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Shephard type problems for general L_p-centroid bodies

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Abstract
Lutwak and Zhang proposed the concept of L_p-centroid bodies. Then Feng and Wang gave the notion of general L_p-centroid bodies. In this article, based on the L_p-dual affine surface area, we address Shephard type problems for the general L_p-centroid bodies.

MSC: 52A20; 52A40; 52A39

Keywords: Shephard problem; L_p-dual affine surface area; general L_p-centroid body

1 Introduction and main results

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in Euclidean space $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interiors, the set of convex bodies whose centroid lie at the origin and the set of origin-symmetric convex bodies in $\mathbb{R}^n$, we write $\mathcal{K}_o^n$, $\mathcal{K}_c^n$, and $\mathcal{K}_{os}^n$, respectively. $\mathcal{S}_o^n$, $\mathcal{S}_{os}^n$, respectively, denote the set of star bodies (about the origin) and the set of origin-symmetric star bodies in $\mathbb{R}^n$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$, and $V(K)$ denotes the n-dimensional volume of body K. For the standard unit ball B in $\mathbb{R}^n$, denote $\omega_n = V(B)$.

In 1997, Lutwak and Zhang [1] gave the concept of an L_p-centroid body as follows: For each compact star-shaped about the origin $K \subset \mathbb{R}^n$, real $p \geq 1$, the L_p-centroid body, $\Gamma_p K$, of K is an origin-symmetric convex body whose support function is defined by

$$h^p_{\Gamma_p K}(u) = \frac{1}{c_{n,p} V(K)} \int_K |u \cdot x|^p \, dx$$

for any $u \in S^{n-1}$. Here

$$c_{n,p} = \omega_{n+p}/\omega_2 \omega_n \omega_{p-1}. \tag{1.1}$$

Meanwhile, they [1] obtained the L_p-centroid affine inequality, which implies the well-known Blaschke-Santaló inequality. Hereafter, associating the L_p-centroid bodies with the L_p-projection bodies, Lutwak *et al.* [2] established the L_p-Busemann-Petty centroid inequality and the L_p-Petty projection inequality. For the studies of L_p-centroid bodies, also see [3–7].

In 2005, Ludwig [8] introduced a function $\varphi_\tau : \mathbb{R} \to [0, +\infty)$ by

$$\varphi_\tau(t) = |t| + \tau t \tag{1.2}$$

for $\tau \in [-1,1]$. Further, in [8] the general L_p-moment bodies and the general L_p-projection bodies were defined by (1.2). In 2009, Haberl and Schuster [9] derived the general L_p-moment body (the general L_p-projection body) is an L_p-Minkowski combination of the asymmetric L_p-moment body (the asymmetric L_p-projection body) and established the general L_p-Busemann-Petty centroid inequality and the general L_p-Petty projection inequality.

Recently, motivated by Ludwig's, and Haberl and Schuster's work, Feng *et al.* [10] defined asymmetric L_p-centroid bodies as follows: For $K \in \mathcal{S}_o^n$, $p \geq 1$, the asymmetric L_p-centroid body, $\Gamma_p^+ K$, of K is a convex body whose support function is defined by

$$h^p_{\Gamma_p^+ K}(u) = \frac{2}{c_{n,p} V(K)} \int_K (u \cdot x)_+^p \, dx$$

for any $u \in S^{n-1}$. Using polar coordinates in the above definition, we easily obtain, for any $u \in S^{n-1}$,

$$h^p_{\Gamma_p^+ K}(u) = \frac{2}{c_{n,p}(n+p) V(K)} \int_{S^{n-1}} (u \cdot v)_+^p \rho_K(v)^{n+p} \, dv, \tag{1.3}$$

where $(u \cdot x)_+ = \max\{u \cdot x, 0\}$, $c_{n,p}$ satisfies (1.1) and the integration is with respect to Lebesgue measure on S^{n-1}. Obviously, $\Gamma_p^+ B = B$. They also defined

$$\Gamma_p^- K = \Gamma_p^+(-K).$$

By (1.2), Feng *et al.* [10] introduced the general L_p-centroid bodies: For $K \in \mathcal{S}_o^n$, $p \geq 1$, and $\tau \in [-1,1]$, the general L_p-centroid body, $\Gamma_p^\tau K$, of K is a convex body whose support function is defined by

$$\begin{aligned} h^p_{\Gamma_p^\tau K}(u) &= \frac{2}{c_{n,p}(\tau) V(K)} \int_K \varphi_\tau (u \cdot x)^p \, dx \\ &= \frac{2}{c_{n,p}(\tau)(n+p) V(K)} \int_{S^{n-1}} \varphi_\tau (u \cdot v)^p \rho_K(v)^{n+p} \, dv, \end{aligned} \tag{1.4}$$

where

$$c_{n,p}(\tau) = c_{n,p}\big[(1+\tau)^p + (1-\tau)^p\big]$$

and $c_{n,p}$ satisfies (1.1).

Obviously, $\Gamma_p^\tau B = B$, and if $\tau = 0$, then $\Gamma_p^\tau K = \Gamma_p K$.

