[\left(\int_0^{\frac{1}{2}} t^{s+1}|t-\theta|\,dt+\int_{\frac{1}{2}}^1 t^s(1-t)|1-\theta-t|\,dt\right)\left|f''(a)\right|^q\right. \\
&\qquad\left.+\left(\int_0^{\frac{1}{2}} t(1-t)^s|t-\theta|\,dt+\int_{\frac{1}{2}}^1(1-t)^{s+1}|1-\theta-t|\,dt\right)\left|f''(b)\right|^q\right] \\
&\quad=\frac{(b-a)}{2}\left(\frac{3\theta-1}{24}\right)^{1-\frac{1}{q}}\left[\left(\int_0^{\frac{1}{2}} t^{s+1}(\theta-t)\,dt+\int_{\frac{1}{2}}^1 t^s(1-t)(\theta+t-1)\,dt\right)\left|f''(a)\right|^q\right. \\
&\qquad\left.+\left(\int_0^{\frac{1}{2}} t(1-t)^s(\theta-t)\,dt+\int_{\frac{1}{2}}^1(1-t)^{s+1}(\theta+t-1)\,dt\right)\left|f''(b)\right|^q\right] \\
&\quad=\frac{(b-a)^2}{2}\left(\frac{3\theta-1}{24}\right)^{1-\frac{1}{q}}\left\{\left[\frac{(s+3)\theta-2}{(s+1)(s+2)(s+3)}+\frac{1-\theta}{2^{s+1}(s+1)(s+2)}\right]\right. \\
&\qquad\left.\times\left(\left|f''(a)\right|^q+\left|f''(b)\right|^q\right)\right\}^{\frac{1}{q}},
\end{aligned}$$

where

$$\begin{aligned}
\int_0^1\left|k(t)\right|dt&=\int_0^{\frac{1}{2}}\left|\frac{1}{2}t(t-\theta)\right|dt+\int_{\frac{1}{2}}^1\left|\frac{1}{2}(1-t)(1-\theta-t)\right|dt \\
&=\frac{1}{2}\left[\int_0^{\frac{1}{2}} t(\theta-t)\,dt+\int_{\frac{1}{2}}^1(1-t)(\theta+t-1)\,dt\right] \\
&=\frac{3\theta-1}{24},
\end{aligned}$$

$$\int_0^{\frac{1}{2}} t^{s+1}(\theta-t)\,dt=\int_{\frac{1}{2}}^1(1-t)^{s+1}(\theta+t-1)\,dt=\frac{(2s+6)\theta-s-2}{2^{s+3}(s+2)(s+3)},$$

and

$$\begin{aligned}
\int_0^{\frac{1}{2}} t(1-t)^s(\theta-t)\,dt&=\int_{\frac{1}{2}}^1(1-t)(\theta+t-1)t^s\,dt \\
&=\frac{2^{s+3}[(s+3)\theta-2]-2(s+3)^2\theta+s^2+7s+14}{2^{s+3}(s+1)(s+2)(s+3)}.
\end{aligned}$$

The proof is thus completed. □

Remark 5 If we take $\theta = 0$ in (28), then we get a midpoint type inequality

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)\right|\\&\quad\le \frac{(b-a)^2}{2}\left(\frac{1}{24}\right)^{1-\frac{1}{q}}\left[\frac{2^{s+2}-s-3}{2^{s+1}(s+1)(s+2)(s+3)}\left(\left|f''(a)\right|^q+\left|f''(b)\right|^q\right)\right]^{\frac{1}{q}}.\end{aligned} \tag{30}$$

If we take $\theta = 1$ in (29), then we get a trapezoid type inequality

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}\right|\\&\quad\le \frac{(b-a)^2}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left[\frac{|f''(a)|^q+|f''(b)|^q}{(s+2)(s+3)}\right]^{\frac{1}{q}}.\end{aligned} \tag{31}$$

If we take $\theta = \frac{1}{3}$ in (28), then we get a Simpson type inequality

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(a)+4f(\frac{a+b}{2})+f(b)}{6}\right|\\&\quad\le \frac{(b-a)^2}{2}\left(\frac{1}{81}\right)^{1-\frac{1}{q}}\\&\qquad\times\left[\frac{2^{s+4}(s+1)+2^{2s+6}(s+7)-6^{s+2}(6-2s)-8(s+3)3^{s+2}}{6^{s+3}(s+1)(s+2)(s+3)}\left(\left|f''(a)\right|^q+\left|f''(b)\right|^q\right)\right]^{\frac{1}{q}}.\end{aligned} \tag{32}$$

