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On some geometric properties of multivalent functions

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Abstract
We prove some new sufficient conditions for a function to be p-valent or p-valently starlike in the unit disc.

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1 Introduction

Let $\mathcal{A}_p$ be the class of analytic functions of the form

$$f(z)=z^p+\sum_{n=1}^{\infty}a_{p+n}z^{p+n}\quad (z\in\mathbb{D}). \tag{1.1}$$

A function $f(z)$ which is analytic in a domain $D\subset C$ is called p-valent in D if for every complex number w, the equation $f(z)=w$ has at most p roots in D, and there will be a complex number w_0 such that the equation $f(z)=w_0$ has exactly p roots in D. Further, a function $f\in\mathcal{A}_p$, $p\in\mathbb{N}\setminus\{1\}$, is said to be *$p$-valently starlike of order α*, $0\le\alpha<p$, if

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha\quad (z\in\mathbb{D}).$$

The class of all such functions is usually denoted by $\mathcal{S}_p^*(\alpha)$. For $p=1$, we receive the well-known class of normalized starlike univalent functions $\mathcal{S}^*(\alpha)$ of order α, $\mathcal{S}_p^*(0)=\mathcal{S}_p^*$. If $zf'(z)\in\mathcal{S}^*(\alpha)$, then $f(z)$ is said to be *p-valently convex of order α*, $0\le\alpha<p$. The class of all such functions is usually denoted by $\mathcal{C}_p(\alpha)$. For $p=1$, we receive the well-known class of normalized convex univalent functions $\mathcal{C}(\alpha)$ of order α, $\mathcal{C}_p(0)=\mathcal{C}_p$.

The well-known Noshiro-Warschawski univalence condition (see [1] and [2]) indicates that if $f(z)$ is analytic in a convex domain $D\subset\mathbb{C}$ and

$$\mathfrak{Re}\left\{e^{i\theta}f(z)\right\}>0\quad (z\in D),$$

where θ is a real number, then $f(z)$ is univalent in D. In [3] Ozaki extended the above result by showing that if $f(z)$ of the form (1.1) is analytic in a convex domain D and for some real

θ we have

$$\mathfrak{Re}\left\{e^{i\theta}f^{(p)}(z)\right\} > 0, \quad |z| < 1,$$

then $f(z)$ is at most p-valent in D. Applying Ozaki's theorem, we find that if $f(z) \in \mathcal{A}_p$ and

$$\mathfrak{Re}\left\{f^{(p)}(z)\right\} > 0 \quad (z \in \mathbb{D}),$$

then $f(z)$ is at most p-valent in $|z| < 1$. In [4] it was proved that if $f(z) \in \mathcal{A}_p$, $p \geq 2$, and

$$\arg\left|f^{(p)}(z)\right| < \frac{3\pi}{4}, \quad |z| < 1,$$

then $f(z)$ is at most p-valent in $|z| < 1$.

2 Preliminary lemmata

Lemma 2.1 [5] *Let $f(z) = z + a_2z^2 + \cdots$ be analytic in the unit disc and suppose that*

$$\left|f''(z)\right| < 1, \quad |z| < 1, \tag{2.1}$$

then $f(z)$ is univalent in $|z| < 1$.

Lemma 2.2 [6] *Let $p(z)$ be an analytic function in $|z| < 1$ with $p(0) = 1$, $p(z) \neq 0$. If there exists a point z_0, $|z_0| < 1$, such that*

$$\left|\arg\left\{p(z)\right\}\right| < \frac{\pi\alpha}{2} \quad \textit{for } |z| < |z_0|$$

and

$$\left|\arg\left\{p(z_0)\right\}\right| = \frac{\pi\alpha}{2}$$

for some $0 < \alpha < 2$, then we have

$$\frac{z_0p'(z_0)}{p(z_0)} = is\alpha,$$

where

$$s \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \quad \textit{when } \arg\left\{p(z_0)\right\} = \frac{\pi\alpha}{2}$$

and

$$s \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \quad \textit{when } \arg\left\{p(z_0)\right\} = -\frac{\pi\alpha}{2},$$

where

$$\left\{p(z_0)\right\}^{1/\alpha} = \pm ia, \ \textit{and } a > 0.$$

Lemma 2.3 [7] *Let p be a positive integer. If* $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ *is analytic in* $\mathbb{D}$ *and if it satisfies*

$$\mathfrak{Re}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > 0, \quad z \in \mathbb{D}, \tag{2.2}$$

then $f(z)$ *is p-valently starlike in* $\mathbb{D}$ *and*

$$\mathfrak{Re}\left\{\frac{zf^{(k)}(z)}{f^{(k-1)}(z)}\right\} > 0, \quad z \in \mathbb{D} \tag{2.3}$$

for $k = 1, 2, \ldots, (p-1)$.

