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On geodesic strongly E-convex sets and geodesic strongly E-convex functions

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Abstract
In this article, geodesic E-convex sets and geodesic E-convex functions on a Riemannian manifold are extended to the so-called geodesic strongly E-convex sets and geodesic strongly E-convex functions. Some properties of geodesic strongly E-convex sets are also discussed. The results obtained in this article may inspire future research in convex analysis and related optimization fields.

MSC: 52A20; 52A41; 53C20; 53C22

Keywords: geodesic E-convex sets; geodesic E-convex functions; Riemannian manifolds

1 Introduction

Convexity and its generalizations play an important role in optimization theory, convex analysis, Minkowski space, and fractal mathematics [1–7]. In order to extend the validity of their results to large classes of optimization, these concepts have been generalized and extended in several directions using novel and innovative techniques. Youness [8] defined E-convex sets and E-convex functions, which have some important applications in various branches of mathematical sciences [9–11]. However, some results given by Youness [8] seem to be incorrect according to Yang [12]. Chen [13] extended E-convexity to a semi-E-convexity and discussed some of there properties. Also, Youness and Emam [14] discussed a new class functions which is called strongly E-convex functions by taking the images of two points x_1 and x_2 under an operator $E\colon \mathbb{R}^n \to \mathbb{R}^n$ besides the two points themselves. Strong E-convexity was extended to a semi-strong E-convexity as well as quasi- and pseudo-semi-strong E-convexity in [15]. The authors investigated the characterization of efficient solutions for multi-objective programming problems involving semi-strong E-convexity [16].

A generalization of convexity on Riemannian manifolds was proposed by Rapcsak [17] and Udriste [18]. Moreover, Iqbal *et al.* [19] introduced geodesic E-convex sets and geodesic E-convex functions on Riemannian manifolds.

Motivated by earlier research works [18, 20–25] and by the importance of the concepts of convexity and generalized convexity, we discuss a new class of sets on Riemannian manifolds and a new class of functions defined on them, which are called geodesic strongly E-convex sets and geodesic strongly E-convex functions, and some of their properties are presented.

2 Preliminaries

In this section, we introduce some definitions and well-known results of Riemannian manifolds, which help us throughout the article. We refer to [18] for the standard material on differential geometry.

Let N be a C^∞ m-dimensional Riemannian manifold, and T_zN be the tangent space to N at z. Also, assume that $\mu_z(x_1,x_2)$ is a positive inner product on the tangent space T_zN ($x_1,x_2 \in T_zN$), which is given for each point of N. Then a C^∞ map $\mu\colon z \to \mu_z$, which assigns a positive inner product μ_z to T_zN for each point z of N is called a Riemannian metric.

The length of a piecewise C^1 curve $\eta\colon [a_1,a_2] \to N$ which is defined as follows:

$$L(\eta) = \int_{a_1}^{a_2} \bigl\|\acute{\eta}(x)\bigr\|\,dx.$$

We define $d(z_1,z_2) = \inf\{L(\eta)\colon \eta \text{ is a piecewise } C^1 \text{ curve joining } z_1 \text{ to } z_2\}$ for any points $z_1,z_2 \in N$. Then d is a distance which induces the original topology on N. As we know on every Riemannian manifold there is a unique determined Riemannian connection, called a Levi-Civita connection, denoted by $\triangledown_X Y$, for any vector fields $X, Y \in N$. Also, a smooth path η is a geodesic if and only if its tangent vector is a parallel vector field along the path η, *i.e.*, η satisfies the equation $\triangledown_{\acute{\eta}(t)}\acute{\eta}(t) = 0$. Any path η joining z_1 and z_2 in N such that $L(\eta) = d(z_1,z_2)$ is a geodesic and is called a minimal geodesic.

Finally, assume that (N,η) is a complete m-dimensional Riemannian manifold with Riemannian connection $\triangledown$. Let $x_1,x_2 \in N$ and $\eta\colon [0,1] \to N$ be a geodesic joining the points x_1 and x_2, which means that $\eta_{x_1,x_2}(0) = x_2$ and $\eta_{x_1,x_2}(1) = x_1$.

