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Inequalities for certain means in two arguments

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Abstract

In this paper, we present the sharp bounds of the ratios $U(a,b)/L_4(a,b)$, $P_2(a,b)/U(a,b)$, $NS(a,b)/P_2(a,b)$ and $B(a,b)/NS(a,b)$ for all $a,b>0$ with $a\neq b$, where $L_4(a,b)=[(b^4-a^4)/(4(\log b-\log a))]^{1/4}$, $U(a,b)=(b-a)/[\sqrt{2}\arctan((b-a)/\sqrt{2ab})]$, $P_2(a,b)=[(b^2-a^2)/(2\arcsin((b^2-a^2)/(b^2+a^2)))]^{1/2}$, $NS(a,b)=(b-a)/[2\sinh^{-1}((b-a)/(b+a))]$, $B(a,b)=Q(a,b)e^{A(a,b)/T(a,b)-1}$, $A(a,b)=(a+b)/2$, $Q(a,b)=\sqrt{(a^2+b^2)/2}$, and $T(a,b)=(a-b)/[2\arctan((a-b)/(a+b))]$.

MSC: 26E60

Keywords: logarithmic mean; Yang mean; first Seiffert mean; Neuman-Sándor mean; Sándor-Yang mean

1 Introduction

For $r\in\mathbb{R}$, the rth power mean $M(a,b;r)$ of two distinct positive real numbers a and b is defined by

$$M(a,b;r)=\begin{cases}\left(\frac{a^r+b^r}{2}\right)^{1/r}, & r\neq 0,\\ \sqrt{ab}, & r=0.\end{cases} \tag{1.1}$$

It is well known that $M(a,b;r)$ is continuous and strictly increasing with respect to $r\in\mathbb{R}$ for fixed $a,b>0$ with $a\neq b$. Many classical means are the special cases of the power mean, for example, $M(a,b;-1)=2ab/(a+b)=H(a,b)$ is the harmonic mean, $M(a,b;0)=\sqrt{ab}=G(a,b)$ is the geometric mean, $M(a,b;1)=(a+b)/2=A(a,b)$ is the arithmetic mean, and $M(a,b;2)=\sqrt{(a^2+b^2)/2}=Q(a,b)$ is the quadratic mean. The main properties for the power mean are given in [1].

Let

$$L(a,b)=\frac{a-b}{\log a-\log b},\qquad P(a,b)=\frac{a-b}{2\arcsin(\frac{a-b}{a+b})},$$

$$U(a,b)=\frac{a-b}{\sqrt{2}\arctan(\frac{a-b}{\sqrt{2ab}})},\qquad NS(a,b)=\frac{a-b}{2\sinh^{-1}(\frac{a-b}{a+b})}, \tag{1.2}$$

$$T(a,b)=\frac{a-b}{2\arctan(\frac{a-b}{a+b})},\qquad B(a,b)=Q(a,b)e^{A(a,b)/T(a,b)-1} \tag{1.3}$$

be, respectively, the logarithmic mean, first Seiffert mean [2], Yang mean [3], Neuman-Sándor mean [4, 5], second Seiffert mean [6], Sándor-Yang mean [3, 7] of two distinct positive real numbers a and b.

Recently, the sharp bounds for certain bivariate means in terms of the power mean have attracted the attention of many mathematicians.

Radó [8] and Lin [9], Jagers [10] and Hästö [11, 12] proved that the double inequalities

$$M(a,b;0) < L(a,b) < M(a,b;1/3), \tag{1.4}$$

$$M(a,b;\log 2/\log\pi) < P(a,b) < M(a,b;2/3) \tag{1.5}$$

hold for all $a,b>0$ and $a\neq b$ with the best possible parameters 0, $1/3$, $\log 2/\log\pi$, and $2/3$.

In [13–17], the authors proved that the double inequalities

$$M(a,b;\alpha) < NS(a,b) < M(a,b;\beta), \tag{1.6}$$

$$M(a,b;\lambda) < U(a,b) < M(a,b;\mu) \tag{1.7}$$

hold for all $a,b>0$ and $a\neq b$ if and only if $\alpha\leq\log 2/\log[2\log(1+\sqrt{2})]$, $\beta\geq 4/3$, $\lambda\leq 2\log 2/(2\log\pi-\log 2)$ and $\mu\geq 4/3$.

