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An inequality for warped product pseudo-slant submanifolds of nearly cosymplectic manifolds

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Abstract

Warped product submanifolds of nearly cosymplectic manifolds were studied in Uddin *et al.* (Math. Probl. Eng. 2011, doi:10.1155/2011/230374), Uddin and Khan (J. Inequal. Appl. 2012:304, 2012) and Uddin *et al.* (Rev. Unión Mat. Argent. 55:55-69, 2014). In this paper, we study warped product submanifolds of nearly cosymplectic manifolds in which the base manifold is slant and thus we derive a sharp relation for the squared norm of the second fundamental form. The equality case is also considered.

MSC: 53C40; 53C42; 53C15

Keywords: slant submanifold; pseudo-slant submanifold; nearly cosymplectic manifold; warped products

1 Introduction

The almost contact manifolds with Killing structures tensors were defined in [1] as nearly cosymplectic manifolds. Later, these manifolds were studied by Blair and Showers from the topological point of view [2]. A totally geodesic hypersurface S^5 of a 6-dimensional sphere S^6 is a nearly cosymplectic manifold. A normal nearly cosymplectic manifold is cosymplectic (see [3]).

On the other hand, pseudo-slant submanifolds of almost contact metric manifolds were studied by Carriazo [4] under the name of anti-slant submanifolds. Later on, Sahin studied these submanifolds for their warped products [5].

Recently, Uddin *et al.* studied warped product semi-invariant and semi-slant submanifolds of nearly cosymplectic manifolds [6–8]. In this paper, we study the warped product pseudo-slant submanifolds of the type $N_\theta \times {}_f N_\perp$ of a nearly cosymplectic manifold, where $N_\perp$ and N_θ are anti-invariant and proper slant submanifolds of a nearly cosymplectic manifold, respectively. We derive an inequality for the second fundamental form of such warped product immersions in terms of the warping function and the slant angle. The equality case is also discussed.

2 Preliminaries

Let $\widetilde{M}$ be a $(2n+1)$-dimensional C^∞ manifold with *almost contact structure* (φ, ξ, η) *i.e.*, a $(1,1)$ tensor field φ, a vector field ξ and a 1-form η on $\widetilde{M}$ such that

$$\varphi^2 = -I + \eta \otimes \xi, \qquad \varphi\xi = 0, \qquad \eta \circ \varphi = 0, \qquad \eta(\xi) = 1. \tag{2.1}$$

There always exists a Riemannian metric g on an almost contact manifold $\widetilde{M}$ satisfying the following compatibility condition:

$$\eta(X) = g(X, \xi), \qquad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

where X and Y are vector fields on $\widetilde{M}$ [2].

An almost contact structure (φ, ξ, η) is said to be *nearly cosymplectic* if φ is Killing, *i.e.*, if

$$(\widetilde{\nabla}_X \varphi)Y + (\widetilde{\nabla}_Y \varphi)X = 0, \tag{2.3}$$

or any X, Y tangent to $\widetilde{M}$, where $\widetilde{\nabla}$ denotes the Riemannian connection of the metric g. Equation (2.3) is equivalent to $(\widetilde{\nabla}_X \varphi)X = 0$, for each X tangent to $\widetilde{M}$. A normal nearly cosymplectic structure is cosymplectic. It is well known that an almost contact metric manifold is *cosymplectic* if and only if $\widetilde{\nabla}\varphi$ vanishes identically, *i.e.*, $(\widetilde{\nabla}_X \varphi)Y = 0$ and $\widetilde{\nabla}_X \xi = 0$.

On a nearly cosymplectic manifold the structure vector field ξ is Killing [9], that is,

$$g(\widetilde{\nabla}_Y \xi, Z) + g(\widetilde{\nabla}_Z \xi, Y) = 0 \tag{2.4}$$

for any Y, Z tangent to $\widetilde{M}$.