Lemma 2.4 ([7], p.282) *Let* $f \in \mathcal{A}_p$. *If there exists a* $(p-k+1)$*-valent starlike function* $g(z) = \sum_{n=p-k+1}^{\infty} b_n z^n$ $(b_{p-k+1} \neq 0)$ *that satisfies*

$$\mathfrak{Re}\left\{\frac{zf^{(k)}(z)}{g(z)}\right\} > 0, \quad |z| < 1, \tag{2.4}$$

then $f(z)$ *is p-valent in* $|z| < 1$.

3 Main results

Now we state and prove the main results. The first theorem poses a growth condition to a higher derivative of an analytic function. Hence, its proof, among others, uses the method of integrating the derivatives, which is frequently used in complex analysis, especially in estimating point evaluation operators (see, *e.g.*, Lemma 7 in [8] and Lemma 4 in [9]).

Theorem 3.1 *Let* $f \in \mathcal{A}_p$ *and suppose that*

$$\left|f^{(p+1)}(z)\right| < \alpha(\beta_0)(p!), \quad |z| < 1, \tag{3.1}$$

where $\beta_0 = 0.38\cdots$ *is the positive root of the equation*

$$2\beta + \frac{2}{\pi}\tan^{-1}\beta = 1, \quad 0 < \beta < 1 \tag{3.2}$$

and

$$\begin{aligned}\alpha(\beta_0) &= \sin\left\{\frac{\pi}{2}\left(\beta_0 + (2/\pi)\tan^{-1}\beta_0\right)\right\} \\ &= \sin\left\{\frac{(1-\beta_0)\pi}{2}\right\} \\ &= 0.82\cdots.\end{aligned}$$

Then we have

$$\mathfrak{Re}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > 0, \quad |z| < 1, \tag{3.3}$$

and, therefore, we have

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad |z| < 1, \tag{3.4}$$

or $f(z)$ is p-valently starlike in $|z| < 1$.

Proof From the hypothesis (3.1), we have

$$\begin{aligned} \left|f^{(p)}(z) - p!\right| &= \left|\int_0^z f^{(p+1)}(t)\,\mathrm{d}t\right| \\ &\le \int_0^{|z|} \left|f^{(p+1)}(t)\right| |\mathrm{d}t| < p! \int_0^{|z|} \alpha(\beta_0)|\mathrm{d}t| \\ &= p!\alpha(\beta_0)|z| < p!\alpha(\beta_0). \end{aligned}$$

This shows that

$$\left|\arg\left\{f^{(p)}(z)\right\}\right| < \sin^{-1}\alpha(\beta_0), \quad |z| < 1. \tag{3.5}$$

Let us put

$$q(z) = \frac{f^{(p-1)}(z)}{p!z}, \qquad q(0) = 1. \tag{3.6}$$

Then it follows that

$$q(z) + zq'(z) = \frac{f^{(p)}(z)}{p!}.$$

If there exists a point z_0, $|z_0| < 1$, such that

$$\left|\arg\left\{q(z)\right\}\right| < \frac{\pi\beta_0}{2} \quad \text{for } |z| < |z_0|$$

and

$$\left|\arg\left\{q(z_0)\right\}\right| = \frac{\pi\beta_0}{2},$$

then by Lemma 2.2 we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = i\beta_0 k,$$

where

$$k \ge 1 \quad \text{when } \arg\left\{q(z_0)\right\} = \frac{\pi\beta_0}{2} \tag{3.7}$$

and

$$k \le -1 \quad \text{when } \arg\left\{q(z_0)\right\} = -\frac{\pi\beta_0}{2}. \tag{3.8}$$

For the case (3.7), we have

$$\begin{aligned}
\arg\{f^{(p)}(z)\} &= \arg\left\{\frac{f^{(p)}(z)}{p!}\right\} \\
&= \arg\{q(z_0) + z_0 q'(z_0)\} \\
&= \arg\left\{q(z_0)\left(1 + \frac{z_0 q'(z_0)}{q(z_0)}\right)\right\} \\
&= \frac{\pi\beta_0}{2} + \arg\{1 + i\beta_0 k\} \\
&\geq \frac{\pi\beta_0}{2} + \tan^{-1}\beta_0 \\
&= \frac{\pi}{2}\{\beta_0 + (2/\pi)\tan^{-1}(\beta_0)\} \\
&= \sin^{-1}\alpha(\beta_0).
\end{aligned}$$