Very recently, Yang and Chu [18] presented that $p=4\log 2/(4+2\log 2-\pi)$ and $q=4/3$ are the best possible parameters such that the double inequality

$$M(a,b;p) < B(a,b) < M(a,b;q) \tag{1.8}$$

holds for all $a,b>0$ and $a\neq b$.

Let

$$L_4(a,b) = L^{1/4}\left(a^4,b^4\right) = \left(\frac{b^4-a^4}{4(\log b-\log a)}\right)^{1/4} \tag{1.9}$$

and

$$P_2(a,b) = P^{1/2}\left(a^2,b^2\right) = \left(\frac{b^2-a^2}{2\arcsin(\frac{b^2-a^2}{b^2+a^2})}\right)^{1/2} \tag{1.10}$$

be, respectively, the fourth-order logarithmic and second-order first Seiffert means of a and b.

Then from (1.4)-(1.10) we clearly see that $M(a,b;4/3)$ is the common sharp upper power mean bound for $L_4(a,b)$, $U(a,b)$, $P_2(a,b)$, $NS(a,b)$, and $B(a,b)$. Therefore, it is natural to ask what are the size relationships among these means? The main purpose of this paper is to answer this question.

2 Lemmas

In order to prove our main results we need several lemmas, which we present in this section.

Lemma 2.1 (See Lemma 7 of [19]) *Let $\{a_k\}_{k=0}^{\infty}$ be a nonnegative real sequence with $a_m>0$ and $\sum_{k=m+1}^{\infty}a_k>0$, and*

$$P(t) = -\sum_{k=0}^{m}a_kt^k + \sum_{k=m+1}^{\infty}a_kt^k$$

be a convergent power series on the interval $(0,\infty)$. *Then there exists* $t_{m+1} \in (0,\infty)$ *such that* $P(t_{m+1}) = 0$, $P(t) < 0$ *for* $t \in (0, t_{m+1})$ *and* $P(t) > 0$ *for* $t \in (t_{m+1}, \infty)$.

Lemma 2.2 *Let* $n \in \mathbb{N}$. *Then*

$$9(n-3)4^{2n} - 8n(4n-11)3^{2n} + 72n(n-1)2^{2n} - 72n(20n-13) > 0$$

for all $n \geq 6$.

Proof Let

$$v_n = 9(n-3)4^{2n} - 8n(4n-11)3^{2n} + 72n(n-1)2^{2n} - 72n(20n-13), \tag{2.1}$$

$$v_n^* = 9 \times \left(\frac{4}{3}\right)^{2n} - \frac{8n(4n-11)}{n-3}.$$

Then we clearly see that

$$v_6^* = \frac{4{,}495{,}024}{59{,}049} > 0, \tag{2.2}$$

$$\begin{aligned} v_n &\geq 9(n-3)4^{2n} - 8n(4n-11)3^{2n} + 72n(n-1)2^{12} - 72n(20n-13) \\ &= 9(n-3)4^{2n} - 8n(4n-11)3^{2n} + 72n(4{,}076n - 4{,}083) \\ &> 9(n-3)4^{2n} - 8n(4n-11)3^{2n} \\ &= (n-3)3^{2n} \times v_n^*, \end{aligned} \tag{2.3}$$

$$v_{n+1}^* - \left(\frac{4}{3}\right)^2 v_n^* = \frac{8(28n^3 - 169n^2 + 334n - 189)}{9(n-2)(n-3)} > 0 \tag{2.4}$$

for $n \geq 6$.

It follows from (2.2) and (2.4) that

$$v_n^* > 0 \tag{2.5}$$

for $n \geq 6$.

Therefore, Lemma 2.2 follows easily from (2.1), (2.3), and (2.5). □

Lemma 2.3 *Let* $t > 0$ *and*

$$g_1(t) = \frac{\sqrt{2}}{2} \arctan\bigl(\sqrt{2}\sinh(t)\bigr) - \frac{4t\sinh^2(2t)}{\sinh(4t)\sinh(t) + 4t\sinh(3t)}. \tag{2.6}$$

Then there exists a unique $t_0 \in (0,\infty)$ *such that* $g_1(t) < 0$ *for* $t \in (0, t_0)$, $g_1(t_0) = 0$, *and* $g_1(t) > 0$ *for* $t \in (t_0, \infty)$.