Let M be submanifold of an almost contact metric manifold $\widetilde{M}$ with induced metric g and let ∇ and $\nabla^\perp$ be the induced connections on the tangent bundle TM and the normal bundle $T^\perp M$ of M, respectively. Denote by $\mathcal{F}(M)$ the algebra of smooth functions on M and by $\Gamma(TM)$ the $\mathcal{F}(M)$-module of smooth sections of TM over M. Then the Gauss and Weingarten formulas are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.5}$$

$$\widetilde{\nabla}_X N = -A_N X + \nabla^\perp_X N, \tag{2.6}$$

for each $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, where h and A_N are the second fundamental form and the shape operator (corresponding to the normal vector field N), respectively, for the immersion of M into $\widetilde{M}$. They are related as

$$g\big(h(X, Y), N\big) = g(A_N X, Y), \tag{2.7}$$

where g denotes the Riemannian metric on $\widetilde{M}$ as well as the one induced on M. The mean curvature vector H of M is given by $H = \frac{1}{m}\sum_{i=1}^{m} h(e_i, e_i)$, where n is the dimension of M and $\{e_1, e_2, \ldots, e_m\}$ is a local orthonormal frame of vector fields on M. A submanifold M of an almost contact metric manifold $\widetilde{M}$ is said to be *totally umbilical* if the second fundamental form satisfies $h(X, Y) = g(X, Y)H$, for all $X, Y \in \Gamma(TM)$. The submanifold M is *totally geodesic* if $h(X, Y) = 0$, for all $X, Y \in \Gamma(TM)$ and minimal if $H = 0$.

Now, let $\{e_1, \dots, e_m\}$ be an orthonormal basis of tangent space TM and e_r belong to the orthonormal basis $\{e_{m+1}, \dots, e_{2n+1}\}$ of the normal bundle $T^\perp M$, we put

$$h^r_{ij} = g\big(h(e_i, e_j), e_r\big) \quad \text{and} \quad \|h\|^2 = \sum_{i,j=1}^{m} g\big(h(e_i, e_j), h(e_i, e_j)\big). \tag{2.8}$$

For a differentiable function φ on M, the gradient $\vec{\nabla}\varphi$ is defined by

$$g(\vec{\nabla}\varphi, X) = X\varphi \tag{2.9}$$

for any $X \in \Gamma(TM)$. As a consequence, we have

$$\|\vec{\nabla}\varphi\|^2 = \sum_{i=1}^{m} \big(e_i(\varphi)\big)^2. \tag{2.10}$$

For any $X \in \Gamma(TM)$, we write

$$\varphi X = PX + FX, \tag{2.11}$$

where PX is the tangential component and FX is the normal component of φX. A submanifold M of an almost contact metric manifold $\widetilde{M}$ is said to be *invariant* if F is identically zero, that is, $\varphi X \in \Gamma(TM)$ and *anti-invariant* if P is identically zero, that is, $\varphi X \in \Gamma(T^\perp M)$, for any $X \in \Gamma(TM)$.

Let M be a submanifold tangent to the structure vector field ξ isometrically immersed into an almost contact metric manifold $\widetilde{M}$. Then M is said to be a contact CR-submanifold if there exists a pair of orthogonal distributions $\mathcal{D} : p \to \mathcal{D}_p$ and $\mathcal{D}^\perp : p \to \mathcal{D}^\perp_p$, $\forall p \in M$ such that:

(i) $TM = \mathcal{D} \oplus \mathcal{D}^\perp \oplus \langle\xi\rangle$, where $\langle\xi\rangle$ is the 1-dimensional distribution spanned by the structure vector field ξ.
(ii) $\mathcal{D}$ is invariant, *i.e.*, $\varphi\mathcal{D} = \mathcal{D}$.
(iii) $\mathcal{D}^\perp$ is anti-invariant, *i.e.*, $\varphi\mathcal{D}^\perp \subseteq T^\perp M$.

Invariant and anti-invariant submanifolds are the special cases of a contact CR-submanifold. If we denote the dimensions of the distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ by d_1 and d_2, respectively. Then M is *invariant* (resp. *anti-invariant*) if $d_2 = 0$ (resp. $d_1 = 0$).

There is another class of submanifolds that is called the slant submanifold. For each non-zero vector X tangent to M at x, such that X is not proportional to ξ_x, we denote by $0 \leq \theta(X) \leq \frac{\pi}{2}$, the angle between φX and T_xM is called the *Wirtinger angle*. If the angle $\theta(X)$ is constant for all nonzero $X \in T_xM - \langle\xi_x\rangle$ and $x \in M$, then M is said to be a *slant* submanifold [10] and the angle θ is the slant angle of M. Obviously if $\theta = 0$, M is invariant and if $\theta = \frac{\pi}{2}$, M is an anti-invariant submanifold. A slant submanifold is said to be *proper slant* if it is neither invariant nor anti-invariant.