From the definition of $\Gamma_p^\pm K$ and (1.4), it follows that if $K \in \mathcal{S}_{os}^n$, $p \geq 1$, and $\tau \in [-1,1]$, then, for any $u \in S^{n-1}$,

$$h\big(\Gamma_p^\tau K, u\big)^p = f_1(\tau) h\big(\Gamma_p^+ K, u\big)^p + f_2(\tau) h\big(\Gamma_p^- K, u\big)^p, \tag{1.5}$$

where

$$f_1(\tau) = \frac{(1+\tau)^p}{(1+\tau)^p + (1-\tau)^p}, \qquad f_2(\tau) = \frac{(1-\tau)^p}{(1+\tau)^p + (1-\tau)^p}. \tag{1.6}$$

From (1.6), we know that

$$f_1(-\tau) = f_2(\tau), \qquad f_2(-\tau) = f_1(\tau), \tag{1.7}$$

$$f_1(\tau) + f_2(\tau) = 1. \tag{1.8}$$

The general L_p-centroid bodies belong to a new and rapidly evolving asymmetric L_p Brunn-Minkowski theory that has its origins in the work of Ludwig, Haberl, and Schuster (see [8, 9, 11–14]). For the further researches of asymmetric L_p Brunn-Minkowski theory, also see [10, 15–27].

In 1996, Lutwak [28] introduced the concept of an L_p-affine surface area as follows: For $K \in \mathcal{K}_o^n$ and $p \geq 1$, the L_p-affine surface area, $\Omega_p(K)$, of K is defined by

$$n^{-\frac{p}{n}} \Omega_p(K)^{\frac{n+p}{n}} = \inf\{nV_p(K, Q^*)V(Q)^{\frac{p}{n}} : Q \in \mathcal{S}_o^n\},$$

where $V_p(M,N)$ denotes the L_p-mixed volume of $M, N \in \mathcal{K}_o^n$.

Further, Wang and Leng [29] defined ith L_p-mixed affine surface area, $\Omega_{p,i}(K)$, of K (for $i = 0$, $\Omega_{p,0}(K)$ is just the L_p-affine surface area $\Omega_p(K)$) and extended some of Lutwak's results. Regarding the studies of an L_p-affine surface area, many results have been obtained (see [28–35]).

Associated with the L_p-dual mixed volumes, Wang and He [36] gave the notion of the L_p-dual affine surface area. For $K \in \mathcal{S}_o^n$ and $1 \leq p < n$, the L_p-dual affine surface area, $\widetilde{\Omega}_{-p}(K)$, of K is defined by

$$n^{\frac{p}{n}} \widetilde{\Omega}_{-p}(K)^{\frac{n-p}{n}} = \inf\{n\widetilde{V}_{-p}(K, Q^*)V(Q)^{-\frac{p}{n}} : Q \in \mathcal{K}_c^n\}, \tag{1.9}$$

where $\widetilde{V}_{-p}(M,N)$ denotes the L_p-dual mixed volume of $M, N \in \mathcal{S}_o^n$.

In 2014, Feng and Wang [37] improved definition (1.9) from $Q \in \mathcal{K}_c^n$ to $Q \in \mathcal{S}_{os}^n$ as follows: For $K \in \mathcal{S}_o^n$ and $1 \leq p < n$, the L_p-dual affine surface area, $\widetilde{\Omega}_{-p}(K)$, of K is defined by

$$n^{\frac{p}{n}} \widetilde{\Omega}_{-p}(K)^{\frac{n-p}{n}} = \inf\{n\widetilde{V}_{-p}(K, Q^*)V(Q)^{-\frac{p}{n}} : Q \in \mathcal{S}_{os}^n\}. \tag{1.10}$$

Let Z_p^* denote the set of polar of all L_p-projection bodies, then $\mathcal{Z}_p^* \subseteq \mathcal{S}_{os}^n$. If $Q \in Z_p^*$ in (1.10), write $\widetilde{\Omega}_{-p}^o(K)$ by

$$n^{\frac{p}{n}} \widetilde{\Omega}_{-p}^o(K) = \inf\{n\widetilde{V}_{-p}(K, Q^*)V(Q)^{\frac{p}{n}} : Q \in \mathcal{Z}_p^*\}. \tag{1.11}$$

According to (1.10) and (1.11), Feng and Wang [37] studied the Shephard type problems for the L_p-centroid bodies. First, they gave an affirmative form of the Shephard type problems for the L_p-centroid bodies as follows.

Theorem 1.A *For $K, L \in \mathcal{S}_o^n$, $1 \leq p < n$, if $\Gamma_p K \subseteq \Gamma_p L$, then*

$$\frac{\widetilde{\Omega}_{-p}^o(K)^{\frac{n-p}{n}}}{V(K)} \leq \frac{\widetilde{\Omega}_{-p}^o(L)^{\frac{n-p}{n}}}{V(L)},$$

with equality if and only if $\Gamma_p K = \Gamma_p L$.

Hereafter, combining with definition (1.10) of the L_p-dual affine surface area, the authors [37] gave an improved form of the Shephard type problems for the L_p-centroid bodies.

Theorem 1.B *For $K \in \mathcal{S}^n_o$, $L \in \mathcal{S}^n_{os}$ and $1 \le p < n$, if $\Gamma_p K = \Gamma_p L$, then*

$$\widetilde{\Omega}_{-p}(K) \le \widetilde{\Omega}_{-p}(L),$$

with equality if and only if $K = L$.