Proof It follows from (2.6) that

$$g_1(0^+) = 0, \qquad \lim_{t\to\infty} g_1(t) = \frac{\sqrt{2}}{4}\pi > 0, \tag{2.7}$$

$$g_1(t) = \frac{\sqrt{2}}{2}\arctan\bigl(\sqrt{2}\sinh(t)\bigr) - \frac{16t\sinh(t)\cosh^2(t)}{\sinh(4t)+16t\cosh^2(t)-4t},$$
$$g_1'(t) = \frac{\cosh(t)}{(1+2\sinh^2(t))(\sinh(4t)+16t\cosh^2(t)-4t)^2}g_2(t), \tag{2.8}$$

where

$$\begin{aligned} g_2(t) ={}& t^2\bigl[128\cosh^2(t)\sinh^2(t) - 512\cosh^4(t)\sinh^2(t) - 64\cosh^2(t) + 256\sinh^4(t) \\ & + 128\sinh^2(t) + 16\bigr] + t\bigl[16\sinh(4t)\cosh^2(t) - 32\sinh(4t)\cosh^2(t)\sinh^2(t) \\ & + 128\cosh(4t)\cosh(t)\sinh^3(t) + 64\cosh(4t)\cosh(t)\sinh(t) \\ & - 64\sinh(4t)\sinh^4(t) - 32\sinh(4t)\sinh^2(t) - 8\sinh(4t)\bigr] \\ & + \sinh^2(4t) - 32\cosh(t)\sinh(4t)\sinh^3(t) - 16\cosh(t)\sinh(4t)\sinh(t) \\ ={}& -\frac{3}{2}\cosh(8t) + 2t\sinh(8t) - 16t^2\cosh(6t) + 12t\sinh(6t) + 16t^2\cosh(4t) \\ & - 4t\sinh(4t) - 80t^2\cosh(2t) + 12t\sinh(2t) + 32t^2 + \frac{3}{2}. \end{aligned} \tag{2.9}$$

Making use of power series formulas, (2.9) gives

$$g_2(t) = \sum_{n=2}^{\infty}\frac{v_n}{18\times(2n)!}(2t)^{2n}, \tag{2.10}$$

where v_n is defined by (2.1).

Note that

$$v_2 = v_3 = 0, \qquad v_4 = -258{,}048, \qquad v_5 = -940{,}032. \tag{2.11}$$

From Lemma 2.1, (2.8), (2.10), and (2.11) we know that there exists $t_1 \in (0,\infty)$ such that $g_1(t)$ is strictly decreasing on $(0,t_1]$ and strictly increasing on $[t_1,\infty)$.

Therefore, Lemma 2.3 follows easily from (2.7) and the piecewise monotonicity of $g_1(t)$. □

Lemma 2.4 *The inequality*

$$-2x^2\cos x + \sin^2 x\cos x + 2x^2\cos^2 x + x\sin x + x^2 - 3x\cos x\sin x > 0$$

holds for all $x \in (0,\pi/2)$.

Proof Simple computations lead to

$$\begin{aligned} & -2x^2\cos x + \sin^2 x\cos x + 2x^2\cos^2 x + x\sin x + x^2 - 3x\cos x\sin x \\ &= x^2\cos(2x) - 2x^2\cos x + \frac{1}{4}\cos x - \frac{1}{4}\cos(3x) + x\sin x - \frac{3}{2}x\sin(2x) + 2x^2 \\ &= \sum_{n=2}^{\infty}(-1)^{n-1}\frac{3^{2n} + 4n(n-2)2^{2n} - 32n^2 + 24n - 1}{4\times(2n)!}x^{2n}. \end{aligned} \tag{2.12}$$

