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A scheme for a solution of a variational inequality for a monotone mapping and a fixed point of a pseudocontractive mapping

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Abstract
We introduce an iterative process which converges strongly to a common point of the solution set of a variational inequality problem for a Lipschitzian monotone mapping and the fixed point set of a continuous pseudocontractive mapping in Hilbert spaces. In addition, a numerical example which supports our main result is presented. Our theorems improve and unify most of the results that have been proved for this important class of nonlinear operators.

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1 Introduction

Let C be a subset of a real Hilbert space H. A mapping $T:C\to H$ is called *Lipschitzian* if there exists $L>0$ such that $\|Tx-Ty\|\leq L\|x-y\|$, $\forall x,y\in C$. If $L=1$ then T is called *nonexpansive* and if $L\in(0,1)$ then T is called a *contraction*. The operator T is called *pseudocontractive* if for each $x,y\in C$ we have

$$\langle Tx-Ty,x-y\rangle\leq\|x-y\|^2. \tag{1.1}$$

T is called *strongly pseudocontractive* if there exists $k\in(0,1)$ such that

$$\langle x-y,Tx-Ty\rangle\leq k\|x-y\|^2,\quad\text{for all } x,y\in C,$$

and T is said to be a *k-strict pseudocontractive* if there exists a constant $0\leq k<1$ such that

$$\langle x-y,Tx-Ty\rangle\leq\|x-y\|^2-k\big\|(I-T)x-(I-T)y\big\|^2,\quad\text{for all } x,y\in C.$$

Observe that the class of pseudocontractive mappings is a more general class of mappings in the sense that it includes the classes of nonexpansive, strongly pseudocontractive, and k-strict pseudocontractive mappings.

Interest in pseudocontractive mappings stems mainly from their firm connection with the important class of nonlinear monotone mappings, where a mapping A with domain

$D(A)$ and range $R(A)$ in H is called *monotone* if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in D(A).$$

A mapping A is called α*-inverse strongly monotone* if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in D(A).$$

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It is obvious to see that the class of monotone mappings includes the class of α-inverse strongly monotone and α-strongly monotone mappings. Furthermore, we observe that any α-inverse strongly monotone mappings A is a monotone and $\frac{1}{\alpha}$-Lipschitzian mapping.

We observe that A is monotone if and only if $T := I - A$ is pseudocontractive and thus a zero of A, $N(A) := \{x \in D(A) : Ax = 0\}$, is a fixed point of T, $F(T) := \{x \in D(T) : Tx = x\}$. It is now well known that if A is monotone then the solutions of the equation $Ax = 0$ correspond to the equilibrium points of some evolution systems. Consequently, considerable research efforts have been devoted to iterative methods for approximating fixed points of T when T is nonexpansive or pseudocontractive (see, *e.g.*, [1–10] and the references therein).

Let C be a nonempty, closed, and convex subset of a real Hilbert space H. The classical variational inequality problem is to find a $u \in C$ such that $\langle v - u, Au\rangle \geq 0$ for all $v \in C$, where A is a nonlinear mapping. The set of solutions of the variational inequality is denoted by $VI(C, A)$. In the context of the variational inequality problem, this implies that $u \in VI(C, A)$ if and only if $u = P_C(u - \lambda Au)$, $\forall \lambda > 0$, where P_C is a metric projection of H into C.

It is now well known that variational inequalities cover disciplines such as partial differential equations, optimal control, optimization, mathematical programming, mechanics and finance. See, for instance, [11–16].

Variational inequalities were introduced and studied by Stampacchia [17] in 1964. Since then, several numerical methods have been developed for solving variational inequalities; see, for instance, [12, 15, 18–23] and the references therein.

In 2003, Takahashi and Toyoda [24] introduced the following iterative scheme under the assumption that a set $C \subset H$ is closed and convex, a mapping T of C into itself is nonexpansive, and a mapping A of C into H is α-inverse strongly monotone:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T P_C(x_n - \lambda_n A x_n), \quad n \geq 0, \end{cases} \tag{1.2}$$

for all $n \geq 0$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They proved that if $F(T) \cap VI(C, A)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.2) *converges weakly* to some $z \in F(T) \cap VI(C, A)$.

In order to obtain a strong convergence theorem, Iiduka and Takahashi [19] reconsidered the common element problem via the following iterative algorithm:

$$\begin{cases} x_0 = x \in C, \\ x_{n+1} = \alpha_n x + (1-\alpha_n) TP_C(x_n - \lambda_n Ax_n), \quad n \geq 0, \end{cases} \tag{1.3}$$

for all $n \geq 0$, where $T : C \to C$ is a nonexpansive mapping, $A : C \to H$ is a α-inverse strongly monotone mapping, $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\lambda_n\}$ is a sequence in $(0,2\alpha)$. They proved that if $F(T) \cap VI(C,A)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.3) *converges strongly* to some $z \in F(T) \cap VI(C,A)$.

In 2006, Nadezhkina and Takahashi [25] introduced the following hybrid method for finding an element of $F(S) \cap VI(C,A)$ and established the following strong convergence theorem for the sequence generated by this process.