This contradicts (3.5), and for the case (3.8), applying the same method as above, we have

$$\arg\{f^{(p)}(z)\} \leq -\sin^{-1}\alpha(\beta_0).$$

This also contradicts (3.5) and, therefore, it shows that

$$\left|\arg\left\{\frac{f^{(p-1)}(z)}{p!z}\right\}\right| < \frac{\pi\beta_0}{2}, \quad |z| < 1. \tag{3.9}$$

Applying (3.5) and (3.9), we have

$$\begin{aligned}
\left|\arg\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\}\right| &= \left|\arg\{f^{(p)}(z)\} + \arg\left\{\frac{z}{f^{(p-1)}(z)}\right\}\right| \\
&\leq \left|\arg\{f^{(p)}(z)\}\right| + \left|\arg\left\{\frac{z}{f^{(p-1)}(z)}\right\}\right| \\
&< \sin^{-1}\alpha(\beta_0) + \frac{\pi\beta_0}{2} \\
&= \frac{\pi}{2}\left(2\beta_0 + \frac{2}{\pi}\tan^{-1}\beta_0\right) \\
&= \frac{\pi}{2}.
\end{aligned}$$

This shows that

$$\mathfrak{Re}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > 0, \quad |z| < 1,$$

and by Lemma 2.3 we have

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad |z| < 1,$$

or $f(z)$ is p-valently starlike in $|z| < 1$. □

Remark Note that if $m(x) = 2x + \frac{2}{\pi}\tan^{-1} x$, then

$$m(0.383) = 0.9988537761\cdots, \qquad m(0.384) = 1.001408771\cdots.$$

Hence, if

$$2\beta_0 + \frac{2}{\pi}\tan^{-1}\beta_0 = 1,$$

then $\beta_0 = 0.38\cdots$. Moreover,

$$0.9673808495\cdots < \sin^{-1}\alpha(\beta_0) = \frac{\pi}{2}\left(\beta_0 + \frac{2}{\pi}\tan^{-1}\beta_0\right) = \frac{(1-\beta_0)\pi}{2} < 0.96982343\cdots$$

and

$$\alpha(\beta_0) = \sin\left\{\frac{\pi}{2}\left(\beta_0 + \frac{2}{\pi}\tan^{-1}\beta_0\right)\right\} = 0.824669\cdots.$$

For $p = 1$, Theorem 3.1 becomes the following corollary which extends the result contained in Lemma 2.1.

Corollary 3.2 *Let* $f \in \mathcal{A}(1)$ *and suppose that*

$$\left|f''(z)\right| < \alpha(\beta_0), \quad |z| < 1, \tag{3.10}$$

where $\beta_0 = 0.38\cdots$ *is the positive root of the equation*

$$2\beta + \frac{2}{\pi}\tan^{-1}\beta = 1, \quad 0 < \beta < 1, \tag{3.11}$$

and

$$\begin{aligned}\alpha(\beta_0) &= \sin\left\{\frac{(1-\beta_0)\pi}{2}\right\}\\ &= 0.82\cdots.\end{aligned}$$

Then we have

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad |z| < 1, \tag{3.12}$$

or $f(z)$ *is starlike univalent in* $|z| < 1$.

An analytic function $f(z)$ is said to be typically real if the inequality $\mathfrak{Im} z \mathfrak{Im} f(z) \geq 0$ holds for all $z \in \mathbb{D}$. From the definition of a typically real function it follows that $z \in \mathbb{D}^+ \Leftrightarrow f(z) \in \mathbb{C}^+$ and $z \in \mathbb{D}^- \Leftrightarrow f(z) \in \mathbb{C}^-$. The symbols $\mathbb{D}^+$, $\mathbb{D}^-$, $\mathbb{C}^+$, $\mathbb{C}^-$ stand for the following sets: the upper and the lower halves of the disk $\mathbb{D}$, the upper half-plane and the lower half-plane, respectively.

