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Some inequalities for warped product pseudo-slant submanifolds of nearly Kenmotsu manifolds

Akram Ali[1]* iD, Wan Ainun Mior Othman[1] and Cenap Ozel[2]

*Correspondence:
akramali133@gmail.com
[1]Institute of Mathematical Sciences,
Faculty of Science, University of
Malaya, Kuala Lumpur, 50603,
Malaysia
Full list of author information is
available at the end of the article

Abstract

In this paper, we study non-trivial warped product pseudo-slant submanifolds of nearly Kenmotsu manifolds. In the beginning, we obtain some lemmas and then develop the general sharp inequalities for mixed totally geodesic warped products pseudo-slant submanifolds. The equality cases are also considered.

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1 Introduction

The geometric inequalities of warped product submanifolds have been studied actively since Chen [1] introduced the notion of a CR-warped product submanifold in a Kaehler manifold and obtained inequalities for the second fundamental form in terms of warping functions. In fact, different types of warped product submanifolds of different structures are studied for the last 14 years (see [2]). Recently, Sahin [3] established a general inequality for warped product pseudo-slant (also named hemi-slant) isometrically immersed in a Kaehler manifold for mixed totally geodesic. Later on, Uddin, *et al.* [4–6] obtained some inequalities of warped product submanifolds in different structures. In the present paper, we extend this idea into a nearly Kenmotsu manifold and derive the geometric inequalities of non-trivial warped product pseudo-slant submanifolds which are the natural extensions of CR-warped product submanifolds. Every CR-warped product submanifold is a non-trivial warped product pseudo-slant submanifold of the forms $M_\perp \times_f M_\theta$ and $M_\theta \times_f M_\perp$ with slant angle $\theta = 0$. First of all we consider non-trivial warped product pseudo-slant submanifolds of the form $M = M_\perp \times_f M_\theta$ and $M_\theta \times_f M_\perp$ such that M_θ and $M_\perp$ are proper-slant and anti-invariant submanifolds. Next we establish inequalities involving the second fundamental form, slant angle, and warping functions.

The paper is organized as follows: In Section 2, we review some preliminary formulas, definitions and address the study of pseudo-slant submanifolds of nearly Kenmotsu manifolds. In Section 3, we study warped product pseudo-slant submanifolds of a nearly Kenmotsu manifold and obtain some lemmas. In Section 4, we define an orthonormal frame

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for warped product pseudo-slant submanifolds and then obtain general sharp inequalities for the second fundamental form in terms of warping functions and slant immersions.

2 Preliminaries

Let $\widetilde{M}$ be a $(2m + 1)$-dimensional almost contact manifold with almost contact structure (φ, ξ, η) where φ is a $(1,1)$ tensor field, ξ is the structure vector field and η is a dual 1-form satisfying the following property:

$$\varphi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \qquad \varphi(\xi) = 0, \qquad \eta \circ \varphi = 0. \tag{2.1}$$

On an almost contact manifold there exists a Riemannian metric g which satisfies the following:

$$g(\varphi U, \varphi V) = g(U, V) - \eta(U)\eta(V), \qquad \eta(U) = g(U, \xi), \tag{2.2}$$

for any U, V tangent to $\widetilde{M}$. Then an almost contact manifold $\widetilde{M}$ equipped with a Riemannian metric g is called an almost contact metric manifold $(\widetilde{M}, g)$. Furthermore, an almost contact metric manifold is known to be a Kenmotsu manifold [7] if

$$(\widetilde{\nabla}_U \varphi)V = g(\varphi U, V)\xi - \eta(V)\varphi U, \tag{2.3}$$

and

$$\widetilde{\nabla}_U \xi = U - \eta(U)\xi, \tag{2.4}$$

for any vector fields U, V on $\widetilde{M}$, where $\widetilde{\nabla}$ denote the Riemannian connection with respect to g. Then an almost contact metric manifold $\widetilde{M}$ is said to be a $n$early Kenmotsu manifolds [7], if

$$(\widetilde{\nabla}_U \varphi)V + (\widetilde{\nabla}_V \varphi)U = -\eta(U)\varphi V - \eta(V)\varphi U, \tag{2.5}$$

for any U, V tangent to $\widetilde{M}$. We shall use the symbol $\Gamma(T\widetilde{M})$ to denote the Lie algebras of vector fields on a manifold $\widetilde{M}$.