Theorem NT [25] *Let C be a closed convex subset of a real Hilbert space H. Let A be a Lipschitzian monotone mapping of C into H with Lipschitz constant L and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(C,A) \neq \emptyset$. Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by*

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n Ax_n), \\ z_n = \alpha_n x_n + (1-\alpha_n) SP_C(x_n - \lambda_n Ay_n), \\ C_n = \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \end{cases}$$

for every $n \geq 0$, where $\{\lambda_n\} \subset [a,b]$ for some $a,b \in (0,\frac{1}{L})$ and $\{\alpha_n\} \subset [0,c]$ for some $c \in [0,1)$. Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to the same element of $P_{F(S)\cap VI(C,A)}x$.

Our concern now is the following: *can an approximation sequence $\{x_n\}$ be constructed which converges to a common point of the solution set of a variational inequality problem for a monotone mapping and the fixed point set of a continuous pseudocontractive mapping?*

In this paper, it is our purpose to introduce an iterative scheme which converges strongly to a common element of the solution set of a variational inequality problem for Lipschitzian monotone mapping and the fixed point set of a continuous pseudocontractive mapping in Hilbert spaces. Our results provide an affirmative answers to our concern. In addition, a numerical example which supports our main result is presented. Our theorems will extend and unify most of the results that have been proved for this important class of nonlinear operators.

2 Preliminaries

Let C be a nonempty, closed, and convex subset of a real Hilbert space H. It is well known that for every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, *i.e.*,

$$\|x - P_C x\| \leq \|x - y\| \quad \text{for all } y \in C. \tag{2.1}$$

The mapping P_C is called the metric projection of H onto C and characterized by the following properties (see, *e.g.*, [26]):

$$P_C x \in C \quad \text{and} \quad \langle x - P_C x, P_C x - y\rangle \geq 0, \quad \text{for all } x \in H, y \in C \text{ and} \tag{2.2}$$

$$\|y - P_C x\|^2 \leq \|x - y\|^2 - \|x - P_C x\|^2, \quad \text{for all } x \in H, y \in C. \tag{2.3}$$

In the sequel we shall make use of the following lemmas.

Lemma 2.1 [27] *Let H be a real Hilbert space. Then, for all $x, y \in H$ and $\alpha \in [0,1]$ the following equality holds*:

$$\left\|\alpha x + (1-\alpha)y\right\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2.$$

Lemma 2.2 *Let H be a real Hilbert space. Then for any given $x, y \in H$, the following inequality holds*:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle.$$

Lemma 2.3 [28] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation*:

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\delta_n, \quad n \geq n_0,$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\delta_n\} \subset \mathbb{R}$ satisfying the following conditions: $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, *and* $\limsup_{n\to\infty}\delta_n \leq 0$. *Then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4 [11] *Let $\{a_n\}$ be sequences of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$, for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all* (*sufficiently large*) *numbers $k \in \mathbb{N}$*:

$$a_{m_k} \leq a_{m_k+1} \quad \textit{and} \quad a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

Lemma 2.5 [29] *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $T : C \to H$ be continuous pseudocontractive mapping. For $r > 0$ and $x \in H$, define a mapping $F_r : H \to C$ as follows*:

$$F_r x := \left\{z \in C : \langle y - z, Tz\rangle - \frac{1}{r}\big\langle y - z, (1+r)z - x\big\rangle \leq 0, \forall y \in C\right\}$$

for all $x \in H$. Then the following hold:

(1) *F_r is single-valued*;

(2) *F_r is firmly nonexpansive type mapping, i.e., for all $x, y \in H$*,

$$\|F_r x - F_r y\|^2 \leq \langle F_r x - F_r y, x - y\rangle;$$

(3) $F(F_r) = F(T)$;
(4) $F(T)$ *is closed and convex.*

Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $T : C \to H$, be a continuous pseudocontractive mapping. Then, in what follows, $T_{r_n} : H \to C$ are defined as follows: For $x \in H$ and $\{r_n\} \subset [e, \infty)$, for some $e > 0$, define

$$T_{r_n}x := \left\{z \in C : \langle y - z, Tz\rangle - \frac{1}{r_n}\bigl\langle y - z, (1 + r_n)z - x\bigr\rangle \leq 0, \forall y \in C\right\}.$$

Now, we prove our main convergence theorem.

3 Main result

Theorem 3.1 *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $T : C \to C$ be a continuous pseudocontractive mapping. Let $A : C \to H$ be a Lipschitzian monotone mapping with Lipschitz constant L. Assume that $\mathcal{F} = F(T) \cap VI(C, A)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0, u \in C$ by*

$$\begin{cases} z_n = P_C[x_n - \gamma_n Ax_n], \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)(a_n x_n + b_n T_{r_n} x_n + c_n P_C[x_n - \gamma_n Az_n]), \end{cases} \tag{3.1}$$

where P_C is a metric projection from H onto C, $\gamma_n \subset [a, b] \subset (0, \frac{1}{L})$, and $\{a_n\}, \{b_n\}, \{c_n\} \subset (a, b) \subset (0, 1)$, $\{\alpha_n\} \subset (0, c) \subset (0, 1)$ satisfying the following conditions: (i) $a_n + b_n + c_n = 1$, (ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum \alpha_n = \infty$. *Then $\{x_n\}$ converges strongly to the point x^* of $\mathcal{F}$ nearest to u.*