We recall the following result for a slant submanifold of an almost contact metric manifold.

Theorem 2.1 [10] *Let M be a submanifold of an almost contact metric manifold $\widetilde{M}$, such that ξ is tangent to M. Then M is slant if and only if there exists a constant $\lambda \in [0,1]$ such* *that*

$$P^2 = \lambda(-I + \eta \otimes \xi). \tag{2.12}$$

Furthermore, if θ is slant angle of M, then $\lambda = \cos^2\theta$.

The following relations are straightforward consequences of (2.12):

$$g(PX, PY) = \cos^2\theta\big(g(X, Y) - \eta(Y)\eta(X)\big), \tag{2.13}$$

$$g(FX, FY) = \sin^2\theta\big(g(X, Y) - \eta(Y)\eta(X)\big), \tag{2.14}$$

for all $X, Y \in \Gamma(TM)$.

Now, we give the brief introduction of pseudo-slant submanifolds introduced by Carriazo in [4] under the name of *anti-slant submanifolds*, which are the generalization of contact CR-submanifolds and slant submanifolds [10]. He defined these submanifolds as follows.

Definition 2.1 A submanifold M of an almost contact metric manifold $\widetilde{M}$ is said to be a pseudo-slant submanifold if there exists a pair of orthogonal distributions $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$ on M such that:

(i) TM admits the orthogonal direct decomposition $TM = \mathcal{D}^\perp \oplus \mathcal{D}^\theta \oplus \langle\xi\rangle$.

(ii) The distribution $\mathcal{D}^\perp$ is anti-invariant, *i.e.*, $\varphi(\mathcal{D}^\perp) \subset T^\perp M$.

(iii) The distribution $\mathcal{D}^\theta$ is slant with angle $\theta \neq \frac{\pi}{2}$.

The normal bundle $T^\perp M$ of a pseudo-slant submanifold is decomposed as

$$T^\perp M = F\mathcal{D}^\theta \oplus \varphi\mathcal{D}^\perp \oplus \mu, \tag{2.15}$$

where μ is an invariant normal subbundle under φ.

3 Warped product pseudo-slant submanifolds

In this section, we discuss the warped product submanifolds of a nearly cosymplectic manifold. These manifolds were studied by Bishop and O'Neill [11]. They defined these manifolds as follows: Let (N_1, g_1) and (N_2, g_2) be two Riemannian manifolds and f a positive differentiable function on N_1. Then their warped product $M = N_1 \times_f N_2$ is the product manifold $N_1 \times N_2$ equipped with the Riemannian structure such that

$$g = g_1 + f^2 g_2.$$

The function f is called the warping function on M. It was proved in [11] that for any $X \in \Gamma(TN_1)$ and $Z \in \Gamma(TN_2)$, the following holds:

$$\nabla_X Z = \nabla_Z X = (X \ln f)Z, \tag{3.1}$$

where ∇ denote the Levi-Civita connection M. A warped product manifold $M = N_1 \times_f N_2$ is said to be *trivial* if the warping function f is constant. If $M = N_1 \times_f N_2$ is a warped product manifold then the base manifold N_1 is totally geodesic and the fiber N_2 is a totally umbilical submanifold of M, respectively [11].

Now, we discuss the warped product pseudo-slant submanifolds of the type $N_\theta \times_f N_\perp$ of a nearly cosymplectic manifold $\widetilde{M}$. We consider the structure vector field ξ tangent to the base manifold N_θ of the warped products. If ξ is tangential to $N_\perp$ then the warped product is trivial [6]. We have the following results for later use.

Lemma 3.1 *Let $M = N_\theta \times_f N_\perp$ be a warped product pseudo-slant submanifold of a nearly cosymplectic manifold $\widetilde{M}$, then*:

(i) $\xi \ln f = 0$,

(ii) $2g(h(X,Y),\varphi Z) = g(h(X,Z),FY) + g(h(Y,Z),FX)$.

for any $X, Y \in \Gamma(TN_\theta)$ and $Z \in \Gamma(TN_\perp)$.

