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A Hilbert-type integral inequality in the whole plane with a non-homogeneous kernel and a few parameters

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Abstract

By using the way of real analysis and estimating the weight functions, we build a new Hilbert-type integral inequality in the whole plane with a non-homogeneous kernel and a few parameters. The constant factor related to the beta function is proved to be the best possible. We also consider the equivalent forms, the reverses, and some particular cases.

MSC: 26D15

Keywords: Hilbert-type integral inequality; weight function; equivalent form; beta function; reverse

1 Introduction

If $f(x), g(y) \geq 0$, satisfying $0 < \int_0^\infty f^2(x)\,dx < \infty$ and $0 < \int_0^\infty g^2(y)\,dy < \infty$, then we have (*cf.* [1])

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy < \pi \left(\int_0^\infty f^2(x)\,dx \int_0^\infty g^2(y)\,dy \right)^{\frac{1}{2}}, \tag{1}$$

where the constant factor π is the best possible. Inequality (1) is known as Hilbert's integral inequality, which is important in analysis and its applications (*cf.* [1, 2]).

In recent years, by using the way of weight functions, a number of extensions of (1) were given by Yang (*cf.* [3]). Noticing that inequality (1) is a homogeneous kernel of degree –1, in 2009, A survey of the study of Hilbert-type inequalities with the homogeneous kernels of degree negative numbers and some parameters is given by [4]. Recently, some inequalities with the homogeneous kernels of degree 0 and non-homogeneous kernels have been studied (*cf.* [5–10]). All of the above integral inequalities are built in the quarter plane.

In 2007, Yang [11] first gave a Hilbert-type integral inequality in the whole plane as follows:

$$\int_{-\infty}^\infty \int_{-\infty}^\infty \frac{f(x)g(y)}{(1+e^{x+y})^\lambda}\,dx\,dy < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left(\int_{-\infty}^\infty e^{-\lambda x} f^2(x)\,dx \int_{-\infty}^\infty e^{-\lambda y} g^2(y)\,dy \right)^{\frac{1}{2}}, \tag{2}$$

where the constant factor $B(\frac{\lambda}{2},\frac{\lambda}{2})$ ($\lambda > 0$) is the best possible, and

$$B(u,v) := \int_0^\infty \frac{t^{u+1}}{(1+t)^{u+v}}\,dt \quad (u,v>0) \tag{3}$$

is the beta function (*cf.* [12]). He *et al.* [13–24] also provided some Hilbert-type integral inequalities in the whole plane.

In this paper, by using the way of real analysis and estimating the weight functions, we build a new Hilbert-type integral inequality in the whole plane with the non-homogeneous kernel and a few parameters. The constant factor related to the beta function is proved to be the best possible. We also consider the equivalent forms, the reverses, and some particular cases.

2 Some lemmas

Lemma 1 *Suppose that* $0 < \alpha_1 \le \alpha_2 < \pi$, $\mu,\sigma > 0$, $\mu+\sigma=\lambda$, $\gamma \in \{\frac{1}{2k+1}, 2k-1\ (k\in\mathbf{N})\}$, $\delta \in \{-1,1\}$. *We define weight functions* $\omega(\sigma,y)$ ($y\in\mathbf{R}$), *and* $\varpi(\sigma,x)$ ($x\in\mathbf{R}$) *as follows*:

$$\omega(\sigma,y) := \int_{-\infty}^\infty \min_{i\in\{1,2\}} \frac{|y|^\sigma |x|^{\delta\sigma-1}}{[|x^\delta y|^\gamma + (x^\delta y)^\gamma \cos\alpha_i + 1]^{\lambda/\gamma}}\,dx, \tag{4}$$

$$\varpi(\sigma,x) := \int_{-\infty}^\infty \min_{i\in\{1,2\}} \frac{|x|^{\delta\sigma} |y|^{\sigma-1}}{[|x^\delta y|^\gamma + (x^\delta y)^\gamma \cos\alpha_i + 1]^{\lambda/\gamma}}\,dy. \tag{5}$$

Then for $y,x\in\mathbf{R}\backslash\{0\}$, *we have*

$$\begin{aligned}\omega(\sigma,y) &= \varpi(\sigma,x) = K(\sigma)\\ &:= \frac{1}{\gamma 2^{\sigma/\gamma}}\left[\left(\sec\frac{\alpha_1}{2}\right)^{\frac{2\sigma}{\gamma}} + \left(\csc\frac{\alpha_2}{2}\right)^{\frac{2\sigma}{\gamma}}\right] B\left(\frac{\mu}{\gamma},\frac{\sigma}{\gamma}\right) \in \mathbf{R}_+. \end{aligned}\tag{6}$$