Proof Let $u_n = P_C(x_n - \gamma_n Az_n)$ and $w_n = T_{r_n} x_n$ for all $n \geq 0$. Let $p \in \mathcal{F}$. Then from Lemma 2.5 we get $\|w_n - p\| \leq \|T_{r_n} x_n - T_{r_n} p\| \leq \|x_n - p\|$. In addition, from (2.3) we have

$$\begin{aligned} \|u_n - p\|^2 &\leq \|x_n - \gamma_n Az_n - p\|^2 - \|x_n - \gamma_n Az_n - u_n\|^2 \\ &= \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2\gamma_n \langle Az_n, p - u_n\rangle \\ &= \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2\gamma_n \bigl(\langle Az_n - Ap, p - z_n\rangle \\ &\quad + \langle Ap, p - z_n\rangle + \langle Az_n, z_n - u_n\rangle\bigr) \\ &\leq \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2\gamma_n \langle Az_n, z_n - u_n\rangle \\ &\leq \|x_n - p\|^2 - \|x_n - z_n\|^2 - 2\langle x_n - z_n, z_n - u_n\rangle \\ &\quad - \|z_n - u_n\|^2 + 2\gamma_n \langle Az_n, z_n - u_n\rangle \\ &= \|x_n - p\|^2 - \|x_n - z_n\|^2 - \|z_n - u_n\|^2 \\ &\quad + 2\langle x_n - \gamma_n Az_n - z_n, u_n - z_n\rangle, \end{aligned} \tag{3.2}$$

and from (2.2), we obtain

$$\begin{aligned} \langle x_n - \gamma_n Az_n - z_n, u_n - z_n\rangle &= \langle x_n - \gamma_n Ax_n - z_n, u_n - z_n\rangle + \langle \gamma_n Ax_n - \gamma_n Az_n, u_n - z_n\rangle \\ &\leq \langle \gamma_n Ax_n - \gamma_n Az_n, u_n - z_n\rangle \\ &\leq \gamma_n L\|x_n - z_n\|\|u_n - z_n\|. \end{aligned}$$

Thus, we get

$$\begin{aligned}
\|u_n - p\|^2 &\le \|x_n - p\|^2 - \|x_n - z_n\|^2 - \|z_n - u_n\|^2 \\
&\quad + 2\gamma_n L\|x_n - z_n\|\|u_n - z_n\| \\
&\le \|x_n - p\|^2 - \|x_n - z_n\|^2 - \|z_n - u_n\|^2 \\
&\quad + \gamma_n L\bigl[\|x_n - z_n\|^2 + \|z_n - u_n\|^2\bigr] \\
&\le \|x_n - p\|^2 + (\gamma_n L - 1)\bigl[\|x_n - z_n\|^2 + \|z_n - u_n\|^2\bigr] &(3.3)\\
&\le \|x_n - p\|^2. &(3.4)
\end{aligned}$$

Furthermore, from (3.1) and Lemma 2.1 we have the following:

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \bigl\|\alpha_n u + (1-\alpha_n)(a_n x_n + b_n w_n + c_n u_n) - p\bigr\|^2 \\
&\le \alpha_n\|u - p\|^2 + (1-\alpha_n)\bigl\|a_n(x_n - p) + b_n(w_n - p) \\
&\quad + c_n(u_n - p)\bigr\|^2 \\
&\le \alpha_n\|u - p\|^2 + (1-\alpha_n)\bigl[a_n\|x_n - p\|^2 + b_n\|w_n - p\|^2 \\
&\quad + c_n\|u_n - p\|^2\bigr] - (1-\alpha_n)a_n b_n\|w_n - x_n\|^2 \\
&\quad - (1-\alpha_n)a_n c_n\|u_n - x_n\|^2 - (1-\alpha_n)b_n c_n\|w_n - u_n\|^2,
\end{aligned}$$

and using (3.3) we get

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \alpha_n\|u - p\|^2 + (1-\alpha_n)a_n\|x_n - p\|^2 + (1-\alpha_n)b_n\|x_n - p\|^2 \\
&\quad + (1-\alpha_n)c_n\bigl[\|x_n - p\|^2 + (\gamma_n L - 1)\bigl[\|x_n - z_n\|^2 + \|z_n - u_n\|^2\bigr]\bigr] \\
&\quad - (1-\alpha_n)a_n b_n\|w_n - x_n\|^2 - (1-\alpha_n)a_n c_n\|u_n - x_n\|^2 \\
&\quad - (1-\alpha_n)b_n c_n\|w_n - u_n\|^2 \\
&\le \alpha_n\|u - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 \\
&\quad + (1-\alpha_n)c_n(\gamma_n L - 1)\bigl[\|x_n - z_n\|^2 + \|z_n - u_n\|^2\bigr] \\
&\quad - (1-\alpha_n)a_n b_n\|w_n - x_n\|^2 - (1-\alpha_n)a_n c_n\|u_n - x_n\|^2 \\
&\quad - (1-\alpha_n)b_n c_n\|w_n - u_n\|^2. &(3.5)
\end{aligned}$$

Since $\gamma_n L < 1$, from (3.5) we get

$$\|x_{n+1} - p\|^2 \le \alpha_n\|u - p\|^2 + (1-\alpha_n)\|x_n - p\|^2. \tag{3.6}$$

Thus, by induction,

$$\|x_{n+1} - p\|^2 \le \max\bigl\{\|u - p\|^2, \|x_0 - p\|^2\bigr\}, \quad \forall n \ge 0,$$

which implies that $\{x_n\}$ and $\{z_n\}$ are bounded.