Finally, they [37] obtained a negative form of the Shephard type problems for the L_p-centroid bodies.

Theorem 1.C *For $L \in \mathcal{S}^n_o$ and $1 \le p < n$, if L is not origin-symmetric star body, then there exists $K \in \mathcal{S}^n_{os}$, such that*

$$\Gamma_p K \subset \Gamma_p L,$$

but

$$\widetilde{\Omega}_{-p}(K) > \widetilde{\Omega}_{-p}(L).$$

In this paper, associated with definition (1.10) of the L_p-dual affine surface area, we will research the Shephard type problems for the general L_p-centroid bodies. For convenience, we improve definition (1.11) as follows: Let $Z^{\tau,*}_p$ denote the set of polar of all general L_p-projection bodies, for $K \in \mathcal{S}^n_o$ and $1 \le p < n$, the L_p-dual affine surface area, $\widetilde{\Omega}^{\star}_{-p}(K)$, of K is given by

$$n^{\frac{p}{n}} \widetilde{\Omega}^{\star}_{-p}(K)^{\frac{n-p}{n}} = \inf\bigl\{n\widetilde{V}_{-p}\bigl(K, Q^*\bigr) V(Q)^{\frac{p}{n}} : Q \in \mathcal{Z}^{\tau,*}_p\bigr\}. \tag{1.12}$$

From definition (1.12), we first give an affirmative form of the Shephard type problems for the general L_p-centroid bodies, *i.e.*, a general form of Theorem 1.A is obtained.

Theorem 1.1 *For $K, L \in \mathcal{S}^n_o$, $1 \le p < n$, and $\tau \in [-1,1]$, if $\Gamma^\tau_p K \subseteq \Gamma^\tau_p L$, then*

$$\frac{\widetilde{\Omega}^{\star}_{-p}(K)^{\frac{n-p}{n}}}{V(K)} \le \frac{\widetilde{\Omega}^{\star}_{-p}(L)^{\frac{n-p}{n}}}{V(L)}, \tag{1.13}$$

with equality if and only if $\Gamma^\tau_p K = \Gamma^\tau_p L$.

Next, corresponding to Theorem 1.B and combining with definition (1.10), we get an improved form of the Shephard type problems for the general L_p-centroid bodies.

Theorem 1.2 *Let $K \in \mathcal{S}^n_o$, $L \in \mathcal{S}^n_{os}$, $1 \le p < n$, and $\tau \in [-1,1]$, if $\Gamma^\tau_p K = \Gamma^\tau_p L$, then*

$$\widetilde{\Omega}_{-p}(K) \le \widetilde{\Omega}_{-p}(L), \tag{1.14}$$

with equality if and only if $K = L$.

Further, we prove a general version of Theorem 1.C, that is, a negative form of the Shephard type problems for the general L_p-centroid bodies is given.

Theorem 1.3 *For $L \in \mathcal{S}_o^n$, $1 \le p < n$, and $\tau \in (-1,1)$, if L is not origin-symmetric star body, then there exists $K \in \mathcal{S}_o^n$ (for $\tau = 0$, $K \in \mathcal{S}_{os}^n$), such that*

$$\Gamma_p^\tau K \subset \Gamma_p^\tau L,$$

but

$$\widetilde{\Omega}_{-p}(K) > \widetilde{\Omega}_{-p}(L).$$

Besides, corresponding to Theorem 1.C, we generalize the scope of negative solutions of the Shephard type problems for the L_p-centroid bodies from $K \in \mathcal{S}_{os}^n$ to $K \in \mathcal{S}_o^n$.

Theorem 1.4 *For $L \in \mathcal{S}_o^n$ and $1 \le p < n$, if L is not origin-symmetric star body, then there exists $K \in \mathcal{S}_o^n$, such that*

$$\Gamma_p K \subset \Gamma_p L,$$

but

$$\widetilde{\Omega}_{-p}(K) > \widetilde{\Omega}_{-p}(L).$$

The proofs of Theorems 1.1-1.4 are completed in Section 4. In order to prove our results, we give two inequalities for the general L_p-harmonic Blaschke bodies in Section 3.

2 Preliminaries

2.1 Support function, radial function and polar

If $K \in \mathcal{K}^n$, then its support function, $h_K = h(K,\cdot) : \mathbb{R}^n \to (-\infty, \infty)$, is defined by (see [38, 39])

$$h(K,x) = \max\{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n,$$

where $x \cdot y$ denotes the standard inner product of x and y.

If K is a compact star-shaped (about the origin) in $\mathbb{R}^n$, then its radial function, $\rho_K = \rho(K,\cdot) : \mathbb{R}^n \setminus \{0\} \to [0, +\infty)$, is defined by (see [38, 39])

$$\rho(K,x) = \max\{\lambda \ge 0 : \lambda x \in K\}, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

Given $c > 0$, we can get, for any $u \in S^{n-1}$,

$$\rho(cK,u) = c\rho(K,u). \tag{2.1}$$

If ρ_K is positive and continuous, K will be called a star body (about the origin). Two star bodies K and L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

If E is a non-empty set in $\mathbb{R}^n$, the polar set, E^*, of E is defined by (see [38, 39])

$$E^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, y \in E\}. \tag{2.2}$$

From (2.2), we easily see that if $K \in \mathcal{S}_o^n$, then $K^* \in \mathcal{K}_o^n$ (see [38]).