Let

$$\omega_n = \frac{3^{2n} + 4n(n-2)2^{2n} - 32n^2 + 24n - 1}{4 \times (2n)!} x^{2n}, \tag{2.13}$$

$$\omega_n^* = 3^{2n} + 4n(n-2)2^{2n} - 32n^2 + 24n - 1. \tag{2.14}$$

Then

$$\omega_2 = 0, \qquad \omega_3 = \frac{4x^6}{9} > 0, \tag{2.15}$$

$$\omega_n^* > 4n(n-2)2^6 - 32n^2 + 24n = 8n(28n - 61) > 0 \quad (n \geq 3), \tag{2.16}$$

$$\omega_{n+1}^* - 9\omega_n^* = -\left(5n^2 - 18n + 4\right)4^{n+1} + 256n(n-1) < 0 \quad (n \geq 4), \tag{2.17}$$

$$\frac{\omega_4}{\omega_3} = \frac{x^2}{56}\frac{\omega_4^*}{\omega_3^*} = \frac{x^2}{56} \times \frac{14{,}336}{1{,}280} = \frac{x^2}{5} < \frac{\pi^2}{20}. \tag{2.18}$$

It follows from (2.13), (2.14), (2.16), and (2.17) that

$$\omega_n > 0 \quad (n \geq 3), \tag{2.19}$$

$$\frac{\omega_{n+1}}{\omega_n} = \frac{x^2}{(2n+1)(2n+2)}\frac{\omega_{n+1}^*}{\omega_n^*} < \frac{9x^2}{(2n+1)(2n+2)} < \frac{\pi^2}{40} \quad (n \geq 4). \tag{2.20}$$

Inequalities (2.18)-(2.20) imply that the sequence $\{\omega_n\}$ is strictly decreasing for $n \geq 3$, $\lim_{n\to\infty} \omega_n = 0$ and $\sum_{n=2}^{\infty}(-1)^{n-1}\omega_n$ is a Leibniz series. Therefore, Lemma 2.4 follows from (2.12), (2.13), and (2.15). □

Lemma 2.5 *The inequality*

$$\frac{\sqrt{2}\sinh(2t)\cosh(t)\arcsin(\tanh(2t))}{\sinh(2t) + \cosh(2t)\arcsin(\tanh(2t))} - \arctan\left(\sqrt{2}\sinh(t)\right) > 0$$

hold for all $t \in (0, \infty)$.

Proof Let $x = \arcsin(\tanh(2t)) \in (0, \pi/2)$ and

$$h_1(t) = \frac{\sqrt{2}\sinh(2t)\cosh(t)\arcsin(\tanh(2t))}{\sinh(2t) + \cosh(2t)\arcsin(\tanh(2t))} - \arctan\left(\sqrt{2}\sinh(t)\right). \tag{2.21}$$

Then

$$\begin{aligned}
&\sinh(2t) = \tan x, \qquad \cosh(2t) = \frac{1}{\cos x}, \qquad \tanh(t) = \frac{1-\cos x}{\sin x}, \\
&h_1\left(0^+\right) = 0, \qquad (2.22) \\
&h_1(t) = \frac{\sqrt{2}x\sin x}{x + \sin x}\cosh(t) - \arctan\left(\sqrt{2}\sinh(t)\right), \\
&h_1'(t) = \frac{d}{dx}\left(\frac{\sqrt{2}x\sin x}{x+\sin x}\right)\frac{d[\arcsin(\tanh(2t))]}{dt}\cosh(t) + \frac{\sqrt{2}x\sin x}{x+\sin x}\sinh(t) - \frac{\sqrt{2}\cosh(t)}{\cosh(2t)} \\
&\qquad = \frac{\sqrt{2}\cosh(t)[2x^2\cos x - 2x\sin x - x^2 + \sin^2 x]}{(x+\sin x)^2\cosh(2t)} + \frac{\sqrt{2}x\sin x}{x+\sin x}\sinh(t)
\end{aligned}$$

$$= \sqrt{2}\cosh(t)\left[\frac{\cos x(2x^2\cos x - 2x\sin x - x^2 + \sin^2 x)}{(x+\sin x)^2} + \frac{x\sin x(1-\cos x)}{\sin x(x+\sin x)}\right]$$
$$= \frac{\sqrt{2}\cosh(t)[-2x^2\cos x + \sin^2 x\cos x + 2x^2\cos^2 x + x\sin x + x^2 - 3x\cos x\sin x]}{(x+\sin x)^2}. \quad (2.23)$$

Therefore, Lemma 2.5 follows easily from (2.21)-(2.23) and Lemma 2.4. □

3 Main results

Theorem 3.1 *The double inequality*

$$\lambda_1 L_4(a,b) \le U(a,b) < \mu_1 L_4(a,b)$$

holds for all $a, b > 0$ *with* $a \neq b$ *if and only if* $\lambda_1 \le c_0$ *and* $\mu_1 = \infty$, *where*

$$c_0 = e^{\log(\sinh(t_0)) - \log(\arctan(\sqrt{2}\sinh(t_0))) - \log(\sinh(4t_0)/t_0)/4 + \log 2}$$

and $t_0 \in (0,\infty)$ *is defined by Lemma* 2.3. *Moreover, numerical computations show that* $t_0 = 1.1336\ldots$ *and* $c_0 = 0.9991\ldots$.

