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The rank inequality for diagonally magic matrices

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Abstract
We study a new class of matrices called diagonally magic matrices. We prove that such a matrix has rank at most 2 and that any square submatrix of a diagonally magic matrix is diagonally magic.

MSC: 15A03; 15A06

Keywords: rank; diagonally magic matrices; eigenvalue; linear equations

1 Introduction

For a positive integer n, let S_n be the set of all $n!$ permutations of $\{1,2,\dots,n\}$. We denote by $\mathbb{C}^{n\times n}$ and $\mathbb{R}^{n\times n}$ the set of $n\times n$ complex matrices and the set of $n\times n$ real matrices, respectively. If $A=(a_{i,j})\in\mathbb{C}^{n\times n}$ and $\sigma\in S_n$, then the sequence $a_{1,\sigma(1)},a_{2,\sigma(2)},\dots,a_{n,\sigma(n)}$ is called a *transversal* of A [1]. In 2012, Professor Xingzhi Zhan defined the following new concept at a seminar and suggested studying its properties.

Definition 1.1 A matrix $A=(a_{i,j})\in\mathbb{C}^{n\times n}$ is called *diagonally magic* if

$$\sum_{i=1}^{n} a_{i,\sigma(i)}=\sum_{i=1}^{n} a_{i,\pi(i)} \tag{1}$$

for all $\sigma,\pi\in S_n$.

Obviously, the zero matrix $0_{n\times n}$ and $J=[1]_{n\times n}$, the matrix of all ones, are diagonally magic matrices. Denote

$$B_n=\begin{pmatrix} 1 & 2 & \cdots & n \\ n+1 & n+2 & \cdots & 2n \\ \vdots & \vdots & \ddots & \vdots \\ (n-1)n+1 & (n-1)n+2 & \cdots & n^2 \end{pmatrix} \tag{2}$$

and

$$C_n=\begin{pmatrix} 1 & 2 & \cdots & n \\ 2 & 3 & \cdots & n+1 \\ \vdots & \vdots & \ddots & \vdots \\ n & n+1 & \cdots & 2n-1 \end{pmatrix}. \tag{3}$$

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We will show that B_n and C_n are diagonally magic matrices. So, there are a lot of diagonally magic matrices. C_n is a Hankel matrix. B_n and C_n are nonnegative matrices which have been a hot research area [2, 3].

For matrix $D = (d_{ij}) \in \mathbb{C}^{m\times n}$, let the columns of D be $d_1, d_2, \dots, d_n$. $\operatorname{vec}(D)$ is a vector defined by $\operatorname{vec}(D) = (d_1^T, d_2^T, \dots, d_n^T)^T$, where the superscript ^{T} denotes the transpose. The matrix $E_{i,j}^n$ denotes the Type 1 *elementary matrix* [4], p.8, which is simply the identity matrix I_n of order n, the i, i and j, j entries replaced by 0 and the i, j entry (respectively j, i entry) replaced by 1 (respectively 1). Given two matrices A and B, their direct sum is written as $A \oplus B$. Given a sequence of matrices A_i, for $i = 1, \dots, k$, one may write their direct sum as

$$A = \bigoplus_{i=1}^{k} A_i = \operatorname{diag}(A_1, \dots, A_k).$$

Each A_i is called a direct summand of A. Let $e_n = (\underbrace{1, \dots, 1}_{n})^T$ and $\widehat{e}_n = (\underbrace{0, \dots, 0}_{n-1}, 1)^T$.

2 Main results

Let $A = (a_{i,j}) \in \mathbb{C}^{n\times n}$ be a diagonally magic matrix with $n \geq 2$. Assume that the sum of every transversal is c. From the definition of diagonally magic matrices (1), we have a system of linear equations

$$\widetilde{A}_n \operatorname{vec}\bigl(A^T\bigr) = ce_{n!}, \tag{4}$$

where $\widetilde{A}_n = (\widetilde{a}_1, \widetilde{a}_2, \dots, \widetilde{a}_{n^2}) \in \mathbb{R}^{n!\times n^2}$ is the *coefficient matrix*. If $n = 2$, from the definition of the diagonally magic matrices and (4), the coefficient matrix $\widetilde{A}_2$ can be chosen to be the 2×4 matrix

$$\widetilde{A}_2 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} = (\widetilde{a}_1, \widetilde{a}_2, \widetilde{a}_3, \widetilde{a}_4),$$

and the augmented matrix is

$$\widehat{A}_2 = \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & c \\ 0 & 1 & 1 & 0 & c \end{array}\right). \tag{5}$$

This augmented matrix is the row-reduced echelon form.