2.2 L_p-Dual mixed volumes

For $K, L \in \mathcal{S}_o^n$, $p \geq 1$, and $\lambda, \mu \geq 0$ (not both zero), the L_p-harmonic radial combination, $\lambda \star K +_{-p} \mu \star L \in \mathcal{S}_o^n$, of K and L is defined by (see [28])

$$\rho(\lambda \star K +_{-p} \mu \star L, \cdot)^{-p} = \lambda\rho(K, \cdot)^{-p} + \mu\rho(L, \cdot)^{-p}, \tag{2.3}$$

where the operation '$+_{-p}$' is called L_p-harmonic radical addition and $\lambda \star K$ denotes the L_p-harmonic radical scalar multiplication. From (2.1) and (2.3), we have $\lambda \star K = \lambda^{-\frac{1}{p}} K$.

Associated with (2.3), Lutwak [28] introduced the notion of an L_p-dual mixed volume as follows: For $K, L \in \mathcal{S}_o^n$, $p \geq 1$, and $\varepsilon > 0$, the L_p-dual mixed volume, $\widetilde{V}_{-p}(K, L)$, of K and L is defined by (see [28])

$$\frac{n}{-p}\widetilde{V}_{-p}(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K +_{-p} \varepsilon \star L) - V(K)}{\varepsilon}.$$

The definition above and Hospital's rule give the following integral representation of an L_p-dual mixed volume (see [28]):

$$\widetilde{V}_{-p}(K, L) = \frac{1}{n}\int_{S^{n-1}} \rho_K^{n+p}(u)\rho_L^{-p}(u)\, du,$$

where the integration is with respect to spherical Lebesgue measure on S^{n-1}.

The L_p-dual Minkowski inequality can be stated as follows (see [28]).

Theorem 2.A *If $K, L \in \mathcal{S}_o^n$, $p \geq 1$, then*

$$\widetilde{V}_{-p}(K, L) \geq V(K)^{\frac{n+p}{n}} V(L)^{-\frac{p}{n}}, \tag{2.4}$$

with equality if and only if K and L are dilates.

2.3 General L_p-projection bodies

The general L_p-projection body was introduced by Ludwig (see [8]). For $K \in \mathcal{K}_o^n$, $p \geq 1$, and $\tau \in [-1, 1]$, the general L_p-projection body, $\Pi_p^\tau K \in \mathcal{K}_o^n$, of K is given by

$$h_{\Pi_p^\tau K}^p(u) = \alpha_{n,p}(\tau) \int_{S^{n-1}} \varphi_\tau(u \cdot v)^p\, dS_p(K, v),$$

where φ_τ satisfies (1.2) and

$$\alpha_{n,p}(\tau) = \frac{\alpha_{n,p}}{(1+\tau)^p + (1-\tau)^p}$$

with $\alpha_{n,p} = 1/c_{n,p}(n+p)\omega_n$.

3 General L_p-harmonic Blaschke bodies

In order to prove our results, we require the notions of L_p-harmonic Blaschke combinations and general L_p-harmonic Blaschke bodies.

For $K, L \in \mathcal{S}_o^n$, $p \geq 1$, and $\lambda, \mu \geq 0$ (not both zero), the L_p-harmonic Blaschke combination, $\lambda * K \hat{+}_p \mu * L \in \mathcal{S}_o^n$, of K and L is defined by (see [37])

$$\frac{\rho(\lambda * K \hat{+}_p \mu * L, \cdot)^{n+p}}{V(\lambda * K \hat{+}_p \mu * L)} = \lambda \frac{\rho(K, \cdot)^{n+p}}{V(K)} + \mu \frac{\rho(L, \cdot)^{n+p}}{V(L)}, \tag{3.1}$$

where the operation '$\hat{+}_p$' is called L_p-harmonic Blaschke addition and $\lambda * K$ denotes L_p-harmonic Blaschke scalar multiplication. From (2.1) and (3.1), we know $\lambda * K = \lambda^{\frac{1}{p}} K$.

Let $\lambda = \mu = \frac{1}{2}$ and $L = -K$ in (3.1), then the L_p-harmonic Blaschke body, $\widehat{\nabla}_p K$, of $K \in \mathcal{S}_o^n$ is given by (see [37])

$$\widehat{\nabla}_p K = \frac{1}{2} * K \hat{+}_p \frac{1}{2} * (-K).$$

According to (3.1), Feng and Wang [15] defined general L_p-harmonic Blaschke bodies as follows: For $K \in \mathcal{S}_o^n$, $p \geq 1$, and $\tau \in [-1,1]$, the general L_p-harmonic Blaschke body, $\widehat{\nabla}_p^\tau K = f_1(\tau) \circ K \hat{+}_p f_2(\tau) \circ (-K)$, of K is defined by

$$\frac{\rho(\widehat{\nabla}_p^\tau K, \cdot)^{n+p}}{V(\widehat{\nabla}_p^\tau K)} = f_1(\tau) \frac{\rho(K, \cdot)^{n+p}}{V(K)} + f_2(\tau) \frac{\rho(-K, \cdot)^{n+p}}{V(-K)}, \tag{3.2}$$

where $f_1(\tau), f_2(\tau)$ satisfy (1.6).