Let $x^* = P_{\mathcal{F}}(u)$. Then, using (3.1), Lemma 2.2, and following the methods used to get (3.5) we obtain that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n u + (1-\alpha_n)(a_n x_n + b_n w_n + c_n u_n) - x^*\|^2 \\
&\le \|\alpha_n(u - x^*) + (1-\alpha_n)[(a_n x_n + b_n w_n + c_n u_n) - x^*]\|^2 \\
&\le (1-\alpha_n)\|a_n x_n + b_n w_n + c_n u_n - x^*\|^2 \\
&\quad + 2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle \\
&\le (1-\alpha_n)a_n\|x_n - x^*\|^2 + (1-\alpha_n)b_n\|w_n - x^*\|^2 \\
&\quad + (1-\alpha_n)c_n\|u_n - x^*\|^2 - (1-\alpha_n)b_n a_n\|w_n - x_n\|^2 \\
&\quad - (1-\alpha_n)b_n c_n\|u_n - w_n\|^2 - (1-\alpha_n)a_n c_n\|x_n - u_n\|^2 \\
&\quad + 2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le (1-\alpha_n)a_n\|x_n - x^*\|^2 + (1-\alpha_n)b_n\|x_n - x^*\|^2 \\
&\quad + (1-\alpha_n)c_n\big[\|x_n - x^*\|^2 + (\gamma_n L - 1)\big[\|x_n - z_n\|^2 + \|z_n - u_n\|^2\big]\big] \\
&\quad - (1-\alpha_n)b_n a_n\|w_n - x_n\|^2 - (1-\alpha_n)b_n c_n\|u_n - w_n\|^2 \\
&\quad - (1-\alpha_n)a_n c_n\|x_n - u_n\|^2 + 2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle \\
&\le (1-\alpha_n)\|x_n - x^*\|^2 + (1-\alpha_n)c_n(\gamma_n L - 1)\big[\|x_n - z_n\|^2 + \|z_n - u_n\|^2\big] \\
&\quad - (1-\alpha_n)b_n a_n\|w_n - x_n\|^2 - (1-\alpha_n)b_n c_n\|u_n - w_n\|^2 \\
&\quad - (1-\alpha_n)a_n c_n\|x_n - u_n\|^2 + 2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle \qquad (3.7) \\
&\le (1-\alpha_n)\|x_n - x^*\|^2 + 2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle. \qquad (3.8)
\end{aligned}$$

Now, we consider two cases.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n - x^*\|\}$ is decreasing for all $n \ge n_0$. Then we get $\{\|x_n - x^*\|\}$ is convergent. Thus, from (3.7) we have

$$x_n - z_n \to 0, \qquad z_n - u_n \to 0 \quad \text{as } n \to \infty, \tag{3.9}$$

and

$$w_n - x_n \to 0, \qquad u_n - w_n \to 0, \qquad x_n - u_n \to 0 \quad \text{as } n \to \infty. \tag{3.10}$$

Moreover, from the fact that $\alpha_n \to 0$, as $n \to \infty$, (3.1), (3.9), and (3.10) we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\alpha_n(u - x_n) + (1-\alpha_n)(b_n w_n + c_n u_n - (1-a_n)x_n)\| \\
&\le \alpha_n\|u - x_n\| + (1-\alpha_n)b_n\|w_n - x_n\| + (1-\alpha_n)c_n\|u_n - x_n\| \to 0, \qquad (3.11)
\end{aligned}$$

as $n \to \infty$.

Furthermore, since $\{x_{n+1}\}$ is bounded subset of H which is reflexive, we can choose a subsequence $\{x_{n_i+1}\}$ of $\{x_{n+1}\}$ such that $x_{n_i+1} \rightharpoonup z$ and $\limsup_{n\to\infty}\langle u - x^*, x_{n+1} - x^*\rangle = \lim_{i\to\infty}\langle u - x^*, x_{n_i+1} - x^*\rangle$. This implies from (3.11) that $x_{n_i} \rightharpoonup z$.

Now, we show that $z \in VI(C,A)$. But, since A is Lipschitz continuous, we have $Az_n - Au_n \to 0$, as $n \to \infty$ and from (3.9) and (3.10) we have $u_{n_i} \rightharpoonup z$ and $z_{n_i} \rightharpoonup z$. Let

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases} \tag{3.12}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C,A)$ (see, *e.g.* [30]). Let $(v,w) \in G(T)$. Then we have $w \in Tv = Av + N_C v$ and hence $w - Av \in N_C v$. So, we have $\langle v - u, w - Av\rangle \geq 0$, for all $u \in C$. On the other hand, from $u_n = P_C(x_n - \gamma_n Az_n)$ and $v \in C$, we have $\langle x_n - \gamma_n Az_n - u_n, u_n - v\rangle \geq 0$, and hence, $\langle v - u_n, (u_n - x_n)/\gamma_n + Az_n\rangle \geq 0$. Therefore, from $w - Av \in N_C v$ and $u_{n_i} \in C$ we have

$$\begin{aligned}
\langle v - u_{n_i}, w\rangle &\geq \langle v - u_{n_i}, Av\rangle \geq \langle v - u_{n_i}, Av\rangle - \bigl\langle v - u_{n_i}, (u_{n_i} - x_{n_i})/\gamma_{n_i} + Az_{n_i}\bigr\rangle \\
&= \langle v - u_{n_i}, Av - Au_{n_i}\rangle + \langle v - u_{n_i}, Au_{n_i} - Az_{n_i}\rangle \\
&\quad - \bigl\langle v - u_{n_i}, (u_{n_i} - x_{n_i})/\gamma_{n_i}\bigr\rangle \\
&\geq \langle v - u_{n_i}, Au_{n_i} - Az_{n_i}\rangle - \bigl\langle v - u_{n_i}, (u_{n_i} - x_{n_i})/\gamma_{n_i}\bigr\rangle.
\end{aligned}$$

Hence, we have $\langle v - z, w\rangle \geq 0$, as $i \to \infty$. Since T is maximal monotone, we have $z \in T^{-1}(0)$ and hence $z \in VI(C,A)$.