Proof (i) For $\delta=1$, $y\in\mathbf{R}\backslash\{0\}$, setting $u=xy$, we find

$$\begin{aligned}\omega(\sigma,y) &= \int_{-\infty}^\infty \min_{i\in\{1,2\}} \frac{1}{(|u|^\gamma + u^\gamma\cos\alpha_i + 1)^{\lambda/\gamma}} |u|^{\sigma-1}\,du\\ &= \int_0^\infty \min_{i\in\{1,2\}} \frac{1}{[u^\gamma(1+\cos\alpha_i)+1]^{\lambda/\gamma}} u^{\sigma-1}\,du\\ &\quad + \int_{-\infty}^0 \min_{i\in\{1,2\}} \frac{1}{[u^\gamma(-1+\cos\alpha_i)+1]^{\lambda/\gamma}} (-u)^{\sigma-1}\,du\\ &= \int_0^\infty \min_{i\in\{1,2\}} \frac{1}{[u^\gamma(1+\cos\alpha_i)+1]^{\lambda/\gamma}} u^{\sigma-1}\,du\\ &\quad + \int_0^\infty \min_{i\in\{1,2\}} \frac{1}{[v^\gamma(1-\cos\alpha_i)+1]^{\lambda/\gamma}} v^{\sigma-1}\,dv\\ &= \int_0^\infty \frac{u^{\sigma-1}\,du}{[u^\gamma(1+\cos\alpha_1)+1]^{\lambda/\gamma}} + \int_0^\infty \frac{v^{\sigma-1}\,dv}{[v^\gamma(1-\cos\alpha_2)+1]^{\lambda/\gamma}}. \end{aligned}\tag{7}$$

Setting $t=u^\gamma(1+\cos\alpha_1)$ ($t=u^\gamma(1-\cos\alpha_2)$) in the above first (second) integral, by (3), it follows that

$$\omega(\sigma,y) = \frac{1}{\gamma}\left[\left(\frac{\sec^2\frac{\alpha_1}{2}}{2}\right)^{\frac{\sigma}{\gamma}} + \left(\frac{\csc^2\frac{\alpha_2}{2}}{2}\right)^{\frac{\sigma}{\gamma}}\right]\int_0^\infty \frac{t^{\frac{\sigma}{\gamma}-1}\,dt}{(t+1)^{\lambda/\gamma}} = K(\sigma).$$

(ii) For $\delta = -1$, setting $\frac{y}{x}$, we still can obtain $\omega(\sigma, y) = K(\sigma)$.

Setting $u = x^{\delta} y$, we also find

$$\varpi(\sigma, x) = \int_{-\infty}^{\infty} \min_{i \in \{1,2\}} \frac{1}{(|u|^{\gamma} + u^{\gamma} \cos \alpha_i + 1)^{\lambda/\gamma}} |u|^{\sigma - 1} \, du = K(\sigma).$$

Hence we have (6). □

Note If we replace $\min_{i\in\{1,2\}}$ by $\max_{i\in\{1,2\}}$ in (4) and (5), then we may exchange α_1 and α_2 in (6).

Lemma 2 *Suppose that* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \alpha_1 \le \alpha_2 < \pi$, $\mu, \sigma > 0$, $\mu + \sigma = \lambda$, $\gamma \in \{\frac{1}{2k+1}, 2k - 1\ (k \in \mathbf{N})\}$, $\delta \in \{-1, 1\}$. *If* $K(\sigma)$ *is indicated by* (6), $f(x)$ *is a non-negative measurable function in* $(-\infty, \infty)$, *then we have*

$$\begin{aligned} J &:= \int_{-\infty}^{\infty} |y|^{p\sigma - 1} \left\{ \int_{-\infty}^{\infty} \min_{i \in \{1,2\}} \frac{1}{[|x^{\delta} y|^{\gamma} + (x^{\delta} y)^{\gamma} \cos \alpha_i + 1]^{\lambda/\gamma}} f(x) \, dx \right\}^p dy \\ &\le K^p(\sigma) \int_{-\infty}^{\infty} |x|^{p(1 - \delta\sigma) - 1} f^p(x) \, dx. \end{aligned} \tag{8}$$