Obviously, if $\tau = 0$, then $\widehat{\nabla}_p^\tau K = \widehat{\nabla}_p K$. In addition, if $\tau = \pm 1$, then we write $\widehat{\nabla}_p^\tau (K) = \widehat{\nabla}_p^\pm K$, and $\widehat{\nabla}_p^+ K = K$, $\widehat{\nabla}_p^- K = -K$.

For the L_p-harmonic Blaschke combination (3.1), Feng and Wang [37] proved the following fact.

Theorem 3.A *If $K, L \in \mathcal{S}_o^n$, $p \geq 1$, $\lambda, \mu \geq 0$ (not both zero), then*

$$V(\lambda * K \hat{+}_p \mu * L)^{\frac{p}{n}} \geq \lambda V(K)^{\frac{p}{n}} + \mu V(L)^{\frac{p}{n}}, \tag{3.3}$$

with equality if and only if K and L are dilates.

From Theorem 3.A, we easily get the following.

Corollary 3.1 *If $K \in \mathcal{S}_o^n$, $p \geq 1$, and $\tau \in [-1,1]$, then*

$$V\big(\widehat{\nabla}_p^\tau K\big) \geq V(K). \tag{3.4}$$

For $\tau \in (-1,1)$, equality holds if and only if K is origin-symmetric. For $\tau = \pm 1$, (3.4) is identic.

Proof For $\tau \in (-1,1)$, taking $\lambda = f_1(\tau)$, $\mu = f_2(\tau)$, and $L = -K$ in (3.3), then by (1.8) we immediately get inequality (3.4). According to the equality condition of inequality (3.3), we see that equality holds in inequality (3.4) if and only if K and $-K$ are dilates, *i.e.*, K is origin-symmetric.

For $\tau = \pm 1$, by $\widehat{\nabla}_p^+ K = K$ and $\widehat{\nabla}_p^- K = -K$, we know that (3.4) is identic. □

Further, according to the L_p-harmonic Blaschke combination (3.1) and definition (1.10) of the L_p-dual affine surface area, Feng and Wang [37] gave the following result.

Theorem 3.B *If $K, L \in \mathcal{S}_o^n$, $\lambda, \mu \geq 0$ (not both zero) and $1 \leq p < n$, then*

$$\frac{\widetilde{\Omega}_{-p}(\lambda * K \hat{+}_p \mu * L)^{\frac{n-p}{n}}}{V(\lambda * K \hat{+}_p \mu * L)} \geq \lambda \frac{\widetilde{\Omega}_{-p}(K)^{\frac{n-p}{n}}}{V(K)} + \mu \frac{\widetilde{\Omega}_{-p}(L)^{\frac{n-p}{n}}}{V(L)}, \tag{3.5}$$

with equality if and only if K and L are dilates.

Corollary 3.2 *If $K \in \mathcal{S}_o^n$, $1 \leq p < n$, and $\tau \in [-1, 1]$, then*

$$\widetilde{\Omega}_{-p}\bigl(\widehat{\nabla}_p^\tau K\bigr) \geq \widetilde{\Omega}_{-p}(K). \tag{3.6}$$

For $\tau \in (-1, 1)$, equality holds if and only if K is origin-symmetric. For $\tau = \pm 1$, (3.6) is identic.

Proof For $\tau \in (-1, 1)$, let $\lambda = f_1(\tau)$, $\mu = f_2(\tau)$, and $L = -K$ in (3.5), we obtain

$$\frac{\widetilde{\Omega}(\widehat{\nabla}_p^\tau K)^{\frac{n-p}{n}}}{V(\widehat{\nabla}_p^\tau K)} \geq f_1(\tau) \frac{\widetilde{\Omega}_{-p}(K)^{\frac{n-p}{n}}}{V(K)} + f_2(\tau) \frac{\widetilde{\Omega}_{-p}(-K)^{\frac{n-p}{n}}}{V(-K)}. \tag{3.7}$$

For any $Q \in \mathcal{S}_{os}^n$, using $\rho_{Q^*}(u) = \rho_{-Q^*}(u)$, for any $u \in S^{n-1}$, we get

$$\widetilde{V}_{-p}\bigl(-K, Q^*\bigr) = \widetilde{V}_{-p}\bigl(K, Q^*\bigr). \tag{3.8}$$

Associated with (1.10) and (3.8), we have

$$\widetilde{\Omega}_{-p}(-K) = \widetilde{\Omega}_{-p}(K). \tag{3.9}$$

Thus by (3.7), (3.9), and (1.8), we know

$$\left(\frac{\widetilde{\Omega}_{-p}(\widehat{\nabla}_p^\tau K)}{\widetilde{\Omega}_{-p}(K)}\right)^{\frac{n-p}{n}} \geq \frac{V(\widehat{\nabla}_p^\tau K)}{V(K)}.$$

This and inequality (3.4) yield inequality (3.6).