Now, we show that $z \in F(T)$. Note that, from the definition of w_{n_i}, we have

$$\langle y - w_{n_i}, Tw_{n_i}\rangle - \frac{1}{r_{n_i}}\bigl\langle y - w_{n_i}, (r_{n_i} + 1)w_{n_i} - x_{n_i}\bigr\rangle \leq 0, \quad \forall y \in C. \tag{3.13}$$

Put $z_t = tv + (1 - t)z$ for all $t \in (0,1]$ and $v \in C$. Consequently, we get $z_t \in C$. From (3.13) and pseudocontractivity of T it follows that

$$\begin{aligned}
\langle w_{n_i} - z_t, Tz_t\rangle &\geq \langle w_{n_i} - z_t, Tz_t\rangle + \langle z_t - w_{n_i}, Tw_{n_i}\rangle - \frac{1}{r_{n_i}}\bigl\langle z_t - w_{n_i}, (1 + r_{n_i})w_{n_i} - x_{n_i}\bigr\rangle \\
&= -\langle z_t - w_{n_i}, Tz_t - Tw_{n_i}\rangle - \frac{1}{r_{n_i}}\langle z_t - w_{n_i}, w_{n_i} - x_{n_i}\rangle - \langle z_t - w_{n_i}, w_{n_i}\rangle \\
&\geq -\|z_t - w_{n_i}\|^2 - \frac{1}{r_{n_i}}\langle z_t - w_{n_i}, w_{n_i} - x_{n_i}\rangle - \langle z_t - w_{n_i}, w_{n_i}\rangle \\
&= \langle w_{n_i} - z_t, z_t\rangle - \left\langle z_t - w_{n_i}, \frac{w_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle.
\end{aligned}$$

Then, since $w_n - x_n \to 0$, as $n \to \infty$ we obtain $\frac{w_{n_i} - x_{n_i}}{r_{n_i}} \to 0$ as $i \to \infty$. Thus, it follows that

$$\langle z - z_t, Tz_t\rangle \geq \langle z - z_t, z_t\rangle \quad \text{as } i \to \infty,$$

and hence

$$-\langle v - z, Tz_t\rangle \geq -\langle v - z, z_t\rangle, \quad \forall v \in C.$$

Letting $t \to 0$ and using the fact that T is continuous we obtain

$$-\langle v - z, Tz\rangle \geq -\langle v - z, z\rangle, \quad \forall v \in C.$$

Now, let $v = Tz$. Then we obtain $z = Tz$ and hence $z \in F(T)$. Therefore, by (2.2) we immediately obtain

$$\begin{aligned}\limsup_{n\to\infty}\langle u - x^*, x_{n+1} - x^*\rangle &= \lim_{i\to\infty}\langle u - x^*, x_{n_i+1} - x^*\rangle \\ &= \langle u - x^*, z - x^*\rangle \leq 0. \end{aligned} \tag{3.14}$$

Then it follows from (3.8), (3.14), and Lemma 2.3 that $\|x_n - x^*\| \to 0$, as $n \to \infty$. Consequently, $\{x_n\}$ converges to the minimum norm point of $\mathcal{F}$.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\|x_{n_i} - x^*\| < \|x_{n_i+1} - x^*\|,$$

for all $i \in \mathbb{N}$. Then, by Lemma 2.4, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$, and

$$\|x_{m_k} - x^*\| \leq \|x_{m_k+1} - x^*\| \quad \text{and} \quad \|x_k - x^*\| \leq \|x_{m_k+1} - x^*\|, \tag{3.15}$$

for all $k \in \mathbb{N}$. Now, from (3.7) we get

$$x_{m_k} - z_{m_k} \to 0, \qquad z_{m_k} - u_{m_k} \to 0 \quad \text{as } k \to \infty, \tag{3.16}$$

and

$$w_{m_k} - x_{m_k} \to 0, \qquad u_{m_k} - w_{m_k} \to 0, \qquad x_{m_k} - u_{m_k} \to 0 \quad \text{as } k \to \infty. \tag{3.17}$$

Thus, like in Case 1, we obtain $x_{m_k+1} - x_{m_k} \to 0$ and

$$\limsup_{k\to\infty}\langle u - x^*, x_{m_k+1} - x^*\rangle \leq 0. \tag{3.18}$$