Proof We set

$$k_{\lambda}^{(\delta)}(x, y) := \min_{i \in \{1,2\}} \frac{1}{[|x^{\delta} y|^{\gamma} + (x^{\delta} y)^{\gamma} \cos \alpha_i + 1]^{\lambda/\gamma}} \quad (x, y \in \mathbf{R}). \tag{9}$$

By Hölder's inequality (*cf.* [25]), we have

$$\begin{aligned} &\left(\int_{-\infty}^{\infty} k_{\lambda}^{(\delta)}(x, y) f(x) \, dx \right)^p \\ &\quad = \left\{ \int_{-\infty}^{\infty} k_{\lambda}^{(\delta)}(x, y) \left[\frac{|x|^{(1-\delta\sigma)/q}}{|y|^{(1-\sigma)/p}} f(x) \right] \left[\frac{|y|^{(1-\sigma)/p}}{|x|^{(1-\delta\sigma)/q}} \right] dx \right\}^p \\ &\quad \le \int_{-\infty}^{\infty} k_{\lambda}^{(\delta)}(x, y) \frac{|x|^{(1-\delta\sigma)(p-1)}}{|y|^{1-\sigma}} f^p(x) \, dx \\ &\qquad \times \left[\int_{-\infty}^{\infty} k_{\lambda}^{(\delta)}(x, y) \frac{|y|^{(1-\sigma)(q-1)}}{|x|^{1-\delta\sigma}} \, dx \right]^{p-1} \\ &\quad = \big(\omega(\sigma, y) \big)^{p-1} |y|^{-p\sigma + 1} \int_{-\infty}^{\infty} k_{\lambda}^{(\delta)}(x, y) \frac{|x|^{(1-\delta\sigma)(p-1)}}{|y|^{(1-\sigma)}} f^p(x) \, dx. \end{aligned} \tag{10}$$

Then by (6) and the Fubini theorem (*cf.* [26]), it follows that

$$\begin{aligned} J &\le K^{p-1}(\sigma) \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} k_{\lambda}^{(\delta)}(x, y) \frac{|x|^{(1-\delta\sigma)(p-1)}}{|y|^{(1-\sigma)}} f^p(x) \, dx \right] dy \\ &= K^{p-1}(\sigma) \int_{-\infty}^{\infty} \varpi(\sigma, x) |x|^{p(1-\delta\sigma)-1} f^p(x) \, dx. \end{aligned}$$

Hence, still in view of (6), inequality (8) follows. □

3 Main results and applications

Theorem 1 *Suppose that* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \alpha_1 \le \alpha_2 < \pi$, $\mu, \sigma > 0$, $\mu + \sigma = \lambda$, $\gamma \in \{\frac{1}{2k+1}, 2k - 1\ (k \in \mathbf{N})\}$, $\delta \in \{-1, 1\}$. *If* $K(\sigma)$ *is indicated by* (6), $f(x), g(y) \ge 0$, *satisfying*

$0 < \int_{-\infty}^{\infty} |x|^{p(1-\delta\sigma)-1} f^p(x)\,dx < \infty$ *and* $0 < \int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y)\,dy < \infty$, *then we have the following equivalent inequalities*:

$$I := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \min_{i\in\{1,2\}} \frac{1}{[|x^\delta y|^\gamma + (x^\delta y)^\gamma \cos\alpha_i + 1]^{\lambda/\gamma}} f(x)g(y)\,dx\,dy$$
$$< K(\sigma)\left[\int_{-\infty}^{\infty} |x|^{p(1-\delta\sigma)-1} f^p(x)\,dx\right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y)\,dy\right]^{\frac{1}{q}}, \quad (11)$$

$$J := \int_{-\infty}^{\infty} |y|^{p\sigma-1} \left\{\int_{-\infty}^{\infty} \min_{i\in\{1,2\}} \frac{1}{[|x^\delta y|^\gamma + (x^\delta y)^\gamma \cos\alpha_i + 1]^{\lambda/\gamma}} f(x)\,dx\right\}^p dy$$
$$< K^p(\sigma) \int_{-\infty}^{\infty} |x|^{p(1-\delta\sigma)-1} f^p(x)\,dx, \quad (12)$$

where the constant factors $K(\sigma)$ *and* $K^p(\sigma)$ *are the best possible.*