From the equality conditions of inequalities (3.4) and (3.5), we see that equality holds in (3.6) if and only if K is origin-symmetric.

For $\tau = \pm 1$, obviously, (3.6) is identic. □

4 Proofs of theorems

In this section, we complete the proofs of Theorems 1.1-1.4. In the proof of Theorem 1.1, we require a lemma as follows.

Lemma 4.1 ([10]) *If $K \in \mathcal{S}_o^n$, $p \geq 1$, $\tau \in [-1,1]$, then, for any $Q \in \mathcal{K}_o^n$,*

$$V_p(Q, \Gamma_p^\tau K) = \frac{\omega_n}{V(K)} \widetilde{V}_{-p}(K, \Pi_p^{\tau,*} Q).$$

Proof of Theorem 1.1 Since $\Gamma_p^\tau K \subseteq \Gamma_p^\tau L$, for any $Q \in \mathcal{K}_o^n$,

$$V_p(Q, \Gamma_p^\tau K) \leq V_p(Q, \Gamma_p^\tau L), \tag{4.1}$$

with equality if and only if $\Gamma_p^\tau K = \Gamma_p^\tau L$.

Therefore, from (4.1) and Lemma 4.1, we have

$$\frac{\widetilde{V}_{-p}(K, \Pi_p^{\tau,*} Q)}{V(K)} \leq \frac{\widetilde{V}_{-p}(L, \Pi_p^{\tau,*} Q)}{V(L)}. \tag{4.2}$$

Let $M = \Pi_p^\tau Q$, then $M \in Z_p^{\tau,*}$. From (1.11) and (4.2), we get

$$\begin{aligned} \frac{n^{\frac{p}{n}} \widetilde{\Omega}^\star_{-p}(K)^{\frac{n-p}{n}}}{V(K)} &= \inf\left\{ \frac{n\widetilde{V}_{-p}(K, M^*)}{V(K)} V(M)^{-\frac{p}{n}} : M \in \mathcal{Z}_p^{\tau,*} \right\} \\ &\leq \inf\left\{ \frac{n\widetilde{V}_{-p}(L, M^*)}{V(L)} V(M)^{-\frac{p}{n}} : M \in \mathcal{Z}_p^{\tau,*} \right\} \\ &= \frac{n^{\frac{p}{n}} \widetilde{\Omega}^\star_{-p}(L)^{\frac{n-p}{n}}}{V(L)}, \end{aligned}$$

i.e., (1.13) is obtained.

According to the equality condition of (4.1), we know that the equality holds in (1.13) if and only if $\Gamma_p^\tau K = \Gamma_p^\tau L$. □

The proof of Theorem 1.2 requires the following lemmas.

Lemma 4.2 ([37]) *For $K, L \in \mathcal{S}_o^n$, $p \geq 1$, if $\Gamma_p K = \Gamma_p L$, then, for any $Q \in \mathcal{S}_{os}^n$,*

$$\frac{\widetilde{V}_{-p}(K, Q)}{V(K)} = \frac{\widetilde{V}_{-p}(L, Q)}{V(L)}.$$

Lemma 4.3 *For $K, L \in \mathcal{S}_o^n$, $p \geq 1$, and $\tau \in [-1,1]$, if $\Gamma_p^\tau K = \Gamma_p^\tau L$, then, for any $Q \in \mathcal{S}_{os}^n$,*

$$\frac{\widetilde{V}_{-p}(K, Q)}{V(K)} = \frac{\widetilde{V}_{-p}(L, Q)}{V(L)}. \tag{4.3}$$

Proof Let $\tau = 0$ in (1.5), we have, for any $u \in S^{n-1}$,

$$h(\Gamma_p K, u)^p = \frac{1}{2} h(\Gamma_p^+ K, u)^p + \frac{1}{2} h(\Gamma_p^- K, u)^p. \tag{4.4}$$

On the other hand, by (1.5), (1.7), (1.8), and (4.4), we see that, for any $u \in S^{n-1}$,

$$\begin{aligned} &\frac{1}{2} h(\Gamma_p^\tau K, u)^p + \frac{1}{2} h(\Gamma_p^{-\tau} K, u)^p \\ &\quad = \frac{1}{2}\left[f_1(\tau) h^p_{\Gamma_p^+ K}(u) + f_2(\tau) h^p_{\Gamma_p^- K}(u)\right] + \frac{1}{2}\left[f_1(-\tau) h^p_{\Gamma_p^+ K}(u) + f_2(-\tau) h^p_{\Gamma_p^- K}(u)\right] \end{aligned}$$

$$= \frac{1}{2}\left[f_1(\tau)h^p_{\Gamma^+_p K}(u) + f_2(\tau)h^p_{\Gamma^-_p K}(u)\right] + \frac{1}{2}\left[f_2(\tau)h^p_{\Gamma^+_p K}(u) + f_1(\tau)h^p_{\Gamma^-_p K}(u)\right]$$
$$= \frac{1}{2}h\left(\Gamma^+_p K, u\right)^p + \frac{1}{2}h\left(\Gamma^-_p K, u\right)^p = h(\Gamma_p K, u)^p,$$

i.e., for any $u \in S^{n-1}$,

$$h(\Gamma_p K, u)^p = \frac{1}{2}h\left(\Gamma^\tau_p K, u\right)^p + \frac{1}{2}h\left(\Gamma^{-\tau}_p K, u\right)^p. \tag{4.5}$$