Definition 2.1 [18] A set B in a Riemannian manifold N is called totally convex if B contains every geodesic η_{x_1,x_2} of N whose endpoints x_1 and x_2 belong to B.

Note the whole of the manifold N is totally convex, and conventionally, so is the empty set. The minimal circle in a hyperboloid is totally convex, but a single point is not. Also, any proper subset of a sphere is not necessarily totally convex.

The following theorem was proved in [18].

Theorem 2.2 [18] *The intersection of any number of a totally convex sets is totally convex.*

Remark 2.3 In general, the union of a totally convex set is not necessarily totally convex.

Definition 2.4 [18] A function $f\colon B \to \mathbb{R}$ is called a geodesic convex function on a totally convex set $B \subset N$ if for every geodesic η_{x_1,x_2}, then

$$f\bigl(\eta_{x_1,x_2}(\gamma)\bigr) \le \gamma f(x_1) + (1-\gamma)f(x_2)$$

holds for all $x_1,x_2 \in B$ and $\gamma \in [0,1]$.

In 2005, strongly E-convex sets and strongly E-convex functions were introduced by Youness and Emam [14] as follows.

Definition 2.5 [14]

(1) A subset $B \subseteq \mathbb{R}^n$ is called a strongly E-convex set if there is a map $E\colon \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\gamma\big(\alpha b_1 + E(b_1)\big) + (1-\gamma)\big(\alpha b_2 + E(b_2)\big) \in B$$

for each $b_1, b_2 \in B$, $\alpha \in [0,1]$ and $\gamma \in [0,1]$.

(2) A function $f\colon B \subseteq \mathbb{R}^n \to \mathbb{R}$ is called a strongly E-convex function on N if there is a map $E\colon \mathbb{R}^n \to \mathbb{R}^n$ such that B is a strongly E-convex set and

$$f\big(\gamma\big(\alpha b_1 + E(b_1)\big) + (1-\gamma)\big(\alpha b_2 + E(b_2)\big)\big) \leq \gamma f\big(E(b_1)\big) + (1-\gamma) f\big(E(b_2)\big)$$

for each $b_1, b_2 \in B$, $\alpha \in [0,1]$ and $\gamma \in [0,1]$.

In 2012, the geodesic E-convex set and geodesic E-convex functions on a Riemannian manifold were introduced by Iqbal *et al.* [19] as follows.

Definition 2.6 [19]

(1) Assume that $E\colon N \to N$ is a map. A subset B in a Riemannian manifold N is called geodesic E-convex iff there exists a unique geodesic $\eta_{E(b_1),E(b_2)}(\gamma)$ of length $d(b_1, b_2)$, which belongs to B, for each $b_1, b_2 \in B$ and $\gamma \in [0,1]$.

(2) A function $f\colon B \subseteq N \to \mathbb{R}$ is called geodesic E-convex on a geodesic E-convex set B if

$$f\big(\eta_{E(b_1),E(b_2)}(\gamma)\big) \leq \gamma f\big(E(b_1)\big) + (1-\gamma) f\big(E(b_2)\big)$$

for all $b_1, b_2 \in B$ and $\gamma \in [0,1]$.

3 Geodesic strongly *E*-convex sets and geodesic strongly *E*-convex functions

In this section, we introduce a geodesic strongly E-convex (GSEC) set and a geodesic strongly E-convex (GSEC) function in a Riemannian manifold N and discuss some of their properties.

Definition 3.1 Assume that $E\colon N \to N$ is a map. A subset B in a Riemannian manifold N is called GSEC if and only if there is a unique geodesic $\eta_{\alpha b_1 + E(b_1), \alpha b_2 + E(b_2)}(\gamma)$ of length $d(b_1, b_2)$, which belongs to B, $\forall b_1, b_2 \in B$, $\alpha \in [0,1]$, and $\gamma \in [0,1]$.