Let M be a Riemannian manifold that is isometrically immersed in an almost contact metric manifold $\widetilde{M}$ and denote by the same symbol g the Riemannian metric induced on M. Let $\Gamma(TM)$ and $\Gamma(T^\perp M)$ be the Lie algebras of vector fields tangent to M and normal to M, respectively, and $\nabla^\perp$ the induced connection on $T^\perp M$. Denote by $\mathcal{F}(M)$ the algebra of smooth functions on M and by $\Gamma(TM)$ the $\mathcal{F}(M)$-module of smooth sections of TM over M. Denote by ∇ the Levi-Civita connection of M then the *Gauss* and *Weingarten* formulas are given by

$$\widetilde{\nabla}_U V = \nabla_U V + h(U, V), \tag{2.6}$$

$$\widetilde{\nabla}_U N = -A_N U + \nabla_U^\perp N, \tag{2.7}$$

for each $U, V \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, where h and A_N are the second fundamental form and the shape operator (corresponding to the normal vector field N), respectively,

for the immersion of M into $\tilde{M}$. They are related as

$$g\big(h(U,V),N\big) = g(A_N U, V). \tag{2.8}$$

Now for any $U \in \Gamma(TM)$, we write

$$\varphi U = PU + FU, \tag{2.9}$$

where PU and FU are the tangential and normal components of φU, respectively. Similarly for any $N \in \Gamma(T^\perp M)$, we have

$$\varphi N = tN + fN, \tag{2.10}$$

where tN (resp. fN) are the tangential (resp. normal) components of φN. A submanifold M is said to be totally geodesic and totally umbilical, if $h(U,V) = 0$ and $h(U,V) = g(U,V)H$, respectively.

There is another class of submanifolds, which is called the class of slant submanifold. For each non-zero vector U tangent to M at p, such that U is not proportional to ξ_p, we denote by $0 \leq \theta(U) \leq \pi/2$, the angle between φU and $T_p M$, which is called the Wirtinger angle. If the angle $\theta(U)$ is constant for all $U \in T_p M - \langle \xi(p) \rangle$ and $p \in M$, then M is called a slant submanifold [8] and the angle θ is called the slant angle of M. Obviously if $\theta = 0$, M is invariant and if $\theta = \pi/2$, M is anti-invariant submanifold. A slant submanifold is said to be proper slant if it is neither invariant nor anti-invariant.

In an almost contact metric manifold, in fact, Cabrerizo *et al.* [9] obtained the following theorem.

Theorem 2.1 *Let M be a submanifold of an almost contact metric manifold $\tilde{M}$ such that $\xi \in TM$. Then M is slant if and only if there exists a constant $\lambda \in [0,1]$ such that*

$$P^2 = \lambda(-I + \eta \otimes \xi). \tag{2.11}$$

Furthermore, in such a case, θ is the slant angle and it satisfies $\lambda = \cos^2 \theta$.

Hence, for a slant submanifold M of an almost contact metric manifold $\widetilde{M}$, we have the following relations which are consequences of Theorem 2.1:

$$g(PU,PV) = \cos^2 \theta \big\{ g(U,V) - \eta(U)\eta(V) \big\}, \tag{2.12}$$

$$g(FU,FV) = \sin^2 \theta \big\{ g(U,V) - \eta(U)\eta(V) \big\}, \tag{2.13}$$

for any $U, V \in \Gamma(TM)$.

Moreover, we define pseudo-slant submanifolds of almost contact manifolds by using the slant distribution given in [10]. However, pseudo-slant submanifolds were defined by Carriazo [11] under the name anti-slant submanifolds as a particular class of bi-slant submanifolds. The definition of pseudo-slant is as follows.

Definition 2.1 A submanifold M of an almost contact manifold $\widetilde{M}$ is said to be pseudo-slant submanifold, if there exist two orthogonal distributions $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$ such that:

(i) $TM = \mathcal{D}^\theta \oplus \mathcal{D}^\perp \oplus \langle \xi \rangle$, where $\langle \xi \rangle$ is a 1-dimensional distribution spanned by ξ.

(ii) $\mathcal{D}^\perp$ is an anti-invariant distribution under φ *i.e.*, $\varphi \mathcal{D}^\perp \subseteq T^\perp M$.

(iii) $\mathcal{D}^\theta$ is a slant distribution with slant angle $\theta \neq 0, \frac{\pi}{2}$.

Let m_1 and m_2 be the dimensions of the distributions $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$, respectively. If $m_2 = 0$, then M is an anti-invariant submanifold. If $m_1 = 0$ and $\theta = 0$, then M is an invariant submanifold. If $m_1 = 0$ and $\theta \neq 0, \frac{\pi}{2}$, then M is a proper-slant submanifold, or if $\theta = \frac{\pi}{2}$, then M is an anti-invariant submanifold and if $\theta = 0$, then M is a semi-invariant submanifold. Since μ is an invariant subspace of a normal bundle $T^\perp M$, then in the case of a pseudo-slant submanifold, the normal bundle $T^\perp M$ can be decomposed as follows:

$$T^\perp M = \varphi \mathcal{D}^\perp \oplus F\mathcal{D}^\theta \oplus \mu, \tag{2.14}$$