From this, if $\Gamma^\tau_p K = \Gamma^\tau_p L$, then $\Gamma^{-\tau}_p K = \Gamma^{-\tau}_p L$. Thus by (4.5) we obtain $\Gamma_p K = \Gamma_p L$. This combined with Lemma 4.2 gives (4.3). □

Proof of Theorem 1.2 According to (1.9), we know

$$\frac{n^{\frac{p}{n}}\widetilde{\Omega}_{-p}(K)^{\frac{n-p}{n}}}{V(K)} = \inf\left\{n\frac{\widetilde{V}_{-p}(K,Q^*)}{V(K)}V(Q)^{-\frac{p}{n}} : Q \in \mathcal{S}^n_{os}\right\}. \tag{4.6}$$

Since $\Gamma^\tau_p K = \Gamma^\tau_p L$, thus from Lemma 4.3, we get, for any $Q \in \mathcal{S}^n_{os}$,

$$\frac{\widetilde{V}_{-p}(K,Q^*)}{V(K)} = \frac{\widetilde{V}_{-p}(L,Q^*)}{V(L)}. \tag{4.7}$$

Thus from (4.6) and (4.7), we have

$$\frac{\widetilde{\Omega}_{-p}(K)^{\frac{n-p}{n}}}{V(K)} = \frac{\widetilde{\Omega}_{-p}(L)^{\frac{n-p}{n}}}{V(L)},$$

i.e.,

$$\left(\frac{\widetilde{\Omega}_{-p}(K)}{\widetilde{\Omega}_{-p}(L)}\right)^{\frac{n-p}{n}} = \frac{V(K)}{V(L)}. \tag{4.8}$$

But $L \in \mathcal{S}^n_{os}$, thus taking $Q = L$ in (4.3), and associated with inequality (2.4), we obtain

$$V(K) = \widetilde{V}_{-p}(K,L) \geq V(K)^{\frac{n+p}{n}}V(L)^{-\frac{p}{n}},$$

i.e.,

$$V(K) \leq V(L).$$

This combined with (4.8), and noticing $n > p$, leads to (1.14).

According to the equality condition of (2.4), we see that equality holds in (1.14) if and only if $K = L$. □

Now we complete the proofs of Theorems 1.3 and 1.4. The following lemmas are required.

Lemma 4.4 *If* $K \in \mathcal{S}^n_o$, $p \geq 1$, $\tau \in [-1,1]$, *then*

$$\Gamma^+_p \widehat{\nabla}^\tau_p K = \Gamma^\tau_p K \tag{4.9}$$

and

$$\Gamma_p^{-}\widehat{\nabla}_p^{\tau}K = \Gamma_p^{-\tau}K. \tag{4.10}$$

Proof From (1.3) and (3.2), we have, for all $u \in S^{n-1}$,

$$\begin{aligned}
h^p_{\Gamma_p^+\widehat{\nabla}_p^\tau K}(u) &= \frac{2}{c_{n,p}(n+p)V(\widehat{\nabla}_p^\tau K)}\int_{S^{n-1}}(u\cdot v)_+^p \rho_{\widehat{\nabla}_p^\tau K}(v)^{n+p}\,dv \\
&= \frac{2}{c_{n,p}(n+p)}\int_{S^{n-1}}(u\cdot v)_+^p\left[f_1(\tau)\frac{\rho_K(v)^{n+p}}{V(K)} + f_2(\tau)\frac{\rho_{-K}(v)^{n+p}}{V(-K)}\right]dv \\
&= f_1(\tau)h^p_{\Gamma_p^+K}(u) + f_2(\tau)h^p_{\Gamma_p^+(-K)}(u) \\
&= f_1(\tau)h^p_{\Gamma_p^+K}(u) + f_2(\tau)h^p_{\Gamma_p^-K}(u) = h^p_{\Gamma_p^\tau K}(u).
\end{aligned}$$

This immediately gives (4.9).

Similarly, we know that, for all $u \in S^{n-1}$,

$$h^p_{\Gamma_p^-\widehat{\nabla}_p^\tau K}(u) = h^p_{\Gamma_p^{-\tau}K}(u).$$

This yields (4.10). □

Lemma 4.5 *For $L \in \mathcal{S}_o^n$, $p \geq 1$, and $\tau \in (-1,1)$, if L is not origin-symmetric, then there exists $K \in \mathcal{S}_o^n$ (for $\tau = 0$, $K \in \mathcal{S}_{os}^n$) such that*

$$\Gamma_p^+K \subset \Gamma_p^\tau L, \qquad \Gamma_p^-K \subset \Gamma_p^{-\tau}L,$$

but

$$\widetilde{\Omega}_{-p}(K) > \widetilde{\Omega}_{-p}(L).$$

Proof Since L is not origin-symmetric and $\tau \in (-1,1)$, thus by Corollary 3.2 we know $\widetilde{\Omega}_{-p}(\widehat{\nabla}_p^\tau L) > \widetilde{\Omega}_{-p}(L)$. From this, choose $\varepsilon > 0$ such that $1-\varepsilon > 0$, and $K = (1-\varepsilon)\widehat{\nabla}_p^\tau L \in \mathcal{S}_o^n$ (if $\tau = 0$ then $K \in \mathcal{S}_{os}^n$) satisfies

$$\widetilde{\Omega}_{-p}(K) = \widetilde{\Omega}_{-p}\bigl((1-\varepsilon)\widehat{\nabla}_p^\tau L\bigr) > \widetilde{\Omega}_{-p}(L).$$