Remark 3.2

(1) Every GSEC set is a GEC set when $\alpha = 0$.

(2) A GEC set is not necessarily a GSEC set. The following example shows this statement.

Example 3.3 Let N^2 be a 2-dimensional simply complete Riemannian manifold of non-positive sectional curvature, and $B \subset N^2$ be an open star-shaped. Let $E\colon N^2 \to N^2$ be a map such that $E(z) = \{y\colon y \in \ker(B), \forall z \in B\}$. Then B is GEC; on the other hand it is not GSEC.

Proposition 3.4 *Every convex set $B \subset N$ is a GSEC set.*

Proof Let us take a map $E\colon N \to N$ such as $E = I$ where I is the identity map and $\alpha = 0$, then we have the required result. □

Note if we take the mapping $E(x) = (1-\alpha)x$, $x \in B$, then the definition of a GSE reduces to the definition of a t-convex set.

Theorem 3.5 *If $B \subset N$ is a GSEC set, then $E(B) \subseteq B$.*

Proof Since B is a GSEC set, we have for each $b_1, b_2 \in B$, $\alpha \in [0,1]$, and $\gamma \in [0,1]$,

$$\eta_{\alpha b_1 + E(b_1), \alpha b_2 + E(b_2)}(\gamma) \in B.$$

For $\gamma = 0$ and $\alpha = 0$, we have $\eta_{E(b_1),E(b_2)}(0) = E(b_2) \in B$, then $E(B) \subseteq B$. □

Theorem 3.6 *If $\{B_j, j \in I\}$ is an arbitrary family of GSEC subsets of N with respect to the mapping $E\colon N \to N$, then the intersection $\bigcap_{j\in I} B_j$ is a GSEC subset of N.*

Proof If $\bigcap_{j\in I} B_j$ is an empty set, then it is obviously a GSEC subset of N. Assume that $b_1, b_2 \in \bigcap_{j\in I} B_j$, then $b_1, b_2 \in B_j$, $\forall j \in I$. By the GSEC of B_j, we get $\eta_{\alpha b_1+E(b_1),\alpha b_2+E(b_2)}(\gamma) \in B_j$, $\forall j \in I$, $\alpha \in [0,1]$, and $\gamma \in [0,1]$. Hence, $\eta_{\alpha b_1+E(b_1),\alpha b_2+E(b_2)}(\gamma) \in \bigcap_{j\in I} B_j$, $\forall \alpha \in [0,1]$ and $\gamma \in [0,1]$. □

Remark 3.7 The above theorem is not generally true for the union of GSEC subsets of N.

Now, we extend the definition of a GEC function on a Riemannian manifold to a GSEC function on a Riemannian manifold.

Definition 3.8 A real-valued function $f\colon B \subset N \to \mathbb{R}$ is said to be a GSEC function on a GSEC set B, if

$$f\bigl(\eta_{\alpha b_1+E(b_1),\alpha b_2+E(b_2)}(\gamma)\bigr) \le \gamma f\bigl(E(b_1)\bigr) + (1-\gamma) f\bigl(E(b_2)\bigr),$$

$\forall b_1, b_2 \in B$, $\alpha \in [0,1]$, and $\gamma \in [0,1]$. If the above inequality is strict for all $b_1, b_2 \in B$, $\alpha b_1 + E(b_1) \neq \alpha b_2 + E(b_2)$, $\alpha \in [0,1]$, and $\gamma \in (0,1)$, then f is called a strictly GSEC function.

Remark 3.9

(1) Every GSEC function is a GEC function when $\alpha = 0$. The following example shows that a GEC function is not necessarily a GSEC function.