In particular, for $\alpha_1 = \alpha_2 = \alpha \in (0,\pi)$, $\gamma = 1$ *in* (11) *and* (12), *we find*

$$K(\sigma) = k(\sigma) := \frac{1}{2^\sigma}\left[\left(\sec\frac{\alpha}{2}\right)^{2\sigma} + \left(\csc\frac{\alpha}{2}\right)^{2\sigma}\right] B(\mu,\sigma), \quad (13)$$

and the following equivalent inequalities:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{(|x^\delta y| + x^\delta y\cos\alpha + 1)^\lambda} f(x)g(y)\,dx\,dy$$
$$< k(\sigma)\left[\int_{-\infty}^{\infty} |x|^{p(1-\delta\sigma)-1} f^p(x)\,dx\right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y)\,dy\right]^{\frac{1}{q}}, \quad (14)$$

$$\int_{-\infty}^{\infty} |y|^{p\sigma-1}\left[\int_{-\infty}^{\infty} \frac{1}{(|x^\delta y| + x^\delta y\cos\alpha + 1)^\lambda} f(x)\,dx\right]^p dy$$
$$< k^p(\sigma) \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)\,dx. \quad (15)$$

Proof If (10) takes the form of equality for $y \in (-\infty,0)\cup(0,\infty)$, then there exist constants A and B, such that they are not all zero, and

$$A\frac{|x|^{(1-\delta\sigma)(p-1)}}{|y|^{(1-\sigma)}} f^p(x) = B\frac{|y|^{(1-\sigma)(q-1)}}{|x|^{(1-\delta\sigma)}} \quad \text{a.e. in } (-\infty,\infty).$$

We suppose $A \neq 0$ (otherwise $B = A = 0$). Then it follows that

$$|x|^{p(1-\delta\sigma)-1} f^p(x) = |y|^{q(1-\sigma)} \frac{B}{A|x|} \quad \text{a.e. in } (-\infty,\infty),$$

which contradicts the fact that $0 < \int_{-\infty}^{\infty} |x|^{p(1-\delta\sigma)-1} f^p(x)\,dx < \infty$. Hence (10) takes the form of a strict inequality. So does (9), and we have (12).

By Hölder's inequality (*cf.* [25]), we find

$$I = \int_{-\infty}^{\infty}\left(|y|^{\sigma-\frac{1}{p}} \int_{-\infty}^{\infty} k_\lambda^{(\delta)}(x,y) f(x)\,dx\right)\left(|y|^{\frac{1}{p}-\sigma} g(y)\,dy\right)$$
$$\le J^{\frac{1}{p}}\left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y)\,dy\right]^{\frac{1}{q}}. \quad (16)$$

Then by (12), we have (11). On the other hand, suppose that (11) is valid. Setting

$$g(y) := |y|^{p\sigma-1}\left(\int_{-\infty}^{\infty} k_{\lambda}^{(\delta)}(x,y)f(x)\,dx\right)^{p-1}, \quad y \in \mathbf{R},$$

then it follows that $J = \int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1}g^q(y)\,dy$. By (9), we have $J < \infty$. If $J = 0$, then (12) is obviously of value; if $0 < J < \infty$, then by (11), we obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1}g^q(y)\,dy \\
&\quad = J = I < K(\sigma)\left[\int_{-\infty}^{\infty} |x|^{p(1-\delta\sigma)-1}f^p(x)\,dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1}g^q(y)\,dy\right]^{\frac{1}{q}}, \qquad (17)
\end{aligned}$$

$$\begin{aligned}
J^{\frac{1}{p}} &= \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1}g^q(y)\,dy\right]^{\frac{1}{p}} \\
&< K(\sigma)\left[\int_{-\infty}^{\infty} |x|^{p(1-\delta\sigma)-1}f^p(x)\,dx\right]^{\frac{1}{p}}. \qquad (18)
\end{aligned}$$

Hence we have (12), which is equivalent to (11).

We set $E_\delta := \{x \in \mathbf{R}; |x|^\delta \geq 1\}$, and $E_\delta^+ := E_\delta \cap \mathbf{R}_+ = \{x \in \mathbf{R}_+; x^\delta \geq 1\}$. For $\varepsilon > 0$, we define functions $\tilde{f}(x)$, $\tilde{g}(y)$ as follows:

$$\tilde{f}(x) := \begin{cases} |x|^{\delta(\sigma-\frac{2\varepsilon}{p})-1}, & x \in E_\delta, \\ 0, & x \in \mathbf{R}\backslash E_\delta, \end{cases}$$

$$\tilde{g}(y) := \begin{cases} 0, & y \in (-\infty,-1) \cup (1,\infty), \\ |y|^{\sigma+\frac{2\varepsilon}{q}-1}, & y \in [-1,1]. \end{cases}$$

