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# Higher order derivatives of approximation polynomials on $\mathbb{R}$

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## Abstract

Leviatan has investigated the behavior of higher order derivatives of approximation polynomials of a differentiable function  $f$  on  $[-1, 1]$ . Especially, when  $P_n$  is the best approximation of  $f$ , he estimates the differences  $\|f^{(k)} - P_n^{(k)}\|_{L_\infty([-1,1])}$ ,  $k = 0, 1, 2, \dots$ . In this paper, we give the analogies for them with respect to the differentiable functions on  $\mathbb{R}$ .

**MSC:** 41A10; 41A50

**Keywords:** the polynomial of best approximation; the exponential-type weight

## 1 Introduction

Let  $\mathbb{R} = (-\infty, \infty)$  and  $\mathbb{R}^+ = [0, \infty)$ . We say that  $f : (0, \infty) \rightarrow \mathbb{R}^+$  is quasi-increasing in  $(0, \infty)$  if there exists  $C > 0$  such that  $f(x) \leq Cf(y)$  for  $0 < x < y$ . The notation  $f(x) \sim g(x)$  means that there are positive constants  $C_1, C_2$  such that for the relevant range of  $x$ ,  $C_1 \leq f(x)/g(x) \leq C_2$ . A similar notation is used for sequences and sequences of functions. Throughout  $C, C_1, C_2, \dots$  denote positive constants independent of  $n, x, t$ . The same symbol does not necessarily denote the same constant in different occurrences. We denote the class of polynomials with degree  $n$  by  $\mathcal{P}_n$ .

First, we introduce some classes of weights. Levin and Lubinsky [1] introduced the class of weights on  $\mathbb{R}$  as follows.

**Definition 1.1** Let  $Q : \mathbb{R} \rightarrow [0, \infty)$  be a continuous even function, and satisfy the following properties:

- $Q'(x) > 0$  for  $x > 0$  and is continuous in  $\mathbb{R}$ , with  $Q(0) = 0$ .
- $Q''(x)$  exists and is positive in  $\mathbb{R} \setminus \{0\}$ .
- $\lim_{x \rightarrow \infty} Q(x) = \infty$ .
- The even function

$$T_w(x) := \frac{xQ'(x)}{Q(x)}, \quad x \neq 0$$

is quasi-increasing in  $(0, \infty)$ , with

$$T_w(x) \geq \Lambda > 1, \quad x \in \mathbb{R} \setminus \{0\}.$$

(e) There exists  $C_1 > 0$  such that

$$\frac{Q''(x)}{|Q'(x)|} \leq C_1 \frac{|Q'(x)|}{Q(x)}, \quad \text{a.e. } x \in \mathbb{R}.$$

Furthermore, if there also exist a compact subinterval  $J (\ni 0)$  of  $\mathbb{R}$  and  $C_2 > 0$  such that

$$\frac{Q''(x)}{|Q'(x)|} \geq C_2 \frac{|Q'(x)|}{Q(x)}, \quad \text{a.e. } x \in \mathbb{R} \setminus J,$$

then we write  $w = \exp(-Q) \in \mathcal{F}(C^2+)$ .

For convenience, we denote  $T$  instead of  $T_w$ , if there is no confusion. Next, we give some typical examples of  $\mathcal{F}(C^2+)$ .

**Example 1.2** [2]

- (1) If  $T(x)$  is bounded, then we call the weight  $w = \exp(-Q(x))$  the Freud-type weight and we write  $w \in \mathcal{F}^* \subset \mathcal{F}(C^2+)$ .
- (2) When  $T(x)$  is unbounded, then we call the weight  $w = \exp(-Q(x))$  the Erdős-type weight: For  $\alpha > 1, l \geq 1$  we define

$$Q(x) := Q_{l,\alpha}(x) = \exp_l(|x|^\alpha) - \exp_l(0),$$

where  $\exp_l(x) = \exp(\exp(\exp \cdots \exp x) \cdots)$  ( $l$  times). More generally, we define

$$Q_{l,\alpha,m}(x) = |x|^m \{ \exp_l(|x|^\alpha) - \tilde{\alpha} \exp_l(0) \}, \quad \alpha + m > 1, m \geq 0, \alpha \geq 0,$$

where  $\tilde{\alpha} = 0$  if  $\alpha = 0$ , and otherwise  $\tilde{\alpha} = 1$ . We note that  $Q_{l,0,m}$  gives a Freud-type weight, and  $Q_{l,\alpha,m}$  ( $\alpha > 0$ ) gives an Erdős-type weight.