Example 3.10 Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ where $f(b) = -|b|$ and suppose that $E\colon \mathbb{R} \to \mathbb{R}$ is given as $E(b) = -b$. We consider the geodesic η such that

$$\begin{aligned} \eta_{\alpha b_1+E(b_1),\alpha b_2+E(b_2)}(\gamma) &= \begin{cases} -[\alpha b_2 + E(b_2) + \gamma(\alpha b_1 + E(b_1) - \alpha b_2 - E(b_2))]; & b_1 b_2 \ge 0, \\ -[\alpha b_2 + E(b_2) + \gamma(\alpha b_2 + E(b_2) - \alpha b_1 - E(b_1))]; & b_1 b_2 < 0 \end{cases} \\ &= \begin{cases} -[(\alpha-1)b_2 + \gamma((\alpha-1)b_1 + (1-\alpha)b_2)]; & b_1 b_2 \ge 0, \\ -[(\alpha-1)b_2 + \gamma((\alpha-1)b_2 + (1-\alpha)b_1)]; & b_1 b_2 < 0. \end{cases} \end{aligned}$$

If $\alpha = 0$, then

$$\eta_{E(b_1),E(b_2)}(\gamma) = \begin{cases} [b_2 + \gamma(b_1 - b_2)]; & b_1 b_2 \geq 0, \\ [b_2 + \gamma(b_2 - b_1)]; & b_1 b_2 < 0. \end{cases}$$

If $b_1, b_2 \geq 0$, then

$$\begin{aligned} f\big(\eta_{E(b_1),E(b_2)}(\gamma)\big) &= f\big(b_2 + \gamma(b_1 - b_2)\big) \\ &= -\big[(1-\gamma)b_2 + \gamma b_1\big]. \end{aligned}$$

On the other hand

$$\gamma f\big(E(b_1)\big) + (1-\gamma) f\big(E(b_2)\big) = \gamma f(-b_1) + (1-\gamma) f(-b_2) = -\big[(1-\gamma)b_2 + \gamma b_1\big].$$

Hence, $f(\eta_{E(b_1),E(b_2)}(\gamma)) \leq \gamma f(E(b_1)) + (1-\gamma) f(E(b_2))$, $\forall \gamma \in [0,1]$.

Similarly, the above inequality holds true when $b_1, b_2 < 0$.

Now, let $b_1 < 0$, $b_2 > 0$, then

$$\begin{aligned} f\big(\eta_{E(b_1),E(b_2)}(\gamma)\big) &= f\big(b_2 + \gamma(b_2 - b_1)\big) \\ &= -\big[(1+\gamma)b_2 - \gamma b_1\big]. \end{aligned}$$

On the other hand

$$\gamma f\big(E(b_1)\big) + (1-\gamma) f\big(E(b_2)\big) = \gamma f(-b_1) + (1-\gamma) f(-b_2) = \gamma b_1 - (1-\gamma) b_2.$$

It follows that

$$f\big(\eta_{E(b_1),E(b_2)}(\gamma)\big) \leq \gamma f\big(E(b_1)\big) + (1-\gamma) f\big(E(b_2)\big)$$

if and only if

$$-\big[(1+\gamma)b_2 - \gamma b_1\big] \leq \gamma b_1 - (1-\gamma) b_2$$

if and only if

$$-2\gamma b_2 \leq 0,$$

which is always true for all $\gamma \in [0,1]$.

Similarly, $f(\eta_{E(b_1),E(b_2)}(\gamma)) \leq \gamma f(E(b_1)) + (1-\gamma) f(E(b_2))$, $\forall \gamma \in [0,1]$ also holds for $b_1 > 0$ and $b_2 < 0$.

Thus, f is a GEC function on $\mathbb{R}$, but it is not a GSEC function because if we take $b_1 = 0$, $b_2 = -1$ and $\gamma = \frac{1}{2}$, then

$$\begin{aligned} f\big(\eta_{\alpha b_1 + E(b_1), \alpha b_2 + E(b_2)}(\gamma)\big) &= f\left(\frac{1}{2}\alpha - \frac{1}{2}\right) \\ &= \frac{1}{2}\alpha - \frac{1}{2} \end{aligned}$$

$$
\begin{aligned}
&> \frac{1}{2} f\bigl(E(0)\bigr) + \frac{1}{2} f\bigl(E(-1)\bigr) \\
&= \frac{-1}{2}, \quad \forall \alpha \in (0,1].
\end{aligned}
$$

(2) Every g-convex function f on a convex set B is a GSEC function when $\alpha = 0$ and E is the identity map.