Proof For any $Z, W \in \Gamma(TN_\perp)$ and ξ tangential to N_θ, we have

$$g(\widetilde{\nabla}_Z \xi, W) = g(\nabla_Z \xi, W).$$

Then from (3.1), we obtain

$$g(\widetilde{\nabla}_Z \xi, W) = \xi \ln f g(Z, W). \tag{3.2}$$

By the polarization identity, we derive

$$g(\widetilde{\nabla}_W \xi, Z) = \xi \ln f g(Z, W). \tag{3.3}$$

Thus the first part follows from (3.2) and (3.3) by using (2.4). For the second part, consider $X, Y \in \Gamma(TN_\theta)$ and $Z \in \Gamma(TN_\perp)$, we have

$$g\big(h(X,Y),\varphi Z\big) = g(\widetilde{\nabla}_X Y, \varphi Z) = -g(\varphi\widetilde{\nabla}_X Y, Z).$$

Then by the covariant derivative property of φ, we derive

$$\begin{aligned} g\big(h(X,Y),\varphi Z\big) &= g\big((\widetilde{\nabla}_X\varphi)Y,Z\big) - g(\widetilde{\nabla}_X\varphi Y,Z) \\ &= g\big((\widetilde{\nabla}_X\varphi)Y,Z\big) - g(\widetilde{\nabla}_X PY,Z) - g(\widetilde{\nabla}_X FY,Z) \\ &= g\big((\widetilde{\nabla}_X\varphi)Y,Z\big) + g(PY,\widetilde{\nabla}_X Z) + g(A_{FY}X,Z). \end{aligned}$$

Using (2.5) and (2.7), we get

$$g\big(h(X,Y),\varphi Z\big) = g\big((\widetilde{\nabla}_X\varphi)Y,Z\big) + g(PY,\nabla_X Z) + g\big(h(X,Z),FY\big).$$

From (3.1), we obtain

$$g\big(h(X,Y),\varphi Z\big) = g\big((\widetilde{\nabla}_X\varphi)Y,Z\big) + (X\ln f)g(PY,Z) + g\big(h(X,Z),FY\big).$$

The second term of right hand side is identically zero by the orthogonality of vector fields, thus we have

$$g\big(h(X,Y),\varphi Z\big) = g\big((\widetilde{\nabla}_X\varphi)Y,Z\big) + g\big(h(X,Z),FY\big). \tag{3.4}$$

Then by the polarization identity, we obtain

$$g\big(h(X,Y),\varphi Z\big)=g\big((\widetilde{\nabla}_Y\varphi)X,Z\big)+g\big(h(Y,Z),FX\big). \tag{3.5}$$

Then from (3.4) and (3.5), we get

$$2g\big(h(X,Y),\varphi Z\big)=g\big((\widetilde{\nabla}_X\varphi)Y+(\widetilde{\nabla}_Y\varphi)X,Z\big)+g\big(h(X,Z),FY\big)+g\big(h(Y,Z),FX\big).$$

The first term of right hand side is identically zero by (2.3), thus we get (ii), which proves the lemma completely. □

Lemma 3.2 *Let $M=N_\theta\times_f N_\perp$ be a warped product pseudo-slant submanifold of a nearly cosymplectic manifold $\widetilde{M}$, where $N_\perp$ and N_θ are anti-invariant and proper slant submanifolds of $\widetilde{M}$, respectively. Then*:

(i) $2g(h(Z,W),FX)=g(h(X,Z),\varphi W)+g(h(X,W),\varphi Z)+2(PX\ln f)g(Z,W)$,

(ii) $2g(h(Z,W),FPX)=g(h(PX,Z),\varphi W)+g(h(PX,W),\varphi Z)-2\cos^2\theta(X\ln f)g(Z,W)$

for any $X\in\Gamma(TN_\theta)$ and $Z,W\in\Gamma(TN_\perp)$.

Proof For any $Z,W\in\Gamma(TN_\perp)$ and $X\in\Gamma(TN_\theta)$, we have

$$\begin{aligned}g\big(h(Z,W),FX\big)&=g(\widetilde{\nabla}_Z W,\varphi X)-g(\widetilde{\nabla}_Z W,PX)\\&=g\big((\widetilde{\nabla}_Z\varphi)W,X\big)-g(\widetilde{\nabla}_Z\varphi W,X)+g(W,\widetilde{\nabla}_Z PX).\end{aligned}$$

Using (2.5) and (2.6), we obtain

$$g\big(h(Z,W),FX\big)=g\big((\widetilde{\nabla}_Z\varphi)W,X\big)+g(A_{\varphi W}Z,X)+g(W,\nabla_Z PX).$$