- (3) For  $\alpha > 1, Q_\alpha(x) = (1 + |x|)^{|x|^\alpha} - 1$  gives also an Erdős-type weight.

For a continuous function  $f : [-1, 1] \rightarrow \mathbb{R}$ , let

$$E_n(f) = \inf_{P \in \mathcal{P}_n} \|f - P\|_{L_\infty([-1,1])} = \inf_{P \in \mathcal{P}_n} \max_{x \in [-1,1]} |f(x) - P(x)|.$$

Leviatan [3] has investigated the behavior of the higher order derivatives of approximation polynomials for the differentiable function  $f$  on  $[-1, 1]$ , as follows.

**Theorem** (Leviatan [3]) *For  $r \geq 0$  we let  $f \in C^{(r)}[-1, 1]$ , and let  $P_n \in \mathcal{P}_n$  denote the polynomial of best approximation of  $f$  on  $[-1, 1]$ . Then for each  $0 \leq k \leq r$  and every  $-1 \leq x \leq 1$ ,*

$$|f^{(k)}(x) - P_n^{(k)}(x)| \leq \frac{C_r}{n^k} \Delta_n^{-k}(x) E_{n-k}(f^{(k)}), \quad n \geq k,$$

where  $\Delta_n(x) := \sqrt{1 - x^2}/n + 1/n^2$  and  $C_r$  is an absolute constant which depends only on  $r$ .

In this paper, we will give an analogy of Leviatan’s theorem for some exponential-type weight. In Section 2, we give the theorems in the space  $L_\infty(\mathbb{R})$ , and we also make a certain assumption and some notations which are needed in order to state the theorems. In Section 3, we give some lemmas and the proofs of the theorems.

## 2 Theorems and preliminaries

First, we introduce some well-known notations. If  $f$  is a continuous function on  $\mathbb{R}$ , then we define

$$\|fw\|_{L_\infty(\mathbb{R})} := \sup_{t \in \mathbb{R}} |f(t)w(t)|,$$

and for  $1 \leq p < \infty$  we denote

$$\|fw\|_{L_p(\mathbb{R})} := \left( \int_{\mathbb{R}} |f(t)w(t)|^p dt \right)^{1/p}.$$

Let  $1 \leq p \leq \infty$ . If  $\|wf\|_{L_p(\mathbb{R})} < \infty$ , then we write  $wf \in L_p(\mathbb{R})$ , and here if  $p = \infty$ , we suppose that  $f \in C(\mathbb{R})$  and  $\lim_{|x| \rightarrow \infty} |w(x)f(x)| = 0$ . We denote the rate of approximation of  $f$  by

$$E_{p,n}(w,f) := \inf_{P \in \mathcal{P}_n} \|(f - P)w\|_{L_p(\mathbb{R})}.$$

The Mhaskar-Rakhmanov-Saff numbers  $a_x$  is defined as follows:

$$x = \frac{2}{\pi} \int_0^1 \frac{a_x u Q'(a_x u)}{\sqrt{1-u^2}} du, \quad x > 0.$$

To write our theorems we need some preliminaries. We need further assumptions.

**Definition 2.1** Let  $w = \exp(-Q) \in \mathcal{F}(C^2+)$  and let  $r \geq 1$  be an integer. Then for  $0 < \lambda < (r + 2)/(r + 1)$  we write  $w \in \mathcal{F}_\lambda(C^{r+2}+)$  if  $Q \in C^{(r+2)}(\mathbb{R} \setminus \{0\})$  and there exist two constants  $C > 1$  and  $K \geq 1$  such that for all  $|x| \geq K$ ,

$$\frac{|Q'(x)|}{Q^\lambda(x)} \leq C \quad \text{and} \quad \left| \frac{Q''(x)}{Q'(x)} \right| \sim \left| \frac{Q^{(k+1)}(x)}{Q^{(k)}(x)} \right|$$

for every  $k = 2, \dots, r$  and also

$$\left| \frac{Q^{(r+2)}(x)}{Q^{(r+1)}(x)} \right| \leq C \left| \frac{Q^{(r+1)}(x)}{Q^{(r)}(x)} \right|.$$

In particular,  $w \in \mathcal{F}_\lambda(C^3+)$  means that  $Q \in C^{(3)}(\mathbb{R} \setminus \{0\})$  and

$$\frac{|Q'(x)|}{Q^\lambda(x)} \leq C \quad \text{and} \quad \left| \frac{Q'''(x)}{Q''(x)} \right| \leq C \left| \frac{Q''(x)}{Q'(x)} \right|$$

hold for  $|x| \geq K$ . In addition, let  $\mathcal{F}_\lambda(C^2+) := \mathcal{F}(C^2+)$ .