where μ is an even dimensional invariant subbundle of $T^\perp M$. A pseudo-slant submanifold is said to be mixed totally geodesic if $h(X, Z) = 0$, for all $X \in \Gamma(\mathcal{D}^\perp)$ and $Z \in \Gamma(\mathcal{D}^\theta)$. Now let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of the tangent space TM and e_r belong to the orthonormal basis $\{e_{n+1}, e_{n+2}, \ldots, e_m\}$ of a normal bundle $T^\perp M$, then we define

$$h_{ij}^r = g\big(h(e_i, e_j), e_r\big) \quad \text{and} \quad \|h\|^2 = \sum_{i,j=1}^{n} g\big(h(e_i, e_j), h(e_i, e_j)\big). \tag{2.15}$$

As a consequence, for a differentiable function λ on M, we have

$$\|\nabla \lambda\|^2 = \sum_{i=1}^{n} \big(e_i(\lambda)\big)^2, \tag{2.16}$$

where the gradient grad λ is defined by $g(\nabla \lambda, X) = X\lambda$, for any $X \in \Gamma(TM)$.

3 Warped product submanifolds

In [12], Bishop and O'Neill, defined the notion of warped product manifolds to construct examples of Riemannian manifolds with a negative curvature. These manifolds are natural generalizations of Riemannian product manifolds. They defined these manifolds as follows: Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and $f : M_1 \to (0, \infty)$ a positive differentiable function on M_1. Consider the product manifold $M_1 \times M_2$ with its canonical projections $\pi_1 : M_1 \times M_2 \to M_1$, $\pi_2 : M_1 \times M_2 \to M_2$ and the projection maps given by $\pi_1(p, q) = p$ and $\pi_2(p, q) = q$ for every $t = (p, q) \in M_1 \times M_2$. The warped product $M = M_1 \times_f M_2$ is the product manifold $M_1 \times M_2$ equipped with the Riemannian structure such that

$$\|U\|^2 = \big\|\pi_1 * (U)\big\|^2 + f^2\big(\pi_1(p)\big)\big\|\pi_2 * (U)\big\|^2$$

for any tangent vector $U \in \Gamma(T_t M)$, where $*$ is the symbol of the tangent maps. Thus we have $g = g_1 + f^2 g_2$. The function f is called the warping function on M. It was defined in [12] and we have the following.

Lemma 3.1 [12] *Let $M = M_1 \times_f M_2$ be warped product manifolds. For any $X, Y \in \Gamma(TM_1)$ and $Z, W \in \Gamma(TM_2)$:*

(i) $\nabla_X Y \in \Gamma(TM_1)$,

(ii) $\nabla_Z X = \nabla_X Z = (X \ln f)Z$,

(iii) $\nabla_Z W = \nabla'_Z W - g(Z, W)\nabla \ln f$,

where ∇ and ∇' denote the Levi-Civita connections on M and M_2, respectively. On the other hand, $\nabla \ln f$, the gradient of $\ln f$, is defined as $g(\nabla \ln f, U) = U \ln f$. A warped product manifold $M = M_1 \times_f M_2$ is said to be trivial if the warping function f is constant. If $M = M_1 \times_f M_2$ is a warped product manifold then M_1 is a totally geodesic and M_2 is a totally umbilical submanifold of M. First of all we give some preparatory lemmas.

Lemma 3.2 *Let $M = M_\perp \times_f M_\theta$ be a warped product pseudo-slant submanifold of a nearly Kenmotsu manifold $\widetilde{M}$ such that the structure vector field ξ is tangent to $M_\perp$. Then*

$$g\big(h(X, PX), \varphi Z\big) = g\big(h(X, Z), FPX\big) + \frac{1}{3}\big\{\eta(Z) - (Z \ln f)\big\}\cos^2 \theta \|X\|^2,$$

for any $X \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$.

Proof Let $M = M_\perp \times_f M_\theta$ be a warped product pseudo-slant submanifold of a nearly Kenmotsu manifold $\widetilde{M}$, then by (2.6), we have

$$g\big(h(X, PX), \varphi Z\big) = -g(\varphi \widetilde{\nabla}_X PX, Z).$$

Thus using the covariant derivatives of φ, we obtain

$$g\big(h(X, PX), \varphi Z\big) = g\big((\widetilde{\nabla}_X \varphi)PX, Z\big) - g(\widetilde{\nabla}_X \varphi PX, Z).$$

From the nearly Kenmotsu structure (2.5) and Theorem 2.1, we derive

$$g\big(h(X, PX), \varphi Z\big) = -g\big((\widetilde{\nabla}_{PX}\varphi)X, Z\big) - \cos^2 \theta g(\widetilde{\nabla}_X Z, X) + g\big(h(X, Z), FPX\big).$$