Then from (2.7) and (3.1), we get

$$g\big(h(Z,W),FX\big)=g\big((\widetilde{\nabla}_Z\varphi)W,X\big)+g\big(h(X,Z),\varphi W\big)+(PX\ln f)g(Z,W). \tag{3.6}$$

Then by the polarization identity we derive

$$g\big(h(Z,W),FX\big)=g\big((\widetilde{\nabla}_W\varphi)Z,X\big)+g\big(h(X,W),\varphi Z\big)+(PX\ln f)g(Z,W). \tag{3.7}$$

From (3.6) and (3.7), we get

$$\begin{aligned}2g\big(h(Z,W),FX\big)&=g\big((\widetilde{\nabla}_W\varphi)Z+(\widetilde{\nabla}_Z\varphi)W,X\big)+g\big(h(X,Z),\varphi W\big)\\&\quad+g\big(h(X,W),\varphi Z\big)+2(PX\ln f)g(Z,W).\end{aligned}$$

Then, from the above relation, (i) holds by using (2.3). If we interchange X by PX in (i) we get (ii) by using Theorem 2.1 and Lemma 3.1(i). Thus, the proof is complete. □

Now, we construct the following frame for a warped product pseudo-slant submanifold $M=N_\theta\times_f N_\perp$ of a $(2n+1)$-dimensional nearly cosymplectic manifold.

Let $M = N_\theta \times {}_f N_\perp$ be a m-dimensional warped product pseudo-slant submanifold of a $(2n+1)$-dimensional nearly cosymplectic manifold $\widetilde{M}$ such that $N_\perp$ is a q-dimensional anti-invariant submanifold and N_θ is a $(2p+1)$-dimensional slant submanifold tangent to the structure vector field ξ of $\widetilde{M}$, respectively. Then the orthonormal frame fields of the tangent spaces of $N_\perp$ and N_θ, respectively, are $\{e_1, \dots, e_q\}$ and $\{e_{q+1} = e_1^*, \dots, e_{q+p} = e_p^*, e_{q+p+1} = e_{p+1}^* = \sec\theta P e_1^*, \dots, e_{q+2p} = e_{2p}^* = \sec\theta P e_p^*, e_{q+2p+1} = e_m = \xi\}$. The orthonormal frames of $\varphi(TN_\perp)$, $F(TN_\theta)$, and μ, respectively, are $\{e_{m+1} = \varphi e_1, \dots, e_{m+q} = \varphi e_q\}$, $\{e_{m+q+1} = \tilde{e}_1 = \csc\theta F e_1^*, \dots, e_{m+p+q} = \tilde{e}_p^* = \csc\theta F e_p^*, e_{m+p+q+1} = \tilde{e}_{p+1} = \csc\theta\sec\theta FPe_1^*, \dots, e_{m+2p+q} = \tilde{e}_{2p} = \csc\theta\sec\theta FPe_p^*\}$ and $\{e_{2m}, \dots, e_{2n+1}\}$. The dimensions of $\varphi(TN_\perp)$, $F(TN_\theta)$, and μ, respectively, are q, $2p$, and $2(n-m+1)$.

Theorem 3.1 *Let $M = N_\theta \times {}_f N_\perp$ be a mixed geodesic warped product pseudo-slant submanifold of a nearly cosymplectic manifold $\widetilde{M}$ such that $N_\perp$ and N_θ are anti-invariant and proper slant submanifolds of $\widetilde{M}$, respectively. Then*:

(i) *The squared norm of the second fundamental form h of M satisfies*

$$\|h\|^2 \geq q\cot^2\theta \left\|\nabla^\theta \ln f\right\|^2$$

where $\nabla^\theta \ln f$ is the gradient of $\ln f$ over N_θ and q is the dimension of $N_\perp$.

(ii) *If the equality holds in* (i), *then $h(Z,W)$ lies in $F(TN_\theta)$ for any $Z, W \in \Gamma(TN_\perp)$ and $h(X,Y)$ lies in $\varphi(TN_\perp)$, for any $X, Y \in \Gamma(TN_\theta)$.*

Proof From (2.8), we have

$$\|h\|^2 = \sum_{i,j=1}^{m} g\big(h(e_i,e_j), h(e_i,e_j)\big) = \sum_{r=m+1}^{2n+1}\sum_{i,j=1}^{m} g\big(h(e_i,e_j), e_r\big)^2.$$