If we take $\theta = \frac{1}{2}$ in (28) or (29), then we get an averaged midpoint-trapezoid type inequality

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]\right|\\&\quad\le \frac{(b-a)^2}{2}\left(\frac{1}{48}\right)^{1-\frac{1}{q}}\left[\frac{2^{s+1}(s-1)+s+3}{2^{s+2}(s+1)(s+2)(s+3)}\left(\left|f''(a)\right|^q+\left|f''(b)\right|^q\right)\right]^{\frac{1}{q}}.\end{aligned} \tag{33}$$

Corollary 2 *Let $I \subset [0,\infty)$, $f : I \to \mathbf{R}$ be a twice differentiable function on I° such that $f'' \in L^1[a,b]$, where $a, b \in I$ with $a < b$. If $|f''|$ is s-convex in the second sense on $[a,b]$ for some fixed $s \in (0,1]$, then the following inequalities hold*:

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)\,dx - (1-\theta)f\left(\frac{a+b}{2}\right) - \theta\frac{f(a)+f(b)}{2}\right|\\&\quad\le \frac{(b-a)^2}{2}\left[\frac{2\theta^{s+3}}{(s+2)(s+3)}\right.\\&\qquad\left.+\frac{2(1-\theta)^{s+2}((s+1)\theta+2)+(s+3)\theta-2}{(s+1)(s+2)(s+3)}-\frac{1-\theta}{2^{s+1}(s+1)(s+2)}\right]\left(\left|f''(a)\right|+\left|f''(b)\right|\right)\end{aligned} \tag{34}$$

for $0 \leq \theta \leq \frac{1}{2}$ *and*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - (1-\theta)f\left(\frac{a+b}{2}\right) - \theta\frac{f(a)+f(b)}{2}\right| \\ \leq \frac{(b-a)^2}{2}\left[\frac{(s+3)\theta-2}{(s+1)(s+2)(s+3)} + \frac{1-\theta}{2^{s+1}(s+1)(s+2)}\right]\left(\left|f''(a)\right| + \left|f''(b)\right|\right) \tag{35}$$

for $\frac{1}{2} \leq \theta \leq 1$.

Proof Inequalities (34) and (35) are immediate by setting $q = 1$ in (28) and (29) of Theorem 4. □

Remark 6 If we take $\theta = 0$ in (34), then we get a midpoint type inequality

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)\right| \leq \frac{2^{s+2}-s-3}{2^{s+2}(s+1)(s+2)(s+3)}(b-a)^2\left(\left|f''(a)\right| + \left|f''(b)\right|\right). \tag{36}$$

If we take $\theta = 1$ in (35), then we get a trapezoid type inequality

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}\right| \leq \frac{1}{2(s+2)(s+3)}(b-a)^2\left(\left|f''(a)\right| + \left|f''(b)\right|\right). \tag{37}$$

If we take $\theta = \frac{1}{3}$ in (34), then we get a Simpson type inequality

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \\ \leq \frac{(s+1)2^{s+3} + (s+7)2^{2s+5} - (3-s)6^{s+2} - 4(s+3)3^{s+2}}{6^{s+3}(s+1)(s+2)(s+3)} \\ \times (b-a)^2\left(\left|f''(a)\right| + \left|f''(b)\right|\right). \tag{38}$$

If we take $\theta = \frac{1}{2}$ in (34) or (35), then we get an averaged midpoint-trapezoid type inequality

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \\ \leq \frac{s+3-2^{s+1}(1-s)}{2^{s+3}(s+1)(s+2)(s+3)}(b-a)^2\left(\left|f''(a)\right| + \left|f''(b)\right|\right). \tag{39}$$

Remark 7 If we put $M = \sup_{x\in[a,b]} |f''|$ in (36)-(39), then we have

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)\right| \leq \frac{M(2^{s+2}-s-3)(b-a)^2}{2^{s+1}(s+1)(s+2)(s+3)}, \tag{40}$$

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}\right| \leq \frac{M(b-a)^2}{(s+2)(s+3)}, \tag{41}$$

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \\ \leq \frac{(s+1)2^{s+3} + (s+7)2^{2s+5} - (3-s)6^{s+2} - 4(s+3)3^{s+2}}{3(s+1)(s+2)(s+3)6^{s+2}}M(b-a)^2 \tag{42}$$

and

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right]\right|$$
$$\leq \frac{s+3-2^{s+1}(1-s)}{2^{s+2}(s+1)(s+2)(s+3)} M(b-a)^2. \tag{43}$$

Remark 8 If we further take $s = 1$ in (40)-(43), *i.e.*, for functions f with convex $|f''|$, then we recapture inequalities (23)-(26).