Thus from Lemma 3.1(ii) and the covariant derivative of an endomorphism φ, we obtain

$$g\big(h(X, PX), \varphi Z\big) = -g(\widetilde{\nabla}_{PX}\varphi X, Z) - g(\widetilde{\nabla}_{PX}X, \varphi Z)$$
$$+ g\big(h(X, Z), FPX\big) - \cos^2 \theta(Z \ln f)\|X\|^2.$$

Using (2.9) and (2.7), we arrive at

$$2g\big(h(X, PX), \varphi Z\big) = g(\nabla_{PX}Z, PX) - g(\widetilde{\nabla}_{PX}FX, Z)$$
$$+ g\big(h(X, Z), FPX\big) - \cos^2 \theta(Z \ln f)\|X\|^2.$$

Finally, using Lemma 3.1(ii) and (2.12), we get

$$2g\big(h(X, PX), \varphi Z\big) = g\big(h(PX, Z), FX)\big) + g\big(h(X, Z), FPX\big). \qquad (3.1)$$

Again for any $X \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$, we have

$$g\big(h(X, Z), FPX\big) = g(\widetilde{\nabla}_Z X, FPX).$$

From the fact that ξ is tangent to $M_\perp$, (2.6), and (2.9), we obtain

$$g\big(h(X,Z),FPX\big) = -g(\varphi\widetilde{\nabla}_Z X, PX) + \cos^2\theta g(\nabla_Z X, X).$$

Then from the definition of the covariant derivative of φ and Lemma 3.1(ii), we can derive

$$g\big(h(X,Z),FPX\big) = g\big((\widetilde{\nabla}_Z\varphi)X, PX\big) - g(\widetilde{\nabla}_Z\phi X, PX) + \cos^2\theta(Z\ln f)\|X\|^2.$$

Thus by using the structure equation (2.5) and (2.5) and (2.12), the above equation takes the form

$$g\big(h(X,Z),FPX\big) = -g\big((\widetilde{\nabla}_X\varphi)Z, PX\big) - \eta(Z)\cos^2\theta\|X\|^2 - g(\nabla_Z PX, PX)$$
$$- g(\widetilde{\nabla}_Z FX, PX) + \cos^2\theta(X\ln f)\|Z\|^2.$$

Using Lemma 3.1(ii), (2.7), and (2.12), we arrive at

$$g\big(h(X,Z),FPX\big) = -g(\widetilde{\nabla}_X\varphi Z, PX) - g(\widetilde{\nabla}_X Z, \varphi PX)$$
$$- \eta(Z)\cos^2\theta\|X\|^2 + g\big(h(Z,PX),FX\big).$$

Equation (2.7) and Theorem 2.1 for a slant submanifold give us

$$g\big(h(X,Z),FPX\big) = g(A_{\varphi Z}X, PX) + \cos^2\theta g(\nabla_X Z, X) - \eta(Z)\cos^2\theta\|X\|^2$$
$$- g(\widetilde{\nabla}_X Z, FPX) + g\big(h(Z,PX),FX\big).$$

Finally, from Lemma 3.1(ii) and (2.6), we derive

$$g\big(h(X,PX),\varphi Z\big) = 2g\big(h(X,Z),FPX\big) - \big\{(Z\ln f) - \eta(Z)\big\}\cos^2\theta\|X\|^2$$
$$- g\big(h(Z,PX),FX\big). \tag{3.2}$$

Thus from (3.1) and (3.2), we get

$$g\big(h(X,PX),\varphi Z\big) - g\big(h(X,Z),FPX\big) = \frac{1}{3}\big\{\eta(Z) - (Z\ln f)\big\}\cos^2\theta\|X\|^2,$$

which is our final result. This completes the proof of the lemma. $\qquad\square$

Lemma 3.3 *Let $M = M_\perp \times_f M_\theta$ be a warped product pseudo-slant submanifold of a nearly Kenmotsu manifold $\widetilde{M}$. Then*:

   (iii) $g(h(X,X),\varphi Z) = g(h(Z,X),FX)$,

   (iv) $g(h(PX,PX),\varphi Z) = g(h(Z,PX),FPX)$,

for any $X \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$.