Proposition 3.11 *Assume that $f: B \to \mathbb{R}$ is a GSEC function on a GSEC set $B \subseteq N$, then $f(\alpha b + E(b)) \leq f(E(b))$, $\forall b \in B$ and $\alpha \in [0,1]$.*

Proof Since $f: B \to \mathbb{R}$ is a GSEC function on a GSEC set $B \subseteq N$, then $\eta_{\alpha b_1 + E(b_1), \alpha b_2 + E(b_2)}(\gamma) \in B$, $\forall b_1, b_2 \in B$, $\alpha \in [0,1]$, and $\gamma \in [0,1]$. Also,

$$
f\bigl(\eta_{\alpha b_1 + E(b_1), \alpha b_2 + E(b_2)}(\gamma)\bigr) \leq \gamma f\bigl(E(b_1)\bigr) + (1-\gamma) f\bigl(E(b_2)\bigr)
$$

thus, for $\gamma = 1$, we get $\eta_{\alpha b_1 + E(b_1), \alpha b_2 + E(b_2)}(\gamma) = \alpha b_1 + E(b_1)$. Then

$$
f\bigl(\alpha b_1 + E(b_1)\bigr) \leq f\bigl(E(b_1)\bigr). \qquad \square
$$

Theorem 3.12 *Consider that $B \subseteq N$ is a GSEC set and $f_1: B \to \mathbb{R}$ is a GSEC function. If $f_2: I \to \mathbb{R}$ is a non-decreasing convex function such that* $\operatorname{rang}(f_1) \subset I$, *then $f_2 \circ f_1$ is a GSEC function on B.*

Proof Since f_1 is a GSEC function, for all $b_1, b_2 \in B$, $\alpha \in [0,1]$, and $\gamma \in [0,1]$,

$$
f_1\bigl(\eta_{\alpha b_1 + E(b_1), \alpha b_2 + E(b_2)}(\gamma)\bigr) \leq \gamma f_1\bigl(E(b_1)\bigr) + (1-\gamma) f_1\bigl(E(b_2)\bigr).
$$

Since f_2 is a non-decreasing convex function,

$$
\begin{aligned}
&f_2 \circ f_2\bigl(\eta_{\alpha b_1 + E(b_1), \alpha b_2 + E(b_2)}(\gamma)\bigr) \\
&\quad = f_2\bigl(f_2\bigl(\eta_{\alpha b_1 + E(b_1), \alpha b_2 + E(b_2)}(\gamma)\bigr)\bigr) \\
&\quad \leq f_2\bigl(\gamma f_1\bigl(E(b_1)\bigr) + (1-\gamma) f_1\bigl(E(b_2)\bigr)\bigr) \\
&\quad \leq \gamma f_2\bigl(f_1\bigl(E(b_1)\bigr)\bigr) + (1-\gamma) f_2\bigl(f_1\bigl(E(b_2)\bigr)\bigr) \\
&\quad = \gamma (f_2 \circ f_1)\bigl(E(b_1)\bigr) + (1-\gamma)(f_2 \circ f_1)\bigl(E(b_2)\bigr),
\end{aligned}
$$

which means that $f_2 \circ f_1$ is a GSEC function on B. Similarly, if f_2 is a strictly non-decreasing convex function, then $f_2 \circ f_1$ is a strictly GSEC function. $\square$

Theorem 3.13 *Assume that $B \subseteq N$ is a GSEC set and $f_j: B \to \mathbb{R}$, $j = 1, 2, \ldots, m$ are GSEC functions. Then the function*

$$
f = \sum_{j=1}^{m} n_j f_j
$$

is GSEC on B, $\forall n_j \in \mathbb{R}$, $n_j \geq 0$.