Now, from (3.8) we have

$$\|x_{m_k+1} - x^*\|^2 \leq (1 - \alpha_{m_k})\|x_{m_k} - x^*\|^2 + 2\alpha_{m_k}\langle u - x^*, x_{m_k+1} - x^*\rangle, \tag{3.19}$$

and hence (3.15) and (3.19) imply that

$$\begin{aligned}\alpha_{m_k}\|x_{m_k} - x^*\|^2 &\leq \|x_{m_k} - x^*\|^2 - \|x_{m_k+1} - x^*\|^2 + 2\alpha_{m_k}\langle u - x^*, x_{m_k+1} - x^*\rangle \\ &\leq +2\alpha_{m_k}\langle u - x^*, x_{m_k+1} - x^*\rangle.\end{aligned}$$

But since that $\alpha_{m_k} > 0$, we obtain

$$\|x_{m_k} - x^*\|^2 \leq +2\langle u - x^*, x_{m_k+1} - x^*\rangle.$$

Then, using (3.18), we get $\|x_{m_k} - x^*\| \to 0$, as $k \to \infty$. This together with (3.19) imply that $\|x_{m_k+1} - x^*\| \to 0$, as $k \to \infty$. But $\|x_k - x^*\| \leq \|x_{m_k+1} - x^*\|$, for all $k \in \mathbb{N}$, thus we obtain $x_k \to x^*$. Therefore, from the above two cases, we can conclude that $\{x_n\}$ converges strongly to the point x^* of $\mathcal{F}$ nearest to u. □

If, in Theorem 3.1, we assume that $T = I$, the identity mapping on C, we obtain the following corollary.

Corollary 3.2 *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $A : C \to H$ be a Lipschitzian monotone mapping with Lipschitz constant L. Assume that $VI(C,A)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0, u \in C$ by*

$$\begin{cases} z_n = P_C[x_n - \gamma_n A x_n], \\ x_{n+1} = \alpha_n u + (1-\alpha_n)(a_n x_n + (1-a_n)P_C[x_n - \gamma_n A z_n]), \end{cases} \tag{3.20}$$

where P_C is a metric projection from H onto C, $\gamma_n \subset [a,b] \subset (0,\frac{1}{L})$, and $\{a_n\} \subset (a,b) \subset (0,1)$, $\{\alpha_n\} \subset (0,c) \subset (0,1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$, $\sum \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to the point $x^* = P_{VI(C,A)}(u)$.

If, in Theorem 3.1, we assume that $A = 0$, we obtain the following corollary, which is Theorem 3.1 of [29].

Corollary 3.3 *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $T : C \to C$ be a continuous pseudocontractive mapping. Assume that $F(T)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0, u \in C$ by*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\big(a_n x_n + (1-a_n)T_{r_n} x_n\big),$$

where $\{a_n\} \subset (a,b) \subset (0,1)$, $\{\alpha_n\} \subset (0,c) \subset (0,1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$, $\sum \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to the point $x^* = P_{F(T)}(u)$.

If, in Theorem 3.1, we assume that A is α-inverse strongly monotone then A is Lipschitzian and we obtain the following corollary.

Corollary 3.4 *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $T : C \to C$ be a continuous pseudocontractive mapping. Let $A : C \to H$ an α-inverse strongly monotone mapping. Assume that $\mathcal{F} = F(T) \cap VI(C,A)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0, u \in C$ by*

$$\begin{cases} z_n = P_C[x_n - \gamma_n A x_n], \\ x_{n+1} = \alpha_n u + (1-\alpha_n)(a_n x_n + b_n T_{r_n} x_n + c_n P_C[x_n - \gamma_n A z_n]), \end{cases} \tag{3.21}$$

where P_C is a metric projection from H onto C, $\gamma_n \subset [a,b] \subset (0,\alpha)$, and $\{a_n\},\{b_n\},\{c_n\} \subset (a,b) \subset (0,1)$, $\{\alpha_n\} \subset (0,c) \subset (0,1)$ satisfying (i) $a_n + b_n + c_n = 1$, (ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum \alpha_n = \infty$. *Then $\{x_n\}$ converges strongly to the point $x^* = P_{\mathcal{F}}(u)$.*

If, in Theorem 3.1, we assume that $C = H$, a real Hilbert space, then P_C becomes identity mapping and $VI(C,A) = A^{-1}(0)$, and hence we get the following corollary.

Corollary 3.5 *Let H be a real Hilbert space. Let $T : H \to H$ be a continuous pseudocontractive mapping. Let $A : H \to H$ be a Lipschitzian monotone mapping with Lipschitz constant L. Assume that $\mathcal{F} = F(T) \cap A^{-1}(0)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0, u \in C$ by*

$$\begin{cases} z_n = x_n - \gamma_n A x_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)(a_n x_n + b_n T_{r_n} x_n + c_n [x_n - \gamma_n A z_n]), \end{cases} \tag{3.22}$$

where $\gamma_n \subset [a,b] \subset (0, \frac{1}{L})$ and $\{a_n\}, \{b_n\}, \{c_n\} \subset (a,b) \subset (0,1)$, $\{\alpha_n\} \subset (0,c) \subset (0,1)$ satisfying the following conditions: (i) $a_n + b_n + c_n = 1$, (ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum \alpha_n = \infty$. *Then $\{x_n\}$ converges strongly to the point x^* of $\mathcal{F}$ nearest to u.*

We also note that the method of proof of Theorem 3.1 provides the following theorem for approximating the common minimum-norm point of the solution set of a variational inequality problem for monotone mapping and fixed point set of a continuous pseudocontractive mapping.