Proof Since $U(a,b)$ and $L_4(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $b > a > 0$. Let $t = \log\sqrt{b/a} > 0$, then (1.2) and (1.9) lead to

$$U(a,b) = \frac{\sqrt{2ab}\sinh(t)}{\arctan(\sqrt{2}\sinh(t))}, \qquad L_4(a,b) = \sqrt{ab}\left(\frac{\sinh(4t)}{4t}\right)^{1/4}, \quad (3.1)$$

$$\log\frac{U(a,b)}{L_4(a,b)} = \log\bigl(\sinh(t)\bigr) - \log\bigl(\arctan\bigl(\sqrt{2}\sinh(t)\bigr)\bigr) - \frac{1}{4}\log\bigl(\sinh(4t)\bigr) + \frac{1}{4}\log t + \log 2. \quad (3.2)$$

Let

$$g(t) = \log\bigl(\sinh(t)\bigr) - \log\bigl(\arctan\bigl(\sqrt{2}\sinh(t)\bigr)\bigr) - \frac{1}{4}\log\bigl(\sinh(4t)\bigr) + \frac{1}{4}\log t + \log 2. \quad (3.3)$$

Then

$$g\bigl(0^+\bigr) = 0, \qquad \lim_{t\to\infty} g(t) = \infty, \quad (3.4)$$

$$g'(t) = \frac{\cosh(t)}{\sinh(t)} - \frac{\sqrt{2}\cosh(t)}{\arctan(\sqrt{2}\sinh(t))\cosh(2t)} - \frac{\cosh(4t)}{\sinh(4t)} + \frac{1}{4t}$$
$$= \frac{\sinh(4t)\sinh(t) + 4t\sinh(3t)}{4t\sinh(4t)\sinh(t)} - \frac{\sqrt{2}\cosh(t)}{\arctan(\sqrt{2}\sinh(t))\cosh(2t)}$$
$$= \frac{\sqrt{2}(\sinh(4t)\sinh(t) + 4t\sinh(3t))}{4t\sinh(4t)\sinh(t)\arctan(\sqrt{2}\sinh(t))} g_1(t), \quad (3.5)$$

where $g_1(t)$ is defined by (2.6).

It follows from Lemma 2.3 and (3.5) that there exists a unique $t_0 \in (0,\infty)$ such that $g_1(t_0) = 0$, $g(t)$ is strictly decreasing on $(0, t_0]$ and strictly increasing on $[t_0,\infty)$.

Therefore, Theorem 3.1 follows from (3.2)-(3.4) and the piecewise monotonicity of $g(t)$. □

Theorem 3.2 *The double inequality*

$$\lambda_2 U(a,b) < P_2(a,b) < \mu_2 U(a,b)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\lambda_2 \leq 1$ and $\mu_2 \geq \sqrt{\pi/2} = 1.2533\ldots$.

Proof Since $U(a,b)$ and $P_2(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $b > a > 0$. Let $t = \log\sqrt{b/a} > 0$, then (1.10) and (3.1) lead to

$$P_2(a,b) = \sqrt{ab}\left(\frac{\sinh(2t)}{\arcsin(\tanh(2t))}\right)^{1/2}, \tag{3.6}$$

$$\log\frac{P_2(a,b)}{U(a,b)} = \log\bigl(\arctan\bigl(\sqrt{2}\sinh(t)\bigr)\bigr) - \frac{1}{2}\log\bigl(\arcsin\bigl(\tanh(2t)\bigr)\bigr) - \frac{1}{2}\log\bigl(\tanh(t)\bigr). \tag{3.7}$$