Proof For $X \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$, we have

$$g\big(h(X,X),\varphi Z\big) = g(\widetilde{\nabla}_X X, \varphi Z) = -g(\varphi\widetilde{\nabla}_X X, Z).$$

From the definition of the covariant derivative of a tensor field φ, we get

$$g\big(h(X,X),\varphi Z\big) = g(\widetilde{\nabla}_X\varphi)X,Z) - g(\widetilde{\nabla}_X\phi X,Z).$$

Then from (2.5) and (2.2), we obtain

$$g\big(h(X,X),\varphi Z\big) = g(\widetilde{\nabla}_X Z, PX) - g(\widetilde{\nabla}_X FX,Z).$$

Thus from Lemma 3.1(ii) and (2.6), the above equation can be written as

$$g\big(h(X,X),\varphi Z\big) = (Z\ln f)g(X,PX) + g(A_{FX}X,Z).$$

Since X and PX are orthogonal vector fields and considering (2.8), we arrive at

$$g\big(h(X,X),\varphi Z\big) = g\big(h(X,Z),FX\big), \tag{3.3}$$

which is the first result of the lemma. Interchanging X by PX in (3.3), we get the last result of the lemma. This completes the proof of the lemma.　□

Lemma 3.4 *Assume we have a $M = M_\theta \times_f M_\perp$ non-trivial warped product pseudo-slant submanifold of a nearly Kenmotsu manifold $\widetilde{M}$. Then:*

(i) $g(h(Z,Z),FPX) = g(h(Z,PX),\varphi Z) + \{\eta(X) - (X\ln f)\}\cos^2\theta\|Z\|^2,$

(ii) $g(h(Z,Z),FX) = g(h(Z,X),\varphi Z) - (PX\ln f)\|Z\|^2,$

for any $X \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$, where the structure vector field ξ is tangent to M_θ.

Proof From (2.6), and (2.9), we get

$$g\big(h(Z,Z),FPX\big) = g(\widetilde{\nabla}_Z Z, FPX) = g(\widetilde{\nabla}_Z Z, \varphi PX) - g(\widetilde{\nabla}_Z Z, P^2 X).$$

From Theorem 2.1, we obtain

$$g\big(h(Z,Z),FPX\big) = -g(\varphi\widetilde{\nabla}_Z Z, PX) + \cos^2\theta\big\{g(\widetilde{\nabla}_Z Z, X) - \eta(X)g(\widetilde{\nabla}_Z Z,\xi)\big\}.$$

Using the property of a Riemannian connection and the covariant derivative of an endomorphism, we derive

$$\begin{aligned}g\big(h(Z,Z),FPX\big) = {}& g\big((\widetilde{\nabla}_Z\varphi)Z,PX\big) - g(\widetilde{\nabla}_Z\phi Z,PX) - \cos^2\theta g(\nabla_Z X,Z)\\ & + \cos^2\theta\,\eta(X)g(\widetilde{\nabla}_Z\xi,Z).\end{aligned}$$

Thus from the fact that in a nearly Kenmotsu manifolds $(\xi\ln f) = 1$ [13] and from Lemma 3.1(ii), we arrive at

$$g\big(h(Z,Z),FPX\big) = g\big(h(Z,PX),\phi Z\big) + \big\{\eta(X) - (X\ln f)\big\}\cos^2\theta\|Z\|^2,$$

which is the first part of the lemma. The second part of the lemma can easily be found by interchanging X by PX in the above equation. This completes the proof of the lemma. □

4 Inequalities of warped product submanifolds

4.1 Inequality for a warped product pseudo-slant submanifold of the form $M_\perp \times_f M_\theta$

In this section, we obtain a geometric inequality of warped product pseudo-slant submanifold in terms of the second fundamental form such that ξ is tangent to the anti-invariant submanifold and the mixed totally geodesic submanifold. First of all we define an orthonormal frame for later use.

Let $M = M_\perp \times_f M_\theta$ be a m-dimensional warped product pseudo-slant submanifold of a $2n+1$-dimensional nearly Kenmotsu manifold $\widetilde{M}$ with M_θ of dimension $d_1 = 2\beta$ and $M_\perp$ of dimension $d_2 = \alpha + 1$, where M_θ and $M_\perp$ are the integral manifolds of $\mathcal{D}^\theta$ and $\mathcal{D}^\perp$, respectively. Then we consider that $\{e_1, e_2, \ldots, e_\alpha, e_{d_2=\alpha+1} = \xi\}$ and $\{e_{\alpha+2} = e_1^*, \ldots, e_{\alpha+\beta+1} = e_\beta^*, e_{\alpha+\beta+2} = e_{\beta+1}^* = \sec\theta P e_1^*, \ldots, e_{\alpha+1+2\beta} = e_{2\beta}^* = \sec\theta P e_\beta^*\}$ are orthonormal frames of $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$, respectively. Thus the orthonormal frames of the normal subbundles $\varphi\mathcal{D}^\perp$, $F\mathcal{D}^\theta$, and μ, respectively, are $\{e_{m+1} = \bar{e}_1 = \varphi e_1, \ldots, e_{m+\alpha} = \bar{e}_\alpha = \varphi e_\alpha\}$, $\{e_{m+\alpha+1} = \bar{e}_{\alpha+1} = \tilde{e}_1 = \csc\theta F e_1^*, \ldots, e_{m+\alpha+\beta} = \bar{e}_{\alpha+\beta} = \tilde{e}_\beta = \csc\theta F e_\beta^*, e_{m+\alpha+\beta+1} = \bar{e}_{\alpha+\beta+1} = \tilde{e}_{\beta+1} = \csc\theta\sec\theta FP e_1^*, \ldots, e_{m+\alpha+2\beta} = \bar{e}_{\alpha+2\beta} = \tilde{e}_{2\beta} = \csc\theta\sec\theta FP e_\beta^*\}$, and $\{e_{2m-1} = \bar{e}_m, \ldots, e_{2n+1} = \bar{e}_{2(n-m+1)}\}$.