Theorem 3.6 *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $T : C \to C$ be a continuous pseudocontractive mapping. Let $A : C \to H$ be a Lipschitzian monotone mapping with Lipschitz constant L. Assume that $\mathcal{F} = F(T) \cap VI(C,A)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0, u \in C$ by*

$$\begin{cases} z_n = P_C[x_n - \gamma_n A x_n], \\ x_{n+1} = P_C[(1 - \alpha_n)(a_n x_n + b_n T_{r_n} x_n + c_n P_C[x_n - \gamma_n A z_n])], \end{cases} \tag{3.23}$$

where P_C is a metric projection from H onto C, $\gamma_n \subset [a,b] \subset (0, \frac{1}{L})$, and $\{a_n\}, \{b_n\}, \{c_n\} \subset (a,b) \subset (0,1)$, $\{\alpha_n\} \subset (0,c) \subset (0,1)$ satisfying the following conditions: (i) $a_n + b_n + c_n = 1$, (ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum \alpha_n = \infty$. *Then $\{x_n\}$ converges strongly to the minimum-norm point x^* of $\mathcal{F}$.*

Remark 3.7 Theorem 3.1 extends Theorem 3.1 of Takahashi and Toyoda [24] and Theorem 3.2 of Yao *et al.* [22], Theorem 3.1 of Iiduka and Takahashi [19] and the results of Nadezhkina and Takahashi [25] in the sense that our scheme provides a common point of the solution set of variational inequalities for a more general class of monotone mappings and/or the fixed point set of a more general class of continuous pseudocontractive mappings. Our results provide an affirmative answer to our concern.

4 Applications to minimization problems

In this section, we study the problem of finding a minimizer of a continuously Fréchet differentiable convex functional in Hilbert spaces. Let f be a continuously Fréchet differentiable convex functionals of H into $(-\infty, \infty)$ such that the gradient of f, $(\triangledown f)$ is continuous and monotone. For $\gamma > 0$, and $x \in H$, let $T_{r_n} x := \{z \in H : \langle y - z, (I - (\triangledown f))z\rangle - \frac{1}{\gamma}\langle y - z, (1 + \gamma)z - x\rangle \leq 0, \forall y \in H\}$. Then the following theorem holds.

Theorem 4.1 *Let H be a real Hilbert space. Let f be a continuously Fréchet differentiable convex functionals of H into $(-\infty, \infty)$ such that the gradient of f, $(\triangledown f)$ is continuous and monotone such that $\mathcal{N} := \arg\min_{y \in C} f(y) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated from an* *arbitrary* $x_0, u \in C$ *by*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)\big(a_n x_n + (1 - a_n)T_{r_n} x_n\big),$$

where $\{a_n\} \subset (a,b) \subset (0,1)$, $\{\alpha_n\} \subset (0,c) \subset (0,1)$ *satisfying* $\lim_{n\to\infty} \alpha_n = 0$, $\sum \alpha_n = \infty$. *Then* $\{x_n\}$ *converges strongly to the point* $x^* \in \mathcal{N}$ *nearest to* u.

Proof We note that $T := (I - \bigtriangledown f)$ is continuous pseudocontractive mapping with $F(T) = (\bigtriangledown f)^{-1}(0)$ and from the convexity and Frećhet differentiability of f we see that the zero of $\bigtriangledown f$ is given by $\mathcal{N} = \arg\min_{y\in C} f(y)$. Thus, the conclusion follows from Corollary 3.3. □

5 Numerical example

In this section, we give an example of a continuous pseudocontractive mapping T and a Lipschitzian monotone mapping with all the conditions of Theorem 3.1 and some numerical experiment results to explain the conclusion of the theorem.

Example 5.1 Let $H = \mathbb{R}$ with Euclidean norm. Let $C = [-2, 6]$ and $T : C \to \mathbb{R}$ be defined by

$$Tx := \begin{cases} -3x, & x \in [-2, 0], \\ x, & (0, 6], \end{cases}$$

and

$$Ax := \begin{cases} 0, & x \in [-2, \frac{1}{2}], \\ 3(x - \frac{1}{2})^2, & x \in (\frac{1}{2}, 6]. \end{cases} \tag{5.1}$$

Then we easily see that T is continuous pseudocontractive with $F(T) = [0, 6]$.

In addition, we observe that A is monotone with $VI(C,A) = [-2, \frac{1}{2}]$. Next, we show that A it is Lipschitzian with $L = 36$. If $x, y \in [-2, \frac{1}{2}]$ then

$$|Ax - Ay| = |0 - 0| \le 36|x - y|.$$

If $x, y \in (\frac{1}{2}, 6]$ then

$$\begin{aligned} |Ax - Ay| &= 3\left|\left(x - \frac{1}{2}\right)^2 - \left(y - \frac{1}{2}\right)^2\right| \\ &= 3\left|\left(\left(x - \frac{1}{2}\right) + \left(y - \frac{1}{2}\right)\right)\left(\left(x - \frac{1}{2}\right) - \left(y - \frac{1}{2}\right)\right)\right| \\ &= 3|x + y - 1||x - y| \le 36|x - y|. \end{aligned}$$

If $x \in [-2, \frac{1}{2}]$ and $y \in (\frac{1}{2}, 6]$ then

$$\begin{aligned} |Ax - Ay| &= \left|0 - 3\left(y - \frac{1}{2}\right)^2\right| = 3\left(y - \frac{1}{2}\right)^2 \\ &= 3\left|\left(y - \frac{1}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2 + \left(x - \frac{1}{2}\right)^2\right| \end{aligned}$$

$$\begin{aligned}&\leq 3|x+y-1||x-y|+(x-y)^2\\&=3\big[|x+y-1|+|x+y|\big]|x-y|\\&\leq 36|x-y|.\end{aligned}$$

Thus, we see that A is a Lipschtzian mapping with $L = 36$. It is also clear that $F(T)\cap VI(C,A) = [0,1]\cap[-2,\frac{1}{2}] = [0,\frac{1}{2}]$.