Then we obtain

$$\begin{aligned}
\tilde{L} &:= \left[\int_{-\infty}^{\infty} |x|^{p(1-\delta\sigma)-1}\tilde{f}^p(x)\,dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1}\tilde{g}^q(y)\,dy\right]^{\frac{1}{q}} \\
&= 2\left(\int_{E_\delta^+} x^{-2\delta\varepsilon-1}\,dx\right)^{\frac{1}{p}}\left(\int_0^1 y^{2\varepsilon-1}\,dy\right)^{\frac{1}{q}} \\
&= \frac{1}{\varepsilon}.
\end{aligned}$$

We find

$$h(x) := \int_{-1}^{1} \min_{i\in\{1,2\}} \frac{|y|^{\sigma+\frac{2\varepsilon}{q}-1}}{[|x^\delta y|^\gamma + (x^\delta y)^\gamma \cos\alpha_i + 1]^{\lambda/\gamma}}\,dy = h(-x).$$

In fact, setting $Y = -y$, we obtain

$$\begin{aligned}
h(-x) &= \int_{-1}^{1} \min_{i\in\{1,2\}} \frac{|y|^{\sigma+\frac{2\varepsilon}{q}-1}}{[|-x^\delta y|^\gamma + (-x^\delta y)^\gamma \cos\alpha_i + 1]^{\lambda/\gamma}}\,dy \\
&= \int_{-1}^{1} \min_{i\in\{1,2\}} \frac{|Y|^{\sigma+\frac{2\varepsilon}{q}-1}}{[|x^\delta Y|^\gamma + (x^\delta Y)^\gamma \cos\alpha_i + 1]^{\lambda/\gamma}}\,dY = h(x).
\end{aligned}$$

It follows that

$$\begin{aligned}
\tilde{I} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k_{\lambda}^{(\delta)}(x,y)\tilde{f}(x)\tilde{g}(y)\,dx\,dy \\
&= \int_{E_\delta} |x|^{\delta(\sigma-\frac{2\varepsilon}{p})-1}h(x)\,dx = 2\int_{E_\delta^+} x^{\delta(\sigma-\frac{2\varepsilon}{p})-1}h(x)\,dx \\
&\overset{u=x^\delta y}{=} 2\int_{E_\delta^+} x^{-2\delta\varepsilon-1}\left\{\int_{-x^\delta}^{x^\delta} \min_{i\in\{1,2\}} \frac{|u|^{\sigma+\frac{2\varepsilon}{q}-1}}{[|u|^\gamma + u^\gamma\cos\alpha_i + 1]^{\lambda/\gamma}}\,du\right\}dx.
\end{aligned}$$

Setting $v = x^\delta$ in the above integral, by the Fubini theorem (*cf.* [26]), we find