Suppose $n \geq 2$ and that, for $n = 2, \dots, k$, we have constructed the coefficient matrix

$$\widetilde{A}_k = (\widetilde{a}_1, \widetilde{a}_2, \dots, \widetilde{a}_{k^2}).$$

Let $n = k + 1$. We use the following method to construct the coefficient matrix $\widetilde{A}_{k+1}$. Firstly, let

$$C_{1,1}^{k+1} = (e_{k!}, 0_{k!\times k}),$$
$$C_{1,m+1}^{k+1} = (0_{k!\times 1}, \widetilde{a}_{(m-1)k+1}, \widetilde{a}_{(m-1)k+2}, \dots, \widetilde{a}_{mk})$$

for $1 \le m \le k$. Secondly, construct

$$C_{i,j}^{k+1} = C_{i-1,j}^{k+1} E_{i-1,i}^{k+1}$$

for $i = 2, 3, \dots, k+1$, $j = 1, 2, 3, \dots, k+1$, where $E_{i,j}^{n}$ denotes the Type 1 *elementary matrix* [4], p.8, which is simply the identity matrix I_n of order n, the i, i and j, j entries replaced by 0 and the i, j entry (respectively j, i entry) replaced by 1 (respectively 1). Then we get the coefficient matrix

$$\widetilde{A}_{k+1} = \begin{pmatrix} C_{1,1}^{k+1} & C_{1,2}^{k+1} & \cdots & C_{1,k+1}^{k+1} \\ C_{2,1}^{k+1} & C_{2,2}^{k+1} & \cdots & C_{2,k+1}^{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ C_{k+1,1}^{k+1} & C_{k+1,2}^{k+1} & \cdots & C_{k+1,k+1}^{k+1} \end{pmatrix}.$$

For example, if $n = 3$, according to the constructing method and $\widetilde{A}_2$, we have

$$\begin{aligned} \widetilde{A}_3 &= \begin{pmatrix} C_{1,1}^{3} & C_{1,2}^{3} & C_{1,3}^{3} \\ C_{2,1}^{3} & C_{2,2}^{3} & C_{2,3}^{3} \\ C_{3,1}^{3} & C_{3,2}^{3} & C_{3,3}^{3} \end{pmatrix} \\ &= \left(\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right) \\ &= (\widetilde{a}_1, \widetilde{a}_2, \widetilde{a}_3, \widetilde{a}_4, \widetilde{a}_5, \widetilde{a}_6, \widetilde{a}_7, \widetilde{a}_8, \widetilde{a}_9). \end{aligned}$$

Assume

$$C_j = \left(C_{j,1}^{k+1}, C_{j,2}^{k+1}, \dots, C_{j,k+1}^{k+1}, ce_{k!}\right)$$

for $j = 1, 2, \dots, k+1$. Consequently, the augmented matrix of $\widetilde{A}_{k+1}$ is

$$\left(C_1^T, C_2^T, \dots, C_{k+1}^T\right)^T.$$

Let

$$D_n = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \in \mathbb{R}^{n \times n},$$

$$F_{(n-1)\times n} = (I_{n-1} \quad -e_{n-1}) \in \mathbb{R}^{(n-1)\times n},$$

$$G_n = \begin{pmatrix} 0 & 1 & \cdots & 1 & 3-n \\ 1 & 0 & \cdots & 1 & 3-n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & 3-n \\ 1 & 1 & \cdots & 1 & 2-n \end{pmatrix} \in \mathbb{R}^{n\times n}.$$