Furthermore, if $x\in(0,6]$, the inequality

$$T_rx=\left\{z\in C:\langle y-z,Tz\rangle-\frac{1}{r}\big\langle y-z,(1+r)z-x\big\rangle\leq 0,\forall y\in C\right\},\tag{5.2}$$

shows that we may take $T_r(x) = x$. If $x\in[-2,0]$, inequality (5.2) gives that

$$r(y-z)(-3z)-(y-z)\big[(1+r)z-x\big]\leq 0,\quad \forall y\in C,$$

which implies that $T_r(x) = z = \frac{x}{4r+1}$ and hence we get

$$T_r(x):=\begin{cases}x, & x\in(0,6],\\ \frac{x}{4r+1}, & x\in[-2,0].\end{cases}$$

Now, if we take, $\alpha_n = \frac{1}{n+100}$, $a_n = b_n = \frac{1}{n+100} + 0.1$, $c_n = 0.8 - \frac{2}{n+100}$; $r_n = 10$, $\forall n\geq 1$ and $\gamma_n = 0.01 + \frac{1}{n+100}$, we observe that the conditions of Theorem 3.1 are satisfied and Scheme (3.1) reduces to

$$\begin{cases}z_n = P_C[x_n-\gamma_nAx_n],\\ x_{n+1} = \alpha_nu+(1-\alpha_n)(a_nx_n+b_nT_{r_n}x_n+c_nP_C[x_n-\gamma_nAz_n]).\end{cases}\tag{5.3}$$

When $u = -0.1$ and $x_0 = 0.8$ we see that Scheme (5.3) converges strongly to $x^* = 0.0$ as shown in Figure 1.

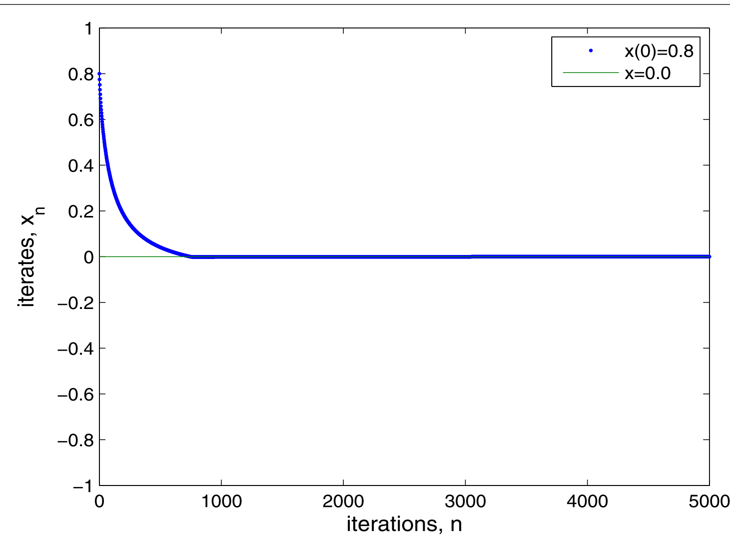

Figure 1 Convergence of $\{x_n\}$ with $u = -0.1$ and $x_0 = 0.8$.

When $u = 0.6$ and $x_0 = -2.0$ we see that Scheme (5.3) converges strongly to $x^* = 0.5$ as shown in Figure 2.

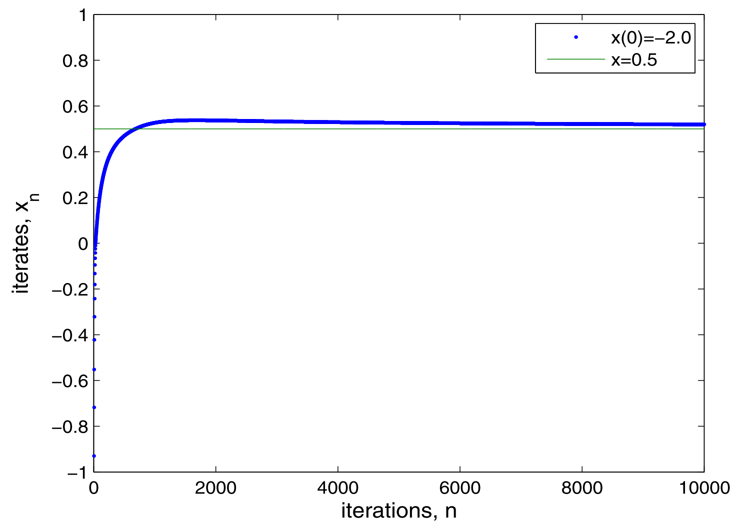

Figure 2 Convergence of $\{x_n\}$ with $u = 0.6$ and $x_0 = -2.0$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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