Then using the frame fields of $TN_\perp$ and TN_θ, we get

$$\begin{aligned}\|h\|^2 = {} & \sum_{r=m+1}^{2n+1}\sum_{i,j=1}^{2p+1} g\big(h(e_i,e_j), e_r\big)^2 + 2\sum_{r=m+1}^{2n+1}\sum_{i=1}^{2p+1}\sum_{j=1}^{q} g\big(h(e_i,e_j), e_r\big)^2 \\ & + \sum_{r=m+1}^{2n+1}\sum_{i,j=1}^{q} g\big(h(e_i^*,e_j^*), e_r\big)^2.\end{aligned}$$

Since M is mixed geodesic, the second term of right hand side is identically zero and break the above relation for the frames of $F(TN_\theta)$, $\varphi(TN_\perp)$, and μ. Then we derive

$$\begin{aligned}\|h\|^2 = {} & \sum_{r=m+1}^{m+q}\sum_{i,j=1}^{2p+1} g\big(h(e_i,e_j), e_r\big)^2 + \sum_{r=m+q+1}^{2m-1}\sum_{i,j=1}^{2p+1} g\big(h(e_i,e_j), e_r\big)^2 \\ & + \sum_{r=m+q+2p+1}^{2n+1}\sum_{i,j=1}^{2p+1} g\big(h(e_i,e_j), e_r\big)^2 + \sum_{r=m+1}^{m+q}\sum_{i,j=1}^{q} g\big(h(e_i,e_j), e_r\big)^2 \\ & + \sum_{r=m+q+1}^{2m-1}\sum_{i,j=1}^{q} g\big(h(e_i,e_j), e_r\big)^2 + \sum_{r=m+q+2p+1}^{2n+1}\sum_{i,j=1}^{q} g\big(h(e_i,e_j), e_r\big)^2.\end{aligned} \tag{3.8}$$

The first term of right hand side is identically zero by Lemma 3.1(ii) for a mixed geodesic warped product submanifold. Also, we have no relation for the μ components with h and $g(h(Z,W),FW')$, for any $Z,W,W'\in\Gamma(TN_\perp)$ in terms of the warping function. Thus, we shall leave all positive terms except the fifth term, then we have

$$\begin{aligned}\|h\|^2 &\geq \sum_{r=1}^{2p}\sum_{i,j=1}^{q} g\big(h(e_i,e_j),\widetilde{e}_r\big)^2\\ &= \sum_{r=1}^{p}\sum_{i,j=1}^{q} g\big(h(e_i,e_j),\sec\theta Fe_r^*\big)^2 + \sum_{r=1}^{p}\sum_{i,j=1}^{q} g\big(h(e_i,e_j),\csc\theta\sec\theta FPe_r^*\big)^2.\end{aligned}$$

Using Lemma 3.2 for mixed geodesic warped products, we derive

$$\begin{aligned}\|h\|^2 \geq\ &\csc^2\theta\sum_{r=1}^{p}\sum_{i,j=1}^{q}\big(Pe_r^*\ln f\big)^2 g(e_i,e_j)^2 + \cot^2\theta\sum_{r=1}^{p}\sum_{i,j=1}^{q}\big(e_r^*\ln f\big)^2 g(e_i,e_j)^2\\ =\ &q\csc^2\theta\sum_{r=1}^{2p+1}\big(Pe_r^*\ln f\big)^2 - q\csc^2\theta\sum_{r=p+1}^{2p}\big(Pe_r^*\ln f\big)^2\\ &- q\csc^2\theta\big(Pe_{2p+1}^*\ln f\big)^2 + q\cot^2\theta\sum_{r=1}^{p}\big(e_r^*\ln f\big)^2.\end{aligned}$$

Since $e_{2p+1}^*\ln f = \xi\ln f = 0$, from (2.10), we derive

$$\begin{aligned}\|h\|^2 \geq\ &q\csc^2\theta\big\|P\nabla^\theta\ln f\big\|^2 - q\csc^2\theta\sum_{r=1}^{p} g\big(e_{r+p}^*, P\nabla^\theta\ln f\big)^2\\ &+ q\cot^2\theta\sum_{r=1}^{p}\big(e_r^*\ln f\big)^2.\end{aligned}$$