Proof Since f_j, $j = 1, 2, \ldots, m$ are GSEC functions, $\forall b_1, b_2 \in B$, $\alpha \in [0,1]$, and $\gamma \in [0,1]$, we have

$$f_j\bigl(\eta_{\alpha b_1+E(b_1),\alpha b_2+E(b_2)}(\gamma)\bigr) \leq \gamma f_j\bigl(E(b_1)\bigr) + (1-\gamma)f_j\bigl(E(b_2)\bigr).$$

It follows that

$$n_j f_j\bigl(\eta_{\alpha b_1+E(b_1),\alpha b_2+E(b_2)}(\gamma)\bigr) \leq \gamma n_j f_j\bigl(E(b_1)\bigr) + (1-\gamma)n_j f_j\bigl(E(b_2)\bigr).$$

Then

$$\begin{aligned}
&\sum_{j=1}^{m} n_j f_j\bigl(\eta_{\alpha b_1+E(b_1),\alpha b_2+E(b_2)}(\gamma)\bigr) \\
&\quad \leq \gamma \sum_{j=1}^{m} n_j f_j\bigl(E(b_1)\bigr) + (1-\gamma)\sum_{j=1}^{m} n_j f_j\bigl(E(b_2)\bigr) \\
&\quad = \gamma f\bigl(E(b_1)\bigr) + (1-\gamma)f\bigl(E(b_2)\bigr).
\end{aligned}$$

Thus, f is a GSEC function. □

Theorem 3.14 *Let $B \subseteq N$ be a GSEC set and $\{f_j, j \in I\}$ be a family of real-valued functions defined on B such that $\sup_{j\in I} f_j(b)$ exists in $\mathbb{R}$, $\forall b \in B$. If $f_j\colon B \to \mathbb{R}$, $j \in I$ are GSEC functions on B, then the function $f\colon B \to \mathbb{R}$, defined by $f(b) = \sup_{j\in I} f_j(b)$, $\forall b \in B$ is GSEC on B.*

Proof Since f_j, $j \in I$ are GSEC functions on a GSEC set B, $\forall b_1, b_2 \in B$, $\alpha \in [0,1]$, and $\gamma \in [0,1]$, we have

$$f_j\bigl(\eta_{\alpha b_1+E(b_1),\alpha b_2+E(b_2)}(\gamma)\bigr) \leq \gamma f_j\bigl(E(b_1)\bigr) + (1-\gamma)f_j\bigl(E(b_2)\bigr).$$

Then

$$\begin{aligned}
&\sup_{j\in I} f_j\bigl(\eta_{\alpha b_1+E(b_1),\alpha b_2+E(b_2)}(\gamma)\bigr) \\
&\quad \leq \sup_{j\in I}\bigl[\gamma f_j\bigl(E(b_1)\bigr) + (1-\gamma)f_j\bigl(E(b_2)\bigr)\bigr] \\
&\quad = \gamma \sup_{j\in I} f_j\bigl(E(b_1)\bigr) + (1-\gamma)\sup_{j\in I} f_j\bigl(E(b_2)\bigr) \\
&\quad = \gamma f\bigl(E(b_1)\bigr) + (1-\gamma)f\bigl(E(b_2)\bigr).
\end{aligned}$$

Hence,

$$f\bigl(\eta_{\alpha b_1+E(b_1),\alpha b_2+E(b_2)}(\gamma)\bigr) \leq \gamma f\bigl(E(b_1)\bigr) + (1-\gamma)f\bigl(E(b_2)\bigr),$$

which means that f is a GSEC function on B. □

Proposition 3.15 *Assume that $h_j\colon N \to \mathbb{R}$, $j = 1, 2, \ldots, m$ are GSEC functions on N, with respect to $E\colon N \to N$. If $E(B) \subseteq B$, then $B = \{b \in N\colon h_j(b) \leq 0, j = 1, 2, \ldots, m\}$ is a GSEC set.*