$$\begin{aligned}
\tilde{I} &= 2\int_1^\infty v^{-2\varepsilon-1}\left\{\int_{-v}^{v} \min_{i\in\{1,2\}} \frac{|u|^{\sigma+\frac{2\varepsilon}{q}-1}}{[|u|^\gamma + u^\gamma\cos\alpha_i + 1]^{\lambda/\gamma}}\,du\right\}dv \\
&= 2\int_1^\infty v^{-2\varepsilon-1}\left\{\int_0^v \left[\min_{i\in\{1,2\}} \frac{1}{[u^\gamma(1+\cos\alpha_i)+1]^{\lambda/\gamma}}\right.\right. \\
&\quad \left.\left. + \min_{i\in\{1,2\}} \frac{1}{[u^\gamma(1-\cos\alpha_i)+1]^{\lambda/\gamma}}\right] u^{\sigma+\frac{2\varepsilon}{q}-1}\,du\right\}dv \\
&= 2\int_1^\infty v^{-2\varepsilon-1}\left\{\int_0^v \left[\frac{1}{[u^\gamma(1+\cos\alpha_1)+1]^{\lambda/\gamma}} + \frac{1}{[u^\gamma(1-\cos\alpha_2)+1]^{\lambda/\gamma}}\right] u^{\sigma+\frac{2\varepsilon}{q}-1}\,du\right\}dv \\
&= 2\int_1^\infty v^{-2\varepsilon-1}\left\{\int_0^1 \left[\frac{u^{\sigma+\frac{2\varepsilon}{q}-1}}{[u^\gamma(1+\cos\alpha_1)+1]^{\frac{\lambda}{\gamma}}} + \frac{u^{\sigma+\frac{2\varepsilon}{q}-1}}{[u^\gamma(1-\cos\alpha_2)+1]^{\frac{\lambda}{\gamma}}}\right]du\right\}dv \\
&\quad + 2\int_1^\infty v^{-2\varepsilon-1}\left\{\int_1^v \left[\frac{u^{\sigma+\frac{2\varepsilon}{q}-1}}{[u^\gamma(1+\cos\alpha_1)+1]^{\frac{\lambda}{\gamma}}} + \frac{u^{\sigma+\frac{2\varepsilon}{q}-1}}{[u^\gamma(1-\cos\alpha_2)+1]^{\frac{\lambda}{\gamma}}}\right]du\right\}dv \\
&= \frac{1}{\varepsilon}\int_0^1 \left\{\frac{u^{\sigma+\frac{2\varepsilon}{q}-1}}{[u^\gamma(1+\cos\alpha_1)+1]^{\lambda/\gamma}} + \frac{u^{\sigma+\frac{2\varepsilon}{q}-1}}{[u^\gamma(1-\cos\alpha_2)+1]^{\lambda/\gamma}}\right\}du \\
&\quad + 2\int_1^\infty \left(\int_u^\infty v^{-2\varepsilon-1}\,dv\right) \\
&\quad \times \left\{\frac{u^{\sigma+\frac{2\varepsilon}{q}-1}}{[u^\gamma(1+\cos\alpha_1)+1]^{\lambda/\gamma}} + \frac{u^{\sigma+\frac{2\varepsilon}{q}-1}}{[u^\gamma(1-\cos\alpha_2)+1]^{\lambda/\gamma}}\right\}du \\
&= \frac{1}{\varepsilon}\left\{\int_0^1 \left[\frac{u^{\sigma+\frac{2\varepsilon}{q}-1}}{[u^\gamma(1+\cos\alpha_1)+1]^{\lambda/\gamma}} + \frac{u^{\sigma+\frac{2\varepsilon}{q}-1}}{[u^\gamma(1-\cos\alpha_2)+1]^{\lambda/\gamma}}\right]du\right. \\
&\quad \left. + \int_1^\infty \left[\frac{u^{\sigma-\frac{2\varepsilon}{p}-1}}{[u^\gamma(1+\cos\alpha_1)+1]^{\lambda/\gamma}} + \frac{u^{\sigma-\frac{2\varepsilon}{p}-1}}{[u^\gamma(1-\cos\alpha_2)+1]^{\lambda/\gamma}}\right]du\right\}.
\end{aligned}$$

If the constant factor $K(\sigma)$ in (11) is not the best possible, then there exists a positive number k, with $K(\sigma) < k$, such that (11) is valid when replacing $K(\sigma)$ by k. Then we have $\varepsilon\tilde{I} < \varepsilon k\tilde{L}$, and

$$\begin{aligned}
&\int_0^1 \left\{\frac{u^{\sigma+\frac{2\varepsilon}{q}-1}}{[u^\gamma(1+\cos\alpha_1)+1]^{\lambda/\gamma}} + \frac{u^{\sigma+\frac{2\varepsilon}{q}-1}}{[u^\gamma(1-\cos\alpha_2)+1]^{\lambda/\gamma}}\right\}du \\
&\quad + \int_1^\infty \left\{\frac{u^{\sigma-\frac{2\varepsilon}{p}-1}}{[u^\gamma(1+\cos\alpha_1)+1]^{\lambda/\gamma}} + \frac{u^{\sigma-\frac{2\varepsilon}{p}-1}}{[u^\gamma(1-\cos\alpha_2)+1]^{\lambda/\gamma}}\right\}du \\
&\quad = \varepsilon\tilde{I} < \varepsilon k\tilde{L} = k. \qquad (19)
\end{aligned}$$