Then by Theorem 2.1, we obtain

$$\begin{aligned}\|h\|^2 \geq\ &q\cot^2\theta\big\{\big\|\nabla^\theta\ln f\big\|^2 - g\big(\nabla^\theta\ln f,\xi\big)^2\big\}\\ &- q\csc^2\theta\sec^2\theta\sum_{r=1}^{p} g\big(Pe_r^*, P\nabla^\theta\ln f\big)^2 + q\cot^2\theta\sum_{r=1}^{p}\big(e_r^*\ln f\big)^2.\end{aligned}$$

Then from (2.9), (2.13), and the trigonometric identities, finally, we get

$$\|h\|^2 \geq q\cot^2\theta\big\|\nabla^\theta\ln f\big\|^2,$$

which is inequality (i). If the equality holds in (i), then from the second and third remaining terms

$$g\big(h(X,Y),FY'\big)=0,\quad \forall X,Y,Y'\in\Gamma(TN_\theta)\quad\Rightarrow\quad h(X,Y)\in\Gamma(\varphi TN_\perp\oplus\mu) \tag{3.9}$$

and

$$g\big(h(X,Y),\zeta\big) = 0, \quad \forall X,Y \in \Gamma(TN_\theta) \text{ and } \zeta \in \Gamma(\mu)$$
$$\Rightarrow \quad h(X,Y) \in \Gamma(\varphi TN_\perp \oplus FTN_\theta). \tag{3.10}$$

Then from (3.9) and (3.10), we get

$$h(X,Y) \in \Gamma(\varphi TN_\perp), \quad \forall X,Y \in \Gamma(TN_\theta). \tag{3.11}$$

Similarly, from the remaining fourth and sixth terms, we conclude that

$$g\big(h(Z,W),\varphi W'\big) = 0, \quad \forall Z,W,W' \in \Gamma(TN_\perp)$$
$$\Rightarrow \quad h(Z,W) \in \Gamma(FTN_\theta \oplus \mu) \tag{3.12}$$

and

$$g\big(h(Z,W),\zeta\big) = 0, \quad \forall Z,W \in \Gamma(TN_\perp) \text{ and } \zeta \in \Gamma(\mu)$$
$$\Rightarrow \quad h(Z,W) \in \Gamma(\varphi TN_\perp \oplus FTN_\theta). \tag{3.13}$$

Then from (3.12) and (3.13), we get

$$h(Z,W) \in \Gamma(FTN_\theta), \quad \forall Z,W \in \Gamma(TN_\perp). \tag{3.14}$$

Thus (ii) follows from (3.11) and (3.14). This completes the proof of the theorem. □

Competing interests
The author declares that he has no competing interests.

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References

1. Blair, DE: Almost contact manifolds with Killing structure tensors I. Pac. J. Math. **39**, 285-292 (1971)
2. Blair, DE, Showers, DK: Almost contact manifolds with Killing structures tensors II. J. Differ. Geom. **9**, 577-582 (1974)
3. Blair, DE, Yano, K: Affine almost contact manifolds and f-manifolds with affine Killing structure tensors. Kodai Math. Semin. Rep. **23**, 473-479 (1971)
4. Carriazo, A: New Developments in Slant Submanifolds Theory. Narosa Publishing House, New Delhi (2002)
5. Sahin, B: Warped product submanifolds of Kaehler manifolds with a slant factor. Ann. Pol. Math. **95**, 207-226 (2009)
6. Uddin, S, Kon, SH, Khan, MA, Singh, K: Warped product semi-invariant submanifolds of nearly cosymplectic manifolds. Math. Probl. Eng. (2011). doi:10.1155/2011/230374
7. Uddin, S, Khan, KA: An inequality for contact CR-warped product submanifolds of nearly cosymplectic manifolds. J. Inequal. Appl. **2012**, 304 (2012)
8. Uddin, S, Mustafa, A, Wong, BR, Ozel, C: A geometric inequality for warped product semi-slant submanifolds of nearly cosymplectic manifolds. Rev. Unión Mat. Argent. **55**, 55-69 (2014)
9. Endo, H: On the curvature tensor of nearly cosymplectic manifolds of constant φ-sectional curvature. An. Ştiinţ. Univ. 'Al.I. Cuza' Iaşi, Mat. **51**, 439-554 (2005)
10. Cabrerizo, JL, Carriazo, A, Fernandez, LM, Fernandez, M: Slant submanifolds in Sasakian manifolds. Glasg. Math. J. **42**, 125-138 (2000)
11. Bishop, RL, O'Neill, B: Manifolds of negative curvature. Trans. Am. Math. Soc. **145**, 1-49 (1969)