From [2], we know that Example 1.2(2), (3) satisfy all conditions of Definition 2.1. Under the same condition as of Definition 2.1 we obtain an interesting theorem as follows.

**Theorem 2.2** ([4], Theorems 4.1, 4.2 and (4.11)) *Let  $r$  be a positive integer,  $0 < \lambda < (r + 2)/(r + 1)$  and let  $w = \exp(-Q) \in \mathcal{F}_\lambda(C^{r+2}+)$ . Then, for any  $\mu, \nu, \alpha, \beta \in \mathbb{R}$ , we can construct a new weight  $w_{\mu, \nu, \alpha, \beta} \in \mathcal{F}_\lambda(C^{r+1}+)$  such that*

$$T_w^\mu(x)(1 + x^2)^\nu (1 + Q(x))^\alpha (1 + |Q'(x)|)^\beta w(x) \sim w_{\mu, \nu, \alpha, \beta}(x)$$

on  $\mathbb{R}$ , and for some  $c \geq 1$ ,

$$a_{n/c}(w) \leq a_n(w_{\mu,v,\alpha,\beta}) \leq a_{cn}(w),$$

$$T_{w_{\mu,v,\alpha,\beta}}(x) \sim T_w(x)$$

hold on  $\mathbb{R} \setminus \{0\}$ .

For a given  $\mu \in \mathbb{R}$  and  $w \in \mathcal{F}_\lambda(C^3+)$  ( $0 < \lambda < 3/2$ ), we let  $w_\mu \in \mathcal{F}(C^2+)$  satisfy  $w_\mu(x) \sim T_w^\mu(x)w(x)$  (see Theorem 4.1 in [4]). Let  $P_{n,f,w_\mu} \in \mathcal{P}_n$  be the best approximation of  $f$  with respect to the weight  $w_\mu$ , that is,

$$\|(f - P_{n,f,w_\mu})w_\mu\|_{L^\infty(\mathbb{R})} = E_n(w_\mu, f) := \inf_{P \in \mathcal{P}_n} \|(f - P)w_\mu\|_{L^\infty(\mathbb{R})}.$$

Then we have the main result as follows.

**Theorem 2.3** *Let  $r \geq 0$  be an integer. Let  $w = \exp(-Q) \in \mathcal{F}_\lambda(C^{r+3}+)$ , where  $0 < \lambda < (r + 3)/(r + 2)$ . Suppose that  $f \in C^{(r)}(\mathbb{R})$  with*

$$\lim_{|x| \rightarrow \infty} T^{1/4}(x)f^{(r)}(x)w(x) = 0.$$

*Then there exists an absolute constant  $C_r > 0$  which depends only on  $r$  such that, for  $0 \leq k \leq r$  and  $x \in \mathbb{R}$ ,*

$$\begin{aligned} |(f^{(k)}(x) - P_{n,f,w}^{(k)}(x))w(x)| &\leq C_r T^{k/2}(x)E_{n-k}(w_{1/4}, f^{(k)}) \\ &\leq C_r T^{k/2}(x) \left(\frac{a_n}{n}\right)^{r-k} E_{n-r}(w_{1/4}, f^{(r)}). \end{aligned}$$

*When  $w \in \mathcal{F}^*$ , we can replace  $w_{1/4}$  with  $cw$  ( $c$  is a constant) in the above.*

Applying Theorem 2.3 with  $w$  or  $w_{-1/4}$ , we have the following corollaries.