But by (4.9) and (4.10), and noticing that $\Gamma_p^{\pm}(cM) = c\Gamma_p^{\pm}M$ ($c > 0$), we, respectively, have

$$\Gamma_p^+K = \Gamma_p^+(1-\varepsilon)\widehat{\nabla}_p^\tau L = (1-\varepsilon)\Gamma_p^+\widehat{\nabla}_p^\tau L = (1-\varepsilon)\Gamma_p^\tau L \subset \Gamma_p^\tau L$$

and

$$\Gamma_p^-K = \Gamma_p^-(1-\varepsilon)\widehat{\nabla}_p^\tau L = (1-\varepsilon)\Gamma_p^-\widehat{\nabla}_p^\tau L = (1-\varepsilon)\Gamma_p^{-\tau}L \subset \Gamma_p^{-\tau}L.$$ □

Proof of Theorem 1.3 Since L is not origin-symmetric and $\tau \in (-1,1)$, thus by Lemma 4.5, there exists $K \in \mathcal{S}_o^n$ such that

$$\Gamma_p^+K \subset \Gamma_p^\tau L, \qquad \Gamma_p^-K \subset \Gamma_p^{-\tau}L,$$

but

$$\widetilde{\Omega}_{-p}(K) > \widetilde{\Omega}_{-p}(L).$$

Because $\tau \in (-1,1)$ is equivalent to $-\tau \in (-1,1)$, we have $\Gamma_p^+ K \subset \Gamma_p^\tau L$, $\Gamma_p^- K \subset \Gamma_p^{-\tau} L$ implying

$$\Gamma_p^+ K \subset \Gamma_p^\tau L, \qquad \Gamma_p^- K \subset \Gamma_p^\tau L.$$

From this together with (1.5) and (1.7), we obtain, for any $u \in S^{n-1}$,

$$\begin{aligned} h\big(\Gamma_p^\tau K, u\big)^p &= f_1(\tau) h\big(\Gamma_p^+ K, u\big)^p + f_2(\tau) h\big(\Gamma_p^- K, u\big)^p \\ &< f_1(\tau) h\big(\Gamma_p^\tau L, u\big)^p + f_2(\tau) h\big(\Gamma_p^\tau L, u\big)^p = h\big(\Gamma_p^\tau L, u\big)^p, \end{aligned}$$

i.e., $\Gamma_p^\tau K \subset \Gamma_p^\tau L$. □

Lemma 4.6 *If* $K \in \mathcal{S}_o^n$, $p \geq 1$, *and* $\tau \in [-1,1]$, *then*

$$\Gamma_p\big(\widehat{\nabla}_p^\tau K\big) = \Gamma_p K. \tag{4.11}$$

Proof From (4.4), (4.9), (4.10), and (4.5), we have, for all $u \in S^{n-1}$,

$$\begin{aligned} h^p_{\Gamma_p \widehat{\nabla}_p^\tau K}(u) &= \frac{1}{2} h^p_{\Gamma_p^+ \widehat{\nabla}_p^\tau K}(u) + \frac{1}{2} h^p_{\Gamma_p^- \widehat{\nabla}_p^\tau K}(u) \\ &= \frac{1}{2} h^p_{\Gamma_p^\tau K}(u) + \frac{1}{2} h^p_{\Gamma_p^{-\tau} K}(u) = h^p_{\Gamma_p K}(u). \end{aligned}$$

So (4.11) is obtained. □

Proof of Theorem 1.4 Since L is not origin-symmetric, for $\tau \in (-1,1)$, by Corollary 3.2 we know

$$\widetilde{\Omega}_{-p}\big(\widehat{\nabla}_p^\tau L\big) > \widetilde{\Omega}_{-p}(L).$$

Choose $\varepsilon > 0$, such that $1 - \varepsilon > 0$ and

$$\widetilde{\Omega}_{-p}\big((1-\varepsilon)\widehat{\nabla}_p^\tau L\big) > \widetilde{\Omega}_{-p}(L).$$

Let $K = (1-\varepsilon)\widehat{\nabla}_p^\tau L$, thus $K \in \mathcal{S}_o^n$ (if $\tau = 0$ then $K \in \mathcal{S}_{os}^n$) and $\widetilde{\Omega}_{-p}(K) > \widetilde{\Omega}_{-p}(L)$.

But from Lemma 4.6 and $\Gamma_p(cM) = c\Gamma_p M$ ($c > 0$), we can get

$$\Gamma_p K = \Gamma_p (1-\varepsilon)\widehat{\nabla}_p^\tau L = (1-\varepsilon)\Gamma_p \widehat{\nabla}_p^\tau L = (1-\varepsilon)\Gamma_p L \subset \Gamma_p L. \qquad \square$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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