We claim that the row-reduced echelon form of the augmented matrix in the system of linear equations (4) has the following form:

$$\widehat{A}_n = \left(\begin{array}{cccccc|c} I_n & \overbrace{D_n & D_n & \cdots & D_n}^{(n-2)n} & G_n & ce_n \\ & F_{(n-1)\times n} & 0 & \cdots & 0 & -F_{(n-1)\times n} & 0 \\ & & F_{(n-1)\times n} & \cdots & 0 & -F_{(n-1)\times n} & 0 \\ & & & \ddots & \vdots & \vdots & \vdots \\ & & \mathbf{0} & & F_{(n-1)\times n} & -F_{(n-1)\times n} & 0 \\ & & & & & 0 & 0 \end{array}\right). \tag{6}$$

We prove this by induction on n. For example, $(\widetilde{A}_2, ce_{2!})$ is row-equivalent to

$$\widehat{A}_2 = \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & c \\ 0 & 1 & 1 & 0 & c \end{array}\right).$$

The row-reduced echelon form of the augmented matrix $(\widetilde{A}_3, ce_{3!})$ has the following form:

$$\begin{aligned} \widehat{A}_3 &= \left(\begin{array}{ccc|c} I_3 & D_3 & G_3 & ce_3 \\ 0_{2\times 3} & F_{2\times 3} & -F_{2\times 3} & 0_{3\times 1} \\ 0_{1\times 3} & 0_{1\times 3} & 0_{1\times 3} & 0 \end{array}\right) \\ &= \left(\begin{array}{ccc|ccc|ccc|c} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & -1 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right). \end{aligned}$$

Obviously, if $n = 2$, $\widehat{A}_2$ in (5) has the form (6). Suppose $n \geq 2$ and that, for $n = 2, \ldots, k$, the assertion has been proved for $n!$-by-n^2 matrix $\widetilde{A}_n$. Let $n = k + 1$,

$$\begin{aligned} &H_{k\times(k+1)} = (e_k, 0_{k\times k}), \qquad J_{k\times(k+1)} = (0_{k\times 1}, I_k), \\ &\widetilde{D}_{k\times(k+1)} = (0_{k\times 1}, D_k), \qquad \widetilde{G}_{k\times(k+1)} = (0_{k\times 1}, G_k), \\ &\widetilde{F}_{(k-1)\times(k+1)} = (0_{k\times 1}, F_{(k-1)\times k}). \end{aligned}$$

By the inductive hypothesis, we can obtain the following matrix after a sequence of elementary operations for C_1:

$$P_1 = \left(\begin{array}{cccccccc|c} H_{k\times(k+1)} & J_{k\times(k+1)} & \widetilde{D}_{k\times(k+1)} & \widetilde{D}_{k\times(k+1)} & \cdots & \widetilde{D}_{k\times(k+1)} & \widetilde{G}_{k\times(k+1)} & ce_k \\ & & \widetilde{F}_{(k-1)\times(k+1)} & 0 & \cdots & 0 & -\widetilde{F}_{(k-1)\times(k+1)} & 0 \\ & & & \widetilde{F}_{(k-1)\times(k+1)} & \cdots & 0 & -\widetilde{F}_{(k-1)\times(k+1)} & 0 \\ & & & & \ddots & \vdots & \vdots & \vdots \\ & & \mathbf{0} & & & \widetilde{F}_{(k-1)\times(k+1)} & -\widetilde{F}_{(k-1)\times(k+1)} & 0 \\ & & & & & & 0 & 0 \end{array}\right)$$

$$\equiv \begin{pmatrix} P_{1,1} \\ P_{1,2} \\ P_{1,3} \\ \vdots \\ P_{1,k-1} \\ 0 \end{pmatrix},$$

where

$$P_{1,1} = \left(\begin{array}{ccccccc|c} H_{k\times(k+1)} & J_{k\times(k+1)} & \widetilde{D}_{k\times(k+1)} & \widetilde{D}_{k\times(k+1)} & \cdots & \widetilde{D}_{k\times(k+1)} & \widetilde{G}_{k\times(k+1)} & ce_k \end{array}\right),$$

$$P_{1,i} = \Bigl(0 \quad \cdots \quad 0 \quad \widetilde{F}_{(k-1)\times(k+1)} \quad \overbrace{0 \quad \cdots \quad 0}^{(k-i-1)(k+1)\text{ columns}} \quad -\widetilde{F}_{(k-1)\times(k+1)} \Bigm| 0\Bigr) \in \mathbb{R}^{(k-1)\times((k+1)^2+1)}$$