Proof Since h_j, $j = 1, 2, \dots m$ are GSEC functions,

$$\begin{aligned} h_j\big(\eta_{\alpha b_1+E(b_1),\alpha b_2+E(b_2)}(\gamma)\big) &\leq \gamma h_j\big(E(b_1)\big) + (1-\gamma)h_j\big(E(b_2)\big) \\ &\leq 0, \end{aligned}$$

$\forall b_1, b_2 \in B$, $\alpha \in [0,1]$, and $\gamma \in [0,1]$. Since $E(B) \subseteq B$, $\eta_{\alpha b_1+E(b_1),\alpha b_2+E(b_2)}(\gamma) \in B$. Hence, B is a GSEC set. □

4 Epigraphs

Youness and Emam [14] defined a strongly $E \times F$-convex set where $E: \mathbb{R}^n \to \mathbb{R}^n$ and $F: \mathbb{R} \to \mathbb{R}$ and studied some of its properties. In this section, we generalize a strongly $E \times F$-convex set to a geodesic strongly $E \times F$-convex set on Riemannian manifolds and discuss GSEC functions in terms of their epigraphs. Furthermore, some properties of GSE sets are given.

Definition 4.1 Let $B \subset N \times \mathbb{R}$, $E: N \to N$ and $F: \mathbb{R} \to \mathbb{R}$. A set B is called geodesic strongly $E \times F$-convex if $(b_1, \beta_1), (b_2, \beta_2) \in B$ implies

$$\big(\eta_{\alpha b_1+E(b_1),\alpha b_2+E(b_2)}(\gamma), \gamma F(\beta_1) + (1-\gamma)F(\beta_2)\big) \in B$$

for all $\alpha \in [0,1]$ and $\gamma \in [0,1]$.

It is not difficult to prove that $B \subseteq N$ is a GSEC set if and only if $B \times \mathbb{R}$ is a geodesic strongly $E \times F$-convex set.

An epigraph of f is given by

$$\operatorname{epi}(f) = \big\{(b,a) : b \in B, a \in \mathbb{R}, f(b) \leq a\big\}.$$

A characterization of a GSEC function in terms of its epi(f) is given by the following theorem.

Theorem 4.2 *Let $E: N \to N$ be a map, $B \subseteq N$ be a GSEC set, $f: B \to \mathbb{R}$ be a real-valued function and $F: \mathbb{R} \to \mathbb{R}$ be a map such that $F(f(b) + a) = f(E(b)) + a$, for each non-negative real number a. Then f is a GSEC function on B if and only if* epi(f) *is geodesic strongly $E \times F$-convex on $B \times \mathbb{R}$.*

Proof Assume that $(b_1, a_1), (b_2, a_2) \in \operatorname{epi}(f)$. If B is a GSEC set, then $\eta_{\alpha b_1+E(b_1),\alpha b_2+E(b_2)}(\gamma) \in B$, $\forall \alpha \in [0,1]$ and $\gamma \in [0,1]$. Since $E(b_1) \in B$ for $\alpha = 0$, $\gamma = 1$, also $E(b_2) \in B$ for $\alpha = 0$, $\gamma = 0$, let $F(a_1)$ and $F(a_2)$ be such that $f(E(b_1)) \leq F(a_1)$ and $f(E(b_2)) \leq F(a_2)$. Then $(E(b_1), F(a_1)), (E(b_2), F(a_2)) \in \operatorname{epi}(f)$.

Let f be GSEC on B, then

$$\begin{aligned} f\big(\eta_{\alpha b_1+E(b_1),\alpha b_2+E(b_2)}(\gamma)\big) &\leq \gamma f\big(E(b_1)\big) + (1-\gamma)f\big(E(b_2)\big) \\ &\leq \gamma F(a_1) + (1-\gamma)F(a_2). \end{aligned}$$

Thus, $(\eta_{\alpha b_1+E(b_1),\alpha b_2+E(b_2)}(\gamma), \gamma F(a_1) + (1-\gamma)F(a_2)) \in \operatorname{epi}(f)$, then epi($f$) is geodesic strongly $E \times F$-convex on $B \times \mathbb{R}$.