Theorem 4.1 *Let $M = M_\perp \times_f M_\theta$ be a m-dimensional mixed totally geodesic warped product pseudo-slant submanifold of a $2n+1$-dimensional nearly Kenmotsu manifold $\widetilde{M}$ such that $\xi \in \Gamma(TM_\perp)$, where $M_\perp$ is an anti-invariant submanifold of dimension d_2 and M_θ is a proper-slant submanifold of dimension d_1 of $\widetilde{M}$. Then*:

(i) *The squared norm of the second fundamental form of M is given by*

$$\|h\|^2 \geq \frac{2\beta}{9}\cos^2\theta\{\|\nabla^\perp \ln f\|^2 - 1\}. \tag{4.1}$$

(ii) *The equality holds in (4.1), if $M_\perp$ is totally geodesic and M_θ is a totally umbilical submanifold into $\widetilde{M}$.*

Proof The squared norm of the second fundamental form is defined as

$$\|h\|^2 = \|h(\mathcal{D}^\theta, \mathcal{D}^\theta)\|^2 + \|h(\mathcal{D}^\perp, \mathcal{D}^\perp)\|^2 + 2\|h(\mathcal{D}^\theta, \mathcal{D}^\perp)\|^2.$$

Since M is mixed totally geodesic,

$$\|h\|^2 = \|h(\mathcal{D}^\perp, \mathcal{D}^\perp)\|^2 + \|h(\mathcal{D}^\theta, \mathcal{D}^\theta)\|^2. \tag{4.2}$$

Then with (2.15), we obtain

$$\|h\|^2 \geq \sum_{l=m+1}^{2n+1}\sum_{i,j=1}^{2\beta} g(h(e_i^*, e_j^*), e_l)^2.$$

Since the above equation can be expressed as in the components of $\varphi\mathcal{D}^\perp$, $F\mathcal{D}^\theta$, and ν, we derive

$$\|h\|^2 \geq \sum_{l=1}^{\alpha}\sum_{i,j=1}^{2\beta}g\big(h\big(e_i^*, e_j^*\big), \bar{e}_l\big)^2 + \sum_{l=\alpha+1}^{2\beta+\alpha}\sum_{i,j=1}^{2\beta}g\big(h\big(e_i^*, e_j^*\big), \bar{e}_l\big)^2$$
$$+ \sum_{l=m}^{2(n-m+1)}\sum_{i,j=1}^{2\beta}g\big(h\big(e_i^*, e_j^*\big), \bar{e}_l\big)^2. \tag{4.3}$$

Leaving all the terms except the first, we get

$$\|h\|^2 \geq \sum_{l=1}^{\alpha}\sum_{i,j=1}^{2\beta}g\big(h\big(e_i^*, e_j^*\big), \bar{e}_l\big)^2.$$

Using another adapted frame for $\mathcal{D}^\theta$, we derive

$$\|h\|^2 \geq \sum_{i=1}^{\alpha}\sum_{r,k=1}^{\beta}g\big(h\big(e_r^*, e_k^*\big), \bar{e}_i\big)^2 + \sec^2\theta\sum_{i=1}^{\alpha}\sum_{r,k=1}^{\beta}g\big(h\big(Pe_r^*, e_k^*\big), \bar{e}_i\big)^2$$
$$+ \sec^2\sum_{i=1}^{\alpha}\sum_{r,k=1}^{\beta}g\big(h\big(e_r^*, Pe_k^*\big), \bar{e}_i\big)^2 + \sec^4\theta\sum_{i=1}^{p}\sum_{r,k=1}^{\beta}g\big(h\big(Pe_r^*, Pe_k^*\big), \bar{e}_i\big)^2.$$

Then for a mixed totally geodesic submanifold, the first and last terms of the right hand side in the above equation vanish identically by using Lemma 3.3, and we obtain

$$\|h\|^2 \geq 2\sec^2\theta\sum_{i=1}^{\alpha}\sum_{r=1}^{\beta}g\big(h\big(Pe_r^*, e_r^*\big), \bar{e}_i\big)^2.$$