Let

$$h(t) = \log\bigl(\arctan\bigl(\sqrt{2}\sinh(t)\bigr)\bigr) - \frac{1}{2}\log\bigl(\arcsin\bigl(\tanh(2t)\bigr)\bigr) - \frac{1}{2}\log\bigl(\tanh(t)\bigr). \tag{3.8}$$

Then simple computations lead to

$$h\bigl(0^+\bigr) = 0, \qquad \lim_{t\to\infty} h(t) = \frac{1}{2}(\log\pi - \log 2), \tag{3.9}$$

$$\begin{aligned} h'(t) &= \frac{\sqrt{2}\cosh(t)}{\cosh(2t)\arctan(\sqrt{2}\sinh(t))} - \frac{1}{\cosh(2t)\arcsin(\tanh(2t))} - \frac{1}{\sinh(2t)} \\ &= \frac{\sinh(2t) + \cosh(2t)\arcsin(\tanh(2t))}{\sinh(2t)\cosh(2t)\arcsin(\tanh(2t))\arctan(\sqrt{2}\sinh(t))} \\ &\quad \times \left[\frac{\sqrt{2}\sinh(2t)\cosh(t)\arcsin(\tanh(2t))}{\sinh(2t) + \cosh(2t)\arcsin(\tanh(2t))} - \arctan\bigl(\sqrt{2}\sinh(t)\bigr)\right]. \end{aligned} \tag{3.10}$$

It follows from Lemma 2.5 and (3.10) that $h(t)$ is strictly increasing on $(0,\infty)$. Therefore, Theorem 3.2 follows easily from (3.7)-(3.9) and the monotonicity of $h(t)$. □

Remark 3.1 Let $b > a > 0$ and $t = \log\sqrt{b/a} > 0$. Then

$$A(a,b) = \sqrt{ab}\cosh(t), \qquad Q(a,b) = \sqrt{ab}\cosh^{1/2}(2t). \tag{3.11}$$

It follows from Lemma 2.5 that

$$\frac{\sqrt{2}\sinh(t)}{\arctan(\sqrt{2}\sinh(t))} > \frac{\frac{\sinh(2t)}{\arcsin(\tanh(2t))} + \cosh(2t)}{2\cosh^2(t)}. \tag{3.12}$$

Equations (3.1), (3.6), and (3.11) together with inequality (3.12) lead to the conclusion that the inequality

$$U(a,b) > \frac{P_2^2(a,b) + Q^2(a,b)}{2A^2(a,b)} G(a,b)$$

holds for all $a, b > 0$ with $a \neq b$.

Theorem 3.3 *The double inequality*

$$\lambda_3 P_2(a,b) < NS(a,b)(a,b) < \mu_3 P_2(a,b)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\lambda_3 \leq 1$ and $\mu_3 \geq \sqrt{\pi}/[2\log(1+\log 2)] = 1.0055\ldots$.

Proof Since $NS(a,b)$ and $P_2(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $b > a > 0$. Let $t = \log\sqrt{b/a} > 0$, then (1.2) and (3.6) lead to

$$NS(a,b) = \sqrt{ab}\frac{\sinh(t)}{\sinh^{-1}(\tanh(t))}, \tag{3.13}$$

$$\begin{aligned} \log\frac{NS(a,b)}{P_2(a,b)} &= \frac{1}{2}\log\bigl(\tanh(t)\bigr) - \log\bigl(\sinh^{-1}\bigl(\tanh(t)\bigr)\bigr) \\ &\quad + \frac{1}{2}\log\bigl(\arcsin\bigl(\tanh(2t)\bigr)\bigr) - \frac{1}{2}\log 2. \end{aligned} \tag{3.14}$$

Let

$$\begin{aligned} h_2(t) &= \frac{1}{2}\log\bigl(\tanh(t)\bigr) - \log\bigl(\sinh^{-1}\bigl(\tanh(t)\bigr)\bigr) \\ &\quad + \frac{1}{2}\log\bigl(\arcsin\bigl(\tanh(2t)\bigr)\bigr) - \frac{1}{2}\log 2. \end{aligned} \tag{3.15}$$