for $i = 2, 3, \ldots, k-1$. C_j is row-equivalent to

$$P_j = P_{j-1}\left(\left(\bigoplus_{i=1}^{k+1} E_{j-1,j}^{k+1}\right) \oplus 1\right) = \left(P_{j,1}^T, \ldots, P_{j,k-1}^T, 0^T\right)^T$$

for $j = 2, \ldots, k+1$. $P_{j,1} \in \mathbb{R}^{k\times((k+1)^2+1)}$ is the first k rows of P_j. $P_{j,i} \in \mathbb{R}^{(k-1)\times((k+1)^2+1)}$ is the rows of P_j from $(k+(i-2)(k-1)+1)$th to $(k+(i-1)(k-1))$th row, $i = 2, \ldots, k-1$. In $P_{1,1}$, multiplying row k by the scalar -1 and adding to row j, for $j = 1, \ldots, k-1$, then $P_{1,1}$ is row-equivalent to

$$\widehat{P}_{1,1} = (\widehat{H}_{k\times(k+1)}, \widehat{J}_{k\times(k+1)}, \widehat{\widetilde{D}}_{k\times(k+1)}, \widehat{\widetilde{D}}_{k\times(k+1)}, \ldots, \widehat{\widetilde{D}}_{k\times(k+1)}, \widehat{\widetilde{G}}_{k\times(k+1)}, \widehat{ce_k}),$$

where

$$\widehat{H}_{k\times(k+1)} = \begin{pmatrix} 0 & 0_{(k-1)\times k} \\ 1 & 0 \end{pmatrix}, \qquad \widehat{J}_{k\times(k+1)} = \begin{pmatrix} 0 & I_{k-1} & -e_{k-1} \\ 0 & 0 & 1 \end{pmatrix},$$

$$\widehat{\widetilde{D}}_{k\times(k+1)} = \begin{pmatrix} 0_{(k-1)\times k} & 0 \\ 0 & 1 \end{pmatrix}, \qquad \widehat{\widetilde{G}}_{k\times(k+1)} = \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 & 1 \\ 0 & 0 & -1 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \\ 0 & 1 & 1 & \cdots & 1 & 2-k \end{pmatrix}.$$

Applying this method to $P_{j,1}$, $j = 2, \ldots, k+1$, then $P_{j,1}$ is row-equivalent to

$$\widehat{P}_{j,1} = \widehat{P}_{j-1,1}\left(\left(\bigoplus_{i=1}^{k+1} E_{j-1,j}^{k+1}\right) \oplus 1\right).$$

Multiplying row $k-1$ of $\widehat{P}_{1,1}$ by the scalar -1 and adding to row k of $\widehat{P}_{k+1,1}$, and multiplying row $k-1$ of $P_{1,i}$ by the scalar -1 and adding to row k of $\widehat{P}_{k+1,1}$ for $i = 2, 3, \ldots, k-1$, then row k of $\widehat{P}_{k+1,1}$ changes to

$$\bigl(\underbrace{\widehat{e}_{k+1}^{T}, \ldots, \widehat{e}_{k+1}^{T}}_{k(k+1)}, \underbrace{1, \ldots, 1}_{k}, 1-k, c\bigr). \tag{7}$$

Picking row k of $\widehat{P}_{j,1}$, $j = 1, 2, \ldots, k$, and (7), we have

$$(I_{k+1}, \underbrace{D_{k+1}, D_{k+1}, \ldots, D_{k+1}}_{(k-1)(k+1)}, G_{k+1}, ce_{k+1}). \tag{8}$$

Combining row 1 of $\widehat{P}_{2,1}$ and row i of $\widehat{P}_{1,1}$, for $i = 1, \ldots, k-1$, we have

$$(0_{k\times(k+1)}, F_{k\times(k+1)}, \underbrace{0_{k\times(k+1)}, \ldots, 0_{k\times(k+1)}}_{(k-3)(k+1)}, -F_{k\times(k+1)}, 0). \tag{9}$$