Thus from Lemma 3.2, for a mixed totally geodesic submanifold and using the fact that $\eta(e_i) = 0$, $i = 1, 2, \ldots, d_2 - 1$ for an orthonormal frame, we arrive at

$$\|h\|^2 \geq \frac{2}{9}\cos^2\theta\sum_{i=1}^{\alpha}\sum_{r=1}^{\beta}(\bar{e}_i\ln f)^2 g\big(e_r^*, e_r^*\big)^2. \tag{4.4}$$

Now we add and subtract the same term $\xi\ln f$ in (4.4), getting

$$\|h\|^2 \geq \frac{2}{9}\cos^2\theta\sum_{i=1}^{\alpha+1}\sum_{r=1}^{\beta}(\bar{e}_j\ln f)^2 g\big(e_r^*, e_r^*\big)^2 - \frac{2}{9}\cos^2\theta\sum_{r=1}^{\beta}(\xi\ln f)^2 g\big(e_r^*, e_r^*\big)^2.$$

It well known that $\xi\ln f = 1$ [13] for a warped product submanifold of a nearly Kenmotsu manifold. Thus the above equation gives

$$\|h\|^2 \geq \frac{2\beta}{9}\cos^2\theta\big\{\big\|\nabla^\perp\ln f\big\|^2 - 1\big\}.$$

If the equality holds above, then from the terms left in (4.2), we obtain the following condition from the first term:

$$h\big(\mathcal{D}^\perp, \mathcal{D}^\perp\big) = 0,$$

which means that $M_\perp$ is totally geodesic in $\widetilde{M}$. Similarly, from the second and third terms in (4.3), we derive

$$g\big(h\big(\mathcal{D}^\theta, \mathcal{D}^\theta\big), F\mathcal{D}^\theta\big) = 0, \qquad g\big(h\big(\mathcal{D}^\theta, \mathcal{D}^\theta\big), \nu\big) = 0,$$

which implies that

$$h\big(\mathcal{D}^\theta, \mathcal{D}^\theta\big) \perp F\mathcal{D}^\theta, \qquad h\big(\mathcal{D}^\theta, \mathcal{D}^\theta\big) \perp \nu \quad \implies \quad h\big(\mathcal{D}^\theta, \mathcal{D}^\theta\big) \subset \phi \mathcal{D}^\perp.$$

We have Lemma 3.2, which shows that M_θ is a totally umbilical into $\widetilde{M}$ due to being totally umbilical in M. So equality holds. This completes the proof of the theorem. □

4.2 Inequality for a warped product pseudo-slant submanifold of the form $M_\theta \times_f M_\perp$

In this section, we obtain an inequality of warped product pseudo-slant subamnifolds such that the structure vector field ξ is tangent to the slant submanifold M_θ. Taking ξ tangent to $\mathcal{D}^\theta$, then we use the last frame.

Theorem 4.2 *Let $M = M_\theta \times_f M_\perp$ be a m-dimensional mixed totally geodesic warped product pseudo-slant submanifold of a $2n + 1$-dimensional nearly Kenmotsu manifold $\widetilde{M}$ such that $\xi \in \Gamma(TM_\theta)$, where $M_\perp$ is an anti-invariant submanifold of dimension $d_2 = \alpha$ and M_θ is a proper-slant submanifold of dimension $d_1 = 2\beta + 1$ of $\widetilde{M}$. Then:*

  (i) *The squared norm of the second fundamental form of M is given by*

$$\|h\|^2 \geq \alpha \left\{ \csc^2 \theta \big(\big\| \nabla^\theta \ln f \big\|^2 - 1 \big) - \sum_{i=1}^{\beta} \big(e_i^* \ln f \big)^2 \right\}. \tag{4.5}$$

  (ii) *If equality holds identically in (4.5), then M_θ is a totally geodesic submanifold and $M_\perp$ is a totally umbilical submanifold of $\widetilde{M}$, respectively.*

Proof We start by the definition of the second fundamental form

$$\|h\|^2 = \big\| h\big(\mathcal{D}^\theta, \mathcal{D}^\theta\big) \big\|^2 + \big\| h\big(\mathcal{D}^\perp, \mathcal{D}^\perp\big) \big\|^2 + 2 \big\| h\big(\mathcal{D}^\theta, \mathcal{D}^\perp\big) \big\|^2.$$

Since M is a mixed totally geodesic, we get

$$\|h\|^2 = \big\| h\big(\mathcal{D}^\perp, \mathcal{D}^\perp\big) \big\|^2 + \big\| h\big(\mathcal{D}^\theta, \mathcal{D}^\theta\big) \big\|^2. \tag{4.6}$$

Thus by (2.15), we obtain

$$\|h\|^2 \geq \sum_{l=m+1}^{2n+1} \sum_{r,k=1}^{\alpha} g\big(h(e_r, e_k), e_l\big)^2.$$