By (7) and the Levi theorem (*cf.* [26]), we have

$$
\begin{aligned}
K(\sigma) &= \int_0^\infty \frac{u^{\sigma-1}\,du}{[u^\gamma(1+\cos\alpha_1)+1]^{\lambda/\gamma}} + \int_0^\infty \frac{u^{\sigma-1}\,du}{[u^\gamma(1-\cos\alpha_2)+1]^{\lambda/\gamma}} \\
&= \int_0^1 \lim_{\varepsilon\to0^+}\left\{\frac{u^{\sigma+\frac{2\varepsilon}{q}-1}}{[u^\gamma(1+\cos\alpha_1)+1]^{\lambda/\gamma}} + \frac{u^{\sigma+\frac{2\varepsilon}{q}-1}}{[u^\gamma(1-\cos\alpha_2)+1]^{\lambda/\gamma}}\right\}du \\
&\quad + \int_1^\infty \lim_{\varepsilon\to0^+}\left\{\frac{u^{\sigma-\frac{2\varepsilon}{p}-1}}{[u^\gamma(1+\cos\alpha_1)+1]^{\lambda/\gamma}} + \frac{u^{\sigma-\frac{2\varepsilon}{p}-1}}{[u^\gamma(1-\cos\alpha_2)+1]^{\lambda/\gamma}}\right\}du \\
&= \lim_{\varepsilon\to0^+}\left\{\int_0^1\left[\frac{u^{\sigma+\frac{2\varepsilon}{q}-1}}{[u^\gamma(1+\cos\alpha_1)+1]^{\lambda/\gamma}} + \frac{u^{\sigma+\frac{2\varepsilon}{q}-1}}{[u^\gamma(1-\cos\alpha_2)+1]^{\lambda/\gamma}}\right]du\right. \\
&\quad \left. + \int_1^\infty\left[\frac{u^{\sigma-\frac{2\varepsilon}{p}-1}}{[u^\gamma(1+\cos\alpha_1)+1]^{\lambda/\gamma}} + \frac{u^{\sigma-\frac{2\varepsilon}{p}-1}}{[u^\gamma(1-\cos\alpha_2)+1]^{\lambda/\gamma}}\right]du\right\} \le k,
\end{aligned}
$$

which contradicts the fact that $k < K(\sigma)$. Hence the constant factor $K(\sigma)$ in (11) is the best possible.

If the constant factor in (12) is not the best possible, then by (16), we may get a contradiction: that the constant factor in (11) is not the best possible. □

Theorem 2 *As the assumptions of Theorem* 1, *replacing* $p > 1$ *by* $0 < p < 1$, *we have the equivalent reverses of* (11) *and* (12) *with the same best constant factors.*

Proof By the reverse Hölder's inequality (*cf.* [25]), we have the reverses of (9) and (16). It is easy to obtain the reverse of (12). In view of the reverses of (12) and (16), we obtain the reverse of (11). On the other hand, suppose that the reverse of (11) is valid. Setting the same $g(y)$ as Theorem 1, by the reverse of (9), we have $J > 0$. If $J = \infty$, then the reverse of (12) is obviously value; if $J < \infty$, then by the reverse of (11), we obtain the reverses of (17) and (18). Hence we have the reverse of (12), which is equivalent to the reverse of (11).

If the constant factor $K(\sigma)$ in the reverse of (11) is not the best possible, then there exists a positive constant k, with $k > K(\sigma)$, such that the reverse of (11) is still valid when replacing $K(\sigma)$ by k. By the reverse of (19), we have

$$
\begin{aligned}
&\int_0^1\left\{\frac{u^{\sigma+\frac{2\varepsilon}{q}-1}}{[u^\gamma(1+\cos\alpha_1)+1]^{\lambda/\gamma}} + \frac{u^{\sigma+\frac{2\varepsilon}{q}-1}}{[u^\gamma(1-\cos\alpha_2)+1]^{\lambda/\gamma}}\right\}du \\
&\quad + \int_1^\infty\left\{\frac{u^{\sigma-\frac{2\varepsilon}{p}-1}}{[u^\gamma(1+\cos\alpha_1)+1]^{\lambda/\gamma}} + \frac{u^{\sigma-\frac{2\varepsilon}{p}-1}}{[u^\gamma(1-\cos\alpha_2)+1]^{\lambda/\gamma}}\right\}du > k. \qquad (20)
\end{aligned}
$$

For $\varepsilon \to 0^+$, by the Levi theorem (*cf.* [26]), we find that

$$
\begin{aligned}
&\int_1^\infty\left\{\frac{u^{\sigma-\frac{2\varepsilon}{p}-1}}{[u^\gamma(1+\cos\alpha_1)+1]^{\lambda/\gamma}} + \frac{u^{\sigma-\frac{2\varepsilon}{p}-1}}{[u^\gamma(1-\cos\alpha_2)+1]^{\lambda/\gamma}}\right\}du \\
&\quad \to \int_1^\infty\left\{\frac{u^{\sigma-1}}{[u^\gamma(1+\cos\alpha_1)+1]^{\lambda/\gamma}} + \frac{u^{\sigma-1}}{[u^\gamma(1-\cos\alpha_2)+1]^{\lambda/\gamma}}\right\}du. \qquad (21)
\end{aligned}
$$