Theorem 3.3 *Let $f(z) \in \mathcal{A}_p$ and suppose that*

$$\left|\arg\left\{\frac{zf^{(p)}(z)}{g(z)}\right\}\right| < \frac{\pi}{2} + \tan^{-1}\frac{1-|z|}{1+|z|}, \quad z \in \mathbb{D}, \tag{3.13}$$

where $g(z)$ is univalent starlike in $\mathbb{D}$ and the functions

$$\frac{zg'(z)}{g(z)} \quad \text{and} \quad \frac{z(\frac{zf^{(p-1)}(z)}{g(z)})'}{\frac{zf^{(p-1)}(z)}{g(z)}} \tag{3.14}$$

are typically real in $\mathbb{D}$. Then we have

$$\left|\arg\frac{zf^{(p-1)}(z)}{g(z)}\right| < \frac{\pi}{2}, \quad z \in \mathbb{D}. \tag{3.15}$$

Proof Let us put

$$q(z) = \frac{zf^{(p-1)}(z)}{p!g(z)}, \qquad q(0) = 1.$$

Then it follows that

$$\frac{zf^{(p)}(z)}{g(z)} = p!q(z)\left(\frac{zg'(z)}{g(z)} + \frac{zq'(z)}{q(z)}\right). \tag{3.16}$$

From the hypothesis

$$\frac{zg'(z)}{g(z)} + \frac{zq'(z)}{q(z)}$$

is typically real in $\mathbb{D}$. If there exists a point z_0, $|z_0| < 1$, such that

$$\left|\arg\left\{q(z)\right\}\right| < \frac{\pi}{2} \quad \text{for } |z| < |z_0|$$

and

$$\left|\arg\left\{q(z_0)\right\}\right| = \frac{\pi}{2}, \quad q(z_0) = \pm ia, \text{ and } a > 0,$$

then by Lemma 2.2 we have

$$\frac{z_0q'(z_0)}{q(z_0)} = is, \tag{3.17}$$

where

$$s \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \quad \text{when } \arg\left\{q(z_0)\right\} = \frac{\pi}{2}$$

and

$$s \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \quad \text{when } \arg\left\{q(z_0)\right\} = -\frac{\pi}{2}.$$

For the case

$$\arg\{q(z_0)\} = \frac{\pi}{2}, \quad q(z_0) = ia, a > 0,$$

we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = is, \quad s \geq \frac{1}{2}\left(a + \frac{1}{a}\right) > 1, \tag{3.18}$$

hence

$$\mathfrak{Im}\{z_0\} > 0 \tag{3.19}$$

because $zq'(z)/q(z)$ is typically real. Therefore, (3.19) yields that

$$\mathfrak{Im}\left\{\frac{z_0 g'(z_0)}{g(z_0)}\right\} > 0, \tag{3.20}$$

because $zg'(z)/g(z)$ is typically real. Moreover,

$$\frac{1-|z_0|}{1+|z_0|} \leq \mathfrak{Re}\left\{\frac{z_0 g'(z_0)}{g(z_0)}\right\} \leq \frac{1+|z_0|}{1-|z_0|} \tag{3.21}$$

because $g(z)$ is a univalent starlike function, see [10]. Applying (3.18), (3.20) and (3.21) in (3.16), we have

$$\begin{aligned}
\arg\left\{\frac{z_0 f^{(p)}(z_0)}{g(z_0)}\right\} &= \arg\{q(z_0)\} + \arg\left\{\frac{z_0 g'(z_0)}{g(z_0)} + \frac{z_0 q'(z_0)}{q(z_0)}\right\} \\
&= \arg\{q(z_0)\} + \arg\left\{\frac{z_0 g'(z_0)}{g(z_0)} + is\right\} \\
&\geq \frac{\pi}{2} + \tan^{-1}\frac{1-|z_0|}{1+|z_0|}.
\end{aligned}$$

This contradicts (3.13), and for the case

$$\arg\{q(z_0)\} = -\frac{\pi}{2},$$

applying the same method as above, we have

$$\begin{aligned}
\arg\left\{\frac{z_0 f^{(p)}(z_0)}{g(z_0)}\right\} &= \arg\{q(z_0)\} + \arg\left\{\frac{z_0 g'(z_0)}{g(z_0)} + \frac{z_0 q'(z_0)}{q(z_0)}\right\} \\
&\leq -\left\{\frac{\pi}{2} + \tan^{-1}\frac{1-|z_0|}{1+|z_0|}\right\}.
\end{aligned}$$

This also contradicts (3.13) and, therefore, it shows that (3.15) holds. □

Corollary 3.4 *Let $f(z) \in \mathcal{A}_p$ and all the coefficients of $f(z)$ are real and suppose that*

$$\left|\arg\left\{\frac{zf^{(p)}(z)}{g(z)}\right\}\right| < \frac{\pi}{2} + \tan^{-1}\frac{1-|z|}{1+|z|}, \quad z \in \mathbb{D},$$

where $g(z)$ is univalent starlike and typically real in $\mathbb{D}$. Then we have

$$\left|\arg \frac{zf^{(p-1)}(z)}{g(z)}\right| < \frac{\pi}{2}, \quad z \in \mathbb{D}.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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