Then simple computations lead to

$$h_2\bigl(0^+\bigr) = 0, \qquad \lim_{t\to\infty} h_2(t) = \log\left(\frac{\sqrt{\pi}}{2\log(1+\sqrt{2})}\right), \tag{3.16}$$

$$\begin{aligned} h_2'(t) &= \frac{1}{\sinh(2t)} - \frac{1}{\cosh(t)\sqrt{\cosh(2t)}\sinh^{-1}(\tanh(t))} + \frac{1}{\cosh(2t)\arcsin(\tanh(2t))} \\ &= \frac{\sinh(2t)\cosh(t) + \cosh(t)\cosh(2t)\arcsin(\tanh(2t))}{\sinh(2t)\cosh(2t)\cosh(t)\sinh^{-1}(\tanh(t))\arcsin(\tanh(2t))} h_3(t), \end{aligned} \tag{3.17}$$

where

$$h_3(t) = \sinh^{-1}\bigl(\tanh(t)\bigr) - \frac{2\sqrt{\cosh(2t)}\sinh(t)\arcsin(\tanh(2t))}{\sinh(2t) + \cosh(2t)\arcsin(\tanh(2t))}, \tag{3.18}$$

$$h_3\bigl(0^+\bigr) = 0, \tag{3.19}$$

$$h_3'(t) = \frac{\sqrt{\cosh(2t)}[\arcsin(\tanh(2t)) - \tanh(2t)][\sinh(2t) - \arcsin(\tanh(2t))]}{\cosh(t)[\sinh(2t) + \cosh(2t)\arcsin(\tanh(2t))]^2}. \tag{3.20}$$

Let $x = \arcsin(\tanh(2t)) \in (0, \pi/2)$. Then

$$\arcsin\bigl(\tanh(2t)\bigr) - \tanh(2t) = x - \sin x > 0, \tag{3.21}$$

$$\sinh(2t) - \arcsin\bigl(\tanh(2t)\bigr) = \tan x - x > 0. \tag{3.22}$$

From (3.17)-(3.22) we clearly see that $h_2(t)$ is strictly increasing on $(0, \infty)$. Therefore, Theorem 3.3 follows from (3.14)-(3.16) and the monotonicity of $h_2(t)$. □

Remark 3.2 From the proof of Theorem 3.2 we know that

$$h_3(t) = \sinh^{-1}\bigl(\tanh(t)\bigr) - \frac{2\sqrt{\cosh(2t)}\sinh(t)\arcsin(\tanh(2t))}{\sinh(2t) + \cosh(2t)\arcsin(\tanh(2t))} > 0,$$

which is equivalent to

$$\frac{\frac{\sinh(2t)}{\arcsin(\tanh(2t))} + \cosh(2t)}{2\sqrt{\cosh(2t)}} > \frac{\sinh(t)}{\sinh^{-1}(\tanh(t))}. \tag{3.23}$$

Equations (3.6), (3.11), and (3.13) together with inequality (3.23) lead to the conclusion that the inequality

$$NS(a,b) < \frac{P_2^2(a,b) + Q^2(a,b)}{2Q(a,b)}$$

holds for all $a, b > 0$ with $a \neq b$.

Theorem 3.4 *The double inequality*

$$\lambda_4 NS(a,b) < B(a,b) < \mu_4 NS(a,b)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\lambda_4 \leq 1$ and $\mu_4 \geq \sqrt{2}e^{\pi/4-1}\log(1+\sqrt{2}) = 1.0057\ldots$.

Proof Since $NS(a,b)$ and $B(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $b > a > 0$. Let $t = \log\sqrt{b/a} > 0$, then (1.3) and (3.13) lead to

$$\begin{aligned} &B(a,b) = \sqrt{ab}\cosh^{1/2}(2t)e^{\arctan(\tanh(t))/\tanh(t)-1}, \\ &\log\frac{B(a,b)}{NS(a,b)} = \frac{1}{2}\log\bigl(\cosh(2t)\bigr) + \frac{\arctan(\tanh(t))}{\tanh(t)} - \log\left(\frac{\sinh(t)}{\sinh^{-1}(\tanh(t))}\right) - 1. \end{aligned} \tag{3.24}$$

Let

$$f(t) = \frac{1}{2}\log\bigl(\cosh(2t)\bigr) + \frac{\arctan(\tanh(t))}{\tanh(t)} - \log\left(\frac{\sinh(t)}{\sinh^{-1}(\tanh(t))}\right) - 1. \tag{3.25}$$