Combining row 1 of $P_{2,i}$ and row j of $P_{1,i}$, for $i, j = 1, 2, \ldots, k-1$, we get

$$\left(\begin{array}{ccccccc|c} 0_{k\times(k+1)} & 0_{k\times(k+1)} & F_{k\times(k+1)} & 0_{k\times(k+1)} & \cdots & 0_{k\times(k+1)} & -F_{k\times(k+1)} & 0 \\ 0_{k\times(k+1)} & 0_{k\times(k+1)} & 0_{k\times(k+1)} & F_{k\times(k+1)} & \cdots & 0_{k\times(k+1)} & -F_{k\times(k+1)} & 0 \\ & & \mathbf{0} & & \ddots & \vdots & \vdots & \vdots \\ & & & & & F_{k\times(k+1)} & -F_{k\times(k+1)} & 0 \end{array}\right). \tag{10}$$

Combining (8), (9) and (10), we have

$$\left(\begin{array}{cccccc|c} I_{k+1} & D_{k+1} & D_{k+1} & \cdots & D_{k+1} & G_{k+1} & ce_{k+1} \\ & F_{k\times(k+1)} & 0 & \cdots & 0 & -F_{k\times(k+1)} & 0 \\ & & F_{k\times(k+1)} & \cdots & 0 & -F_{k\times(k+1)} & 0 \\ & \mathbf{0} & & \ddots & \vdots & \vdots & \vdots \\ & & & & F_{k\times(k+1)} & -F_{k\times(k+1)} & 0 \end{array}\right).$$

The other rows depend linearly on some rows of the above matrix. From the *row-reduced echelon form* (6), we get $\operatorname{rank}(\widetilde{A}_n) = n^2 - 2n + 2$. $a_{j,n}$, $a_{n,i}$, for $2 \le j \le n$, $1 \le i \le n-1$, are free variables. The other entries of matrix $A = (a_{i,j}) \in \mathbb{C}^{n\times n}$ are the pivot variables. The pivot variables are completely determined in terms of free variables.

Theorem 2.1 *Let $A \in \mathbb{C}^{n\times n}$ be a diagonally magic matrix. Then* $\operatorname{rank}(A) \le 2$.

Proof Let $A = (a_{i,j}) \in \mathbb{C}^{n\times n}$ be a diagonally magic matrix. If $n = 1$, the conclusion is trivial. Next, we prove that it is true for $n \ge 2$. Assume unknowns $a_{j,n}$, $a_{n,i}$, for $2 \le j \le n$, $1 \le i \le n-1$, are free variables. According to (6), we have

$$a_{1,i} = -\sum_{j=2}^{n-1} a_{j,n} - \sum_{j\neq i}^{n-1} a_{n,j} + (n-3)a_{n,n} + c$$

for $i = 1, \dots, n-1$, and

$$a_{1,n} = -\sum_{j=2}^{n-1} a_{j,n} - \sum_{j=1}^{n-1} a_{n,j} + (n-2)a_{n,n} + c.$$

We also have

$$a_{i,j} = a_{i,n} + a_{n,j} - a_{n,n}$$

for $2 \leq i, j \leq n-1$. That is,

$$A = \begin{pmatrix} -\sum_{j=2}^{n-1} a_{j,n} - \sum_{j\neq 1}^{n-1} a_{n,j} + (n-3)a_{n,n} + c & \cdots & -\sum_{j=2}^{n-1} a_{j,n} - \sum_{j=1}^{n-1} a_{n,j} + (n-2)a_{n,n} + c \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix}. \tag{11}$$

Using row elementary operations, A is row-equivalent to

$$\begin{pmatrix} c - \sum_{j=1}^{n} a_{n,j} & c - \sum_{j=1}^{n} a_{n,j} & \cdots & c - \sum_{j=1}^{n} a_{n,j} & c - \sum_{j=1}^{n} a_{n,j} \\ a_{2,n} - a_{n,n} & a_{2,n} - a_{n,n} & \cdots & a_{2,n} - a_{n,n} & a_{2,n} - a_{n,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,n} - a_{n,n} & a_{n-1,n} - a_{n,n} & \cdots & a_{n-1,n} - a_{n,n} & a_{n-1,n} - a_{n,n} \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n-1} & a_{n,n} \end{pmatrix}. \tag{12}$$