There exists a constant $\delta_0 > 0$, such that $\sigma - \frac{1}{2}\delta_0 > 0$, and then $K(\sigma - \frac{\delta_0}{2}) < \infty$. For $0 < \varepsilon < \frac{\delta_0|q|}{4}$ ($q < 0$), since $u^{\sigma+\frac{2\varepsilon}{q}-1} \le u^{\sigma-\frac{\delta_0}{2}-1}$, $u \in (0,1]$, and

$$0 < \int_0^1 \left\{ \frac{u^{\sigma-\frac{\delta_0}{2}-1}}{[u^\gamma(1+\cos\alpha_1)+1]^{\lambda/\gamma}} + \frac{u^{\sigma-\frac{\delta_0}{2}-1}}{[u^\gamma(1-\cos\alpha_2)+1]^{\lambda/\gamma}} \right\} du \le K\left(\sigma - \frac{\delta_0}{2}\right) < \infty,$$

then by the Lebesgue control convergence theorem (*cf.* [26]), for $\varepsilon \to 0^+$, we have

$$\int_0^1 \left\{ \frac{u^{\sigma+\frac{2\varepsilon}{q}-1}}{[u^\gamma(1+\cos\alpha_1)+1]^{\lambda/\gamma}} + \frac{u^{\sigma+\frac{2\varepsilon}{q}-1}}{[u^\gamma(1-\cos\alpha_2)+1]^{\lambda/\gamma}} \right\} du \to \int_0^1 \left\{ \frac{u^{\sigma-1}}{[u^\gamma(1+\cos\alpha_1)+1]^{\lambda/\gamma}} + \frac{u^{\sigma-1}}{[u^\gamma(1-\cos\alpha_2)+1]^{\lambda/\gamma}} \right\} du. \tag{22}$$

By (20), (21), and (22), for $\varepsilon \to 0^+$, we find $K(\sigma) \ge k$, which contradicts the fact that $k > K(\sigma)$. Hence, the constant factor $K(\sigma)$ in the reverse of (11) is the best possible.

If the constant factor in reverse of (12) is not the best possible, then by the reverse of (16), we may get a contradiction that the constant factor in the reverse of (11) is not the best possible. □

Remarks For $\delta = -1$ in (11) and (12), replacing $|x|^\lambda f(x)$ by $f(x)$, we obtain the following equivalent inequalities with the homogeneous kernel and the best possible constant factors:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \min_{i\in\{1,2\}} \frac{1}{(|y|^\gamma + \operatorname{sgn}(x)y^\gamma\cos\alpha_i + |x|^\gamma)^{\lambda/\gamma}} f(x)g(y)\,dx\,dy < K(\sigma)\left[\int_{-\infty}^{\infty} |x|^{p(1-\mu)-1} f^p(x)\,dx\right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y)\,dy\right]^{\frac{1}{q}}, \tag{23}$$

$$\int_{-\infty}^{\infty} |y|^{p\sigma-1}\left[\int_{-\infty}^{\infty} \min_{i\in\{1,2\}} \frac{1}{(|y|^\gamma + \operatorname{sgn}(x)y^\gamma\cos\alpha_i + |x|^\gamma)^{\lambda/\gamma}} f(x)\,dx\right]^p dy < K^p(\sigma)\int_{-\infty}^{\infty} |x|^{p(1-\mu)-1} f^p(x)\,dx. \tag{24}$$

In particular, for $\alpha_1 = \alpha_2 = \alpha \in (0,\pi)$, $\gamma = 1$ in (23) and (24), we obtain the following equivalent inequalities:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{(|y| + \operatorname{sgn}(x)y\cos\alpha + |x|)^\lambda} f(x)g(y)\,dx\,dy < k(\sigma)\left[\int_{-\infty}^{\infty} |x|^{p(1-\mu)-1} f^p(x)\,dx\right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y)\,dy\right]^{\frac{1}{q}}, \tag{25}$$

$$\int_{-\infty}^{\infty} |y|^{p\sigma-1}\left[\int_{-\infty}^{\infty} \frac{1}{(|y| + \operatorname{sgn}(x)y\cos\alpha + |x|)^\lambda} f(x)\,dx\right]^p dy < k^p(\sigma)\int_{-\infty}^{\infty} |x|^{p(1-\mu)-1} f^p(x)\,dx, \tag{26}$$

where $k(\sigma)$ is indicated by (13).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
BY carried out the mathematical studies, participated in the sequence alignment and drafted the manuscript.
ZG participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

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