Then simple computations lead to

$$f(0^+) = 0, \qquad \lim_{t\to\infty} f(t) = \frac{\pi}{4} - 1 + \frac{1}{2}\log 2 + \log\bigl[\log(1+\sqrt{2})\bigr], \tag{3.26}$$

$$f'(t) = \frac{f_1(t)}{\sinh^2(t)\sinh^{-1}(\tanh(t))}, \tag{3.27}$$

where

$$f_1(t) = \frac{\sinh^2(t)}{\sqrt{\cosh(2t)}\cosh(t)} - \sinh^{-1}\big(\tanh(t)\big)\arctan\big(\tanh(t)\big).$$

Let $x = \tanh(t) \in (0,1)$. Then

$$f_1(t) = \frac{x^2}{\sqrt{1+x^2}} - \sinh^{-1}(x)\arctan(x) := f_2(x), \tag{3.28}$$

$$f_2\big(0^+\big) = 0, \tag{3.29}$$

$$f_2'(x) = \frac{1}{\sqrt{x^2+1}}\left[x\frac{x^2+2}{x^2+1} - \frac{\sinh^{-1}(x)}{\sqrt{x^2+1}} - \arctan(x)\right] := \frac{f_3(x)}{\sqrt{x^2+1}}, \tag{3.30}$$

$$f_3\big(0^+\big) = 0, \tag{3.31}$$

$$f_3'(x) = \frac{x}{(x^2+1)^{3/2}}\left[\sinh^{-1}(x) - \frac{x-x^3}{\sqrt{x^2+1}}\right] := \frac{x}{(x^2+1)^{3/2}}f_4(x), \tag{3.32}$$

$$f_4\big(0^+\big) = 0, \tag{3.33}$$

$$f_4'(x) = \frac{2x^2(x^2+2)}{(x^2+1)^{3/2}} > 0 \tag{3.34}$$

for $x \in (0,1)$.

It follows from (3.27)-(3.34) that $f(t)$ is strictly increasing on $(0,\infty)$. Therefore, Theorem 3.4 follows easily from (3.24)-(3.26) and the monotonicity of $f(t)$. □

Remark 3.3 From the proof of Theorem 3.4 we know that the inequalities

$$\frac{x^2}{\sqrt{1+x^2}} > \sinh^{-1}(x)\arctan(x), \tag{3.35}$$

$$x\frac{x^2+2}{x^2+1} > \frac{\sinh^{-1}(x)}{\sqrt{x^2+1}} + \arctan(x), \tag{3.36}$$

$$\sinh^{-1}(x) > \frac{x-x^3}{\sqrt{x^2+1}} \tag{3.37}$$

hold for all $x \in (0,\infty)$. Inequalities (3.35)-(3.37) lead to the conclusion that the inequalities

$$NS(a,b)T(a,b) > A(a,b)Q(a,b),$$

$$\frac{A^2(a,b)}{G^2(a,b)} > \frac{NS(a,b)}{Q(a,b)},$$

$$\frac{A(a,b)}{Q(a,b)} + \frac{Q(a,b)}{A(a,b)} > \frac{A(a,b)}{NS(a,b)} + \frac{Q(a,b)}{T(a,b)}$$

hold for all $a, b > 0$ with $a \neq b$.

Remark 3.4 Let $I(a,b) = (b^b/a^a)^{1/(b-a)}/e$ be the identric mean of two distinct positive real numbers a and b, and $I_2(a,b) = I^{1/2}(a^2,b^2)$ be the second-order identric mean. Then from Theorems 3.1-3.4 and the inequalities $M(a,b;2/3) < I(a,b) < M(a,b;\log 2)$ [20, 21] and

$P_2(a,b) > L_4(a,b)$ [22] we get two inequalities chains as follows:

$$\begin{aligned}\frac{999}{1{,}000}L_4(a,b) &< U(a,b) < P_2(a,b) < NS(a,b)\\ &< B(a,b) < M(a,b;4/3) < I_2(a,b)\\ &< M(a,b;2\log 2)\end{aligned}$$

and

$$\begin{aligned}L_4(a,b) &< P_2(a,b) < NS(a,b) < B(a,b)\\ &< M(a,b;4/3) < I_2(a,b)\\ &< M(a,b;2\log 2)\end{aligned}$$

for all $a,b > 0$ with $a \neq b$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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