3 Applications to special means

We consider the means for arbitrary positive numbers α, β ($\alpha \neq \beta$) as follows:

(1) The arithmetic mean

$$A(\alpha,\beta) = \frac{\alpha+\beta}{2}.$$

(2) The geometric mean

$$G(\alpha,\beta) = \sqrt{\alpha\beta}.$$

(3) The harmonic mean

$$H(\alpha,\beta) = \frac{2\alpha\beta}{\alpha+\beta}.$$

(4) The logarithmic mean

$$L(\alpha,\beta) = \frac{\beta-\alpha}{\ln\beta - \ln\alpha}.$$

(5) The generalized log-mean

$$L_p(\alpha,\beta) = \left[\frac{\beta^{p+1}-\alpha^{p+1}}{(p+1)(\beta-\alpha)}\right]^{\frac{1}{p}}, \quad p \neq -1, 0.$$

(6) The identric mean

$$I(\alpha,\beta) = \frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right)\frac{1}{\beta-\alpha}.$$

Proposition 1 *Let* $0 < a < b$ *and* $s \in (0,1)$. *Then we have*

$$\left|A^s(a,b) - L_s^s(a,b)\right| \leq \frac{s(1-s)(b-a)^2}{24a^{2-s}},$$
$$\left|A\left(a^s,b^s\right) - L_s^s(a,b)\right| \leq \frac{s(1-s)(b-a)^2}{12a^{2-s}},$$
$$\left|\frac{2A^s(a,b) + A(a^s,b^s)}{3} - L_s^s(a,b)\right| \leq \frac{s(1-s)(b-a)^2}{81a^{2-s}},$$

and

$$\left|\frac{A^s(a,b)+A(a^s,b^s)}{2}-L_s^s(a,b)\right|\le\frac{s(1-s)(b-a)^2}{48a^{2-s}}.$$

Proof The assertion follows from applying inequalities (23)-(26) to the mapping $f(x)=x^s$, $x\in[a,b]$, which implies that $|f''(x)|=s(1-s)x^{s-2}$ is convex on $[a,b]$, and we may take $M=\frac{s(1-s)}{a^{2-s}}$. □

Proposition 2 *Let* $0<a<b$. *Then we have*

$$\left|A^{-1}(a,b)-L^{-1}(a,b)\right|\le\frac{(b-a)^2}{12a^3},$$
$$\left|H^{-1}(a,b)-L^{-1}(a,b)\right|\le\frac{(b-a)^2}{6a^3},$$
$$\left|\frac{2A^{-1}(a,b)+H^{-1}(a,b)}{3}-L^{-1}(a,b)\right|\le\frac{2(b-a)^2}{81a^3}$$

and

$$\left|\frac{A^{-1}(a,b)+H^{-1}(a,b)}{2}-L^{-1}(a,b)\right|\le\frac{(b-a)^2}{24a^3}.$$

Proof The assertion follows from applying inequalities (23)-(26) to the mapping $f(x)=\frac{1}{x}$, $x\in[a,b]$, which implies that $|f''(x)|=\frac{2}{x^3}$ is convex on $[a,b]$, and we may take $M=\frac{2}{a^3}$. □

Proposition 3 *Let* $a,b\in\mathbf{R}$, $0<a<b$. *Then we have*

$$\left|\ln A(a,b)-\ln I(a,b)\right|\le\frac{(b-a)^2}{24a^2},$$
$$\left|\ln G(a,b)-\ln I(a,b)\right|\le\frac{(b-a)^2}{12a^2},$$
$$\left|\frac{2\ln A(a,b)+\ln G(a,b)}{3}-\ln I(a,b)\right|\le\frac{(b-a)^2}{81a^2}$$

and

$$\left|\frac{\ln A(a,b)+\ln G(a,b)}{2}-\ln I(a,b)\right|\le\frac{(b-a)^2}{48a^2}.$$

Proof The assertion follows from applying inequalities (23)-(26) to the mapping $f(x)=\ln x$, $x\in[a,b]$, which implies that $|f''(x)|=\frac{1}{x^2}$ is convex on $[a,b]$, and we may take $M=\frac{1}{a^2}$. □

Competing interests
The author declares that he has no competing interests.

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