The above equation can be expressed in the components of $\phi \mathcal{D}^\perp$, $F\mathcal{D}^\theta$, and ν as

$$\|h\|^2 \geq \sum_{l,r,k=1}^{\alpha} g\big(h(e_r, e_k), \bar{e}_l\big)^2 + \sum_{l=\alpha+1}^{2\beta+\alpha} \sum_{r,k=1}^{\alpha} g\big(h(e_r, e_k), \bar{e}_l\big)^2$$

$$+ \sum_{l=m}^{2(n-m+1)} \sum_{r,k=1}^{\alpha} g\big(h(e_r, e_k), \bar{e}_l\big)^2. \tag{4.7}$$

We shall leave all the terms, except the second term, and we get

$$\|h\|^2 \geq \sum_{l=1}^{2\beta} \sum_{r,k=1}^{\alpha} g\big(h(e_r, e_k), \tilde{e}_l\big)^2.$$

Thus using the adapted frame for $F\mathcal{D}^\theta$, we derive

$$\|h\|^2 \geq \csc^2\theta \sum_{j=1}^{\beta} \sum_{r=1}^{\alpha} g\big(h(e_r, e_r), Fe_j^*\big)^2 + \csc^2\theta\sec^2\theta \sum_{j=1}^{\beta} \sum_{r=1}^{\alpha} g\big(h(e_r, e_r), FPe_j^*\big)^2.$$

Using Lemma 3.4 for a mixed totally geodesic warped product submanifold and the fact that $\eta(e_j) = 0$, $1 \leq j \leq d_1 - 1$ for an orthonormal frame, we arrive at

$$\|h\|^2 \geq \csc^2\theta \sum_{j=1}^{\beta} \sum_{r=1}^{\alpha} \big(Pe_j^* \ln f\big)^2 g(e_r, e_r) + \csc^2\theta\cos^2\theta \sum_{j=1}^{\beta} \sum_{r=1}^{\alpha} \big(e_j^* \ln f\big)^2 g(e_r, e_r).$$

Hence by hypothesis, we obtain

$$\|h\|^2 \geq \alpha\csc^2\theta \sum_{j=1}^{\beta} \big(Pe_j^* \ln f\big)^2 + \alpha\cot^2\theta \sum_{j=1}^{\beta} \big(e_j^* \ln f\big)^2.$$

By the property of trigonometric functions, we arrive at

$$\|h\|^2 \geq \alpha\csc^2\theta \sum_{j=1}^{2\beta} \big(e_j^* \ln f\big)^2 - \alpha \sum_{j=1}^{\beta} \big(e_j^* \ln f\big)^2.$$

Then by adding and subtracting the same terms $\xi \ln f$ in the above equation, we get

$$\|h\|^2 \geq \alpha\csc^2\theta \sum_{j=1}^{2\beta+1} \big(e_j^* \ln f\big)^2 - \alpha \sum_{j=1}^{\beta} \big(e_j^* \ln f\big)^2 - \alpha\csc^2\theta(\xi \ln f)^2.$$

$\xi \ln f = 1$ [13] for a warped product submanifold of a nearly Kenmotsu manifold. Thus the above equation gives

$$\|h\|^2 \geq \alpha\left[\csc^2\theta\big\{\big\|\nabla^\theta \ln f\big\|^2 - 1\big\} - \sum_{j=1}^{\beta} \big(e_j^* \ln f\big)^2\right].$$

If the equality holds, from the terms left in (4.6) and (4.7), we obtain the following conditions from the first and third terms:

$$\big\|h(\mathcal{D}, \mathcal{D})\big\|^2 = 0, \qquad g\big(h(\mathcal{D}^\perp, \mathcal{D}^\perp), \phi\mathcal{D}^\perp\big) = 0,$$

and

$$g\big(h(\mathcal{D}^\perp, \mathcal{D}^\perp), v\big) = 0,$$

where $\mathcal{D} = \mathcal{D}^{\theta} \oplus \xi$, this means that M_{θ} is a totally geodesic in $\widetilde{M}$ and $h(\mathcal{D}^{\perp}, \mathcal{D}^{\perp}) \subseteq F\mathcal{D}^{\theta}$. Now from Lemma 3.4 for a mixed totally geodesic submanifold we find

$$g\bigl(h(Z, W), FX\bigr) = -(PX \ln f) g(Z, W),$$

for any $Z, W \in \Gamma(TM_{\perp})$ and $X \in \Gamma(TM_{\theta})$. The above equations imply that $M_{\perp}$ is totally umbilical in $\widetilde{M}$, so the equality holds. This completes the proof of the theorem. □

Author details
[1] Institute of Mathematical Sciences, Faculty of Science, University of Malaya, Kuala Lumpur, 50603, Malaysia.
[2] Department of Mathematics, Dokuz Eylul University, Tinaztepe Campus Buca, Izmir, 35160, Turkey.

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