From (12), we can easily get

$$\operatorname{rank}(A) \leq 2.$$

This completes the proof. □

According to (11), we know that the matrices B_n in (2) and C_n in (3) are diagonally magic matrices. It is easy to verify that B_n is row-equivalent to

$$B_n \longrightarrow \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 0 & 1 & 2 & \cdots & n-1 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

C_n is row-equivalent to

$$C_n \longrightarrow \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 0 & 1 & 2 & \cdots & n-1 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Now it is clear that there are diagonally magic matrices of ranks 0, 1, 2. Indeed, $\mathrm{rank}(0_{n\times n}) = 0$, $\mathrm{rank}([1]_{n\times n}) = 1$, and $\mathrm{rank}(B_n) = \mathrm{rank}(C_n) = 2$.

Theorem 2.2 *If the diagonally magic matrix $A \in \mathbb{C}^{n\times n}$ has a form* (11), *then the characteristic polynomial of A is*

$$p_A(\lambda) = \lambda^{n-2}\bigl(\lambda^2 - c\lambda + d\bigr), \tag{13}$$

where $d = (\sum_{j=1}^{n} a_{n,j} - c)\sum_{j=2}^{n}(a_{n,1} - a_{n,j}) - n\sum_{j=2}^{n-1}(a_{n,n} - a_{j,n})(a_{n,1} - a_{n,j})$.

From (13), we can see that *the algebraic multiplicity of the eigenvalue* 0 of the diagonally magic matrix A is *at least* $n - 2$.

Theorem 2.3 *Let A and B be diagonally magic matrices of the same order. Then $A \pm B$, kA, PAQ and A^* are diagonally magic matrices, where k is a constant, P and Q are the square matrices every row and every column of which has at most one nonzero entry, and A^* denotes the conjugate transpose of A.*

Proof This can be easily checked from the definition. □

Let $A \in \mathbb{C}^{n\times n}$, $1 \le i_1 \le i_2 \le \cdots \le i_k \le n$, $1 \le j_1 \le j_2 \le \cdots \le j_s \le n$. We denote by $A[i_1, i_2, \ldots, i_k | j_1, j_2, \ldots, j_s]$ the $k \times s$ submatrix of A that lies in the rows $i_1, i_2, \ldots, i_k$ and columns $j_1, j_2, \ldots, j_s$. Denote by $A(i_1, i_2, \ldots, i_k | j_1, j_2, \ldots, j_s)$ the $(n - k) \times (n - s)$ submatrix of A obtained by deleting the rows $i_1, i_2, \ldots, i_k$ and columns $j_1, j_2, \ldots, j_s$.

Theorem 2.4 *Any square submatrix of a diagonally magic matrix is diagonally magic.*

Proof Let B be a $k \times k$ submatrix of a diagonally magic matrix $A = (a_{i,j})$. Then there are row and column indices $\alpha = (i_1, i_2, \ldots, i_k)$ and $\beta = (j_1, j_2, \ldots, j_k)$ such that $B = A[\alpha|\beta]$. Note that the union of a transversal of B and a transversal of $A(\alpha|\beta)$ is a transversal of A. Choose an arbitrary but fixed transversal T of the square matrix $A(\alpha|\beta)$. For any $\sigma, \pi \in S_k$, $a_{i_1,j_{\sigma(1)}}, \ldots, a_{i_k,j_{\sigma(k)}}$ and the entries in T constitute a transversal of A, while $a_{i_1,j_{\pi(1)}}, \ldots, a_{i_k,j_{\pi(k)}}$ and the entries in T also constitute a transversal of A. Let b be the sum of the entries in T. Since A is diagonally magic, we have

$$\sum_{t=1}^{k} a_{i_t,j_{\sigma(t)}} + b = \sum_{t=1}^{k} a_{i_t,j_{\pi(t)}} + b,$$

which yields

$$\sum_{t=1}^{k} a_{i_t,j_{\sigma(t)}} = \sum_{t=1}^{k} a_{i_t,j_{\pi(t)}}.$$

This shows that B is diagonally magic. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions

All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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