Conversely, assume that epi(f) is geodesic strongly $E \times F$-convex on $B \times \mathbb{R}$. Let $b_1, b_2 \in B$, $\alpha \in [0,1]$, and $\gamma \in [0,1]$, then $(b_1, f(b_1)) \in \operatorname{epi}(f)$ and $(b_2, f(b_2)) \in \operatorname{epi}(f)$. Now, since epi($f$) is geodesic strongly $E \times F$-convex on $B \times \mathbb{R}$, we obtain $(\eta_{\alpha b_1+E(b_1),\alpha b_2+E(b_2)}(\gamma), \gamma F(f(b_1)) + (1-\gamma)F(f(b_2))) \in \operatorname{epi}(f)$, then

$$\begin{aligned} f\bigl(\eta_{\alpha b_1+E(b_1),\alpha b_2+E(b_2)}(\gamma)\bigr) &\le \gamma F\bigl(f(b_1)\bigr) + (1-\gamma)F\bigl(f(b_2)\bigr) \\ &= \gamma f\bigl(E(b_1)\bigr) + (1-\gamma)f\bigl(E(b_2)\bigr). \end{aligned}$$

This shows that f is a GSEC function on B. □

Theorem 4.3 *Assume that $\{B_j, j \in I\}$ is a family of geodesic strongly $E \times F$-convex sets. Then the intersection $\bigcap_{j\in I} B_j$ is a geodesic strongly $E \times F$-convex set.*

Proof Assume that $(b_1, a_1), (b_2, a_2) \in \bigcap_{j\in I} B_j$, so $\forall j \in I$, $(b_1, a_1), (b_2, a_2) \in B_j$. Since B_j is the geodesic strongly $E \times F$-convex sets $\forall j \in I$, we have

$$\bigl(\eta_{\alpha b_1+E(b_1),\alpha b_2+E(b_2)}(\gamma), \gamma F(a_1) + (1-\gamma)F(a_2)\bigr) \in B_j,$$

$\forall \alpha \in [0,1]$ and $\gamma \in [0,1]$. Therefore,

$$\bigl(\eta_{\alpha b_1+E(b_1),\alpha b_2+E(b_2)}(\gamma), \gamma F(a_1) + (1-\gamma)F(a_2)\bigr) \in \bigcap_{j\in I} B_j,$$

$\forall \alpha \in [0,1]$ and $\gamma \in [0,1]$. Then $\bigcap_{j\in I} B_j$ is a geodesic strongly $E \times F$-convex set. □

Theorem 4.4 *Assume that $E\colon N \to N$ and $F\colon \mathbb{R} \to \mathbb{R}$ are two maps such that $F(f(b) + a) = f(E(b)) + a$ for each non-negative real number a. Suppose that $\{f_j, j \in I\}$ is a family of real-valued functions defined on a GSEC set $B \subseteq N$ which are bounded from above. If* epi(f_j) *are geodesic strongly $E \times F$-convex sets, then the function f which is given by $f(b) = \sup_{j\in I} f_j(b)$, $\forall b \in B$, is a GSEC function on B.*

Proof If each f_j, $j \in I$ is a GSEC function on a GSEC geodesic set B, then

$$\operatorname{epi}(f_j) = \bigl\{(b,a)\colon b \in B, a \in \mathbb{R}, f_j(b) \le a\bigr\}$$

are geodesic strongly $E \times F$-convex on $B \times \mathbb{R}$. Therefore,

$$\begin{aligned} \bigcap_{j\in I} \operatorname{epi}(f_j) &= \bigl\{(b,a)\colon b \in B, a \in \mathbb{R}, f_j(b) \le a, j \in I\bigr\} \\ &= \bigl\{(b,a)\colon b \in B, a \in \mathbb{R}, f(b) \le a\bigr\} \end{aligned}$$

is geodesic strongly $E \times F$-convex set. Then, according to Theorem 4.2 we see that f is a GSEC function on B. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions
All authors jointly worked on deriving the results and approved the final manuscript.

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