**Corollary 2.4**

- (1) *Let  $w = \exp(-Q) \in \mathcal{F}_\lambda(C^{r+3}+)$  and  $0 < \lambda < (r + 3)/(r + 2)$ ,  $r \geq 0$ . We suppose that  $f \in C^{(r)}(\mathbb{R})$  with*

$$\lim_{|x| \rightarrow \infty} T^{1/4}(x)f^{(r)}(x)w(x) = 0,$$

*then for  $0 \leq k \leq r$  we have*

$$\begin{aligned} \|(f^{(k)} - P_{n,f,w}^{(k)})w_{-k/2}\|_{L^\infty(\mathbb{R})} &\leq C_r E_{n-k}(w_{1/4}, f^{(k)}) \\ &\leq C_r \left(\frac{a_n}{n}\right)^{r-k} E_{n-r}(w_{1/4}, f^{(r)}). \end{aligned}$$

- (2) *Let  $w = \exp(-Q) \in \mathcal{F}_\lambda(C^{r+4}+)$ ,  $0 < \lambda < (r + 4)/(r + 3)$ ,  $r \geq 0$ . We suppose that  $f \in C^{(r)}(\mathbb{R})$  with*

$$\lim_{|x| \rightarrow \infty} f^{(r)}(x)w(x) = 0,$$

then for  $0 \leq k \leq r$  we have

$$\begin{aligned} \|(f^{(k)} - P_{nf, w_{-1/4}}^{(k)})w_{-(2k+1)/4}\|_{L_\infty(\mathbb{R})} &\leq C_r E_{n-k}(w, f^{(k)}) \\ &\leq C_r \left(\frac{a_n}{n}\right)^{r-k} E_{n-r}(w, f^{(r)}). \end{aligned}$$

When  $w \in \mathcal{F}^*$ , we can replace  $w_\mu$  ( $\mu = -k/2, \mu = -(2k + 1)/4, 0 \leq k \leq r$ , and  $\mu = 1/4$ ) with  $cw$  ( $c$  is a constant) in the above.

**Corollary 2.5** *Let  $r \geq 0$  be an integer. Let  $w = \exp(-Q) \in \mathcal{F}_\lambda(C^{r+4}_+)$ ,  $0 < \lambda < (r + 4)/(r + 3)$ , and let  $w_{(2r+1)/4} f^{(r)} \in L_\infty(\mathbb{R})$ . Then, for each  $k$  ( $0 \leq k \leq r$ ) and the best approximation polynomial  $P_{nf, w_{k/2}}$ :*

$$\|(f - P_{nf, w_{k/2}})w_{k/2}\|_{L_\infty(\mathbb{R})} = E_n(w_{k/2}, f),$$

we have

$$\begin{aligned} \|(f^{(k)} - P_{nf, w_{k/2}}^{(k)})w\|_{L_\infty(\mathbb{R})} &\leq C_r E_{n-k}(w_{(2k+1)/4}, f^{(k)}) \\ &\leq C_r \left(\frac{a_n}{n}\right)^{r-k} E_{n-r}(w_{(2k+1)/4}, f^{(r)}). \end{aligned}$$

When  $w \in \mathcal{F}^*$ , we can replace  $w_\mu$  ( $\mu = k/2, \mu = (2k + 1)/4, 0 \leq k \leq r$ ) with  $cw$  ( $c$  is a constant) in the above.

### 3 Proofs of theorems

We give the proofs of the theorems. First, we give some lemmas to prove the theorems. We construct the orthonormal polynomials  $p_n(x) = p_n(w^2, x)$  of degree  $n$  for  $w^2(x)$ , that is,

$$\int_{-\infty}^{\infty} p_n(w^2, x)p_m(w^2, x)w^2(x) dx = \delta_{nm} \quad (\text{Kronecker delta}).$$

Let  $f \in L_2(\mathbb{R})$ . The Fourier-type series of  $f$  is defined by

$$\tilde{f}(x) := \sum_{k=0}^{\infty} a_k(w^2, f)p_k(w^2, x), \quad a_k(w^2, f) := \int_{-\infty}^{\infty} f(t)p_k(w^2, t)w^2(t) dt.$$

We denote the partial sum of  $\tilde{f}(x)$  by

$$s_n(f, x) := s_n(w^2, f, x) := \sum_{k=0}^{n-1} a_k(w^2, f)p_k(w^2, x).$$

Moreover, we define the de la Vallée Poussin means by

$$v_n(f, x) := \frac{1}{n} \sum_{j=n+1}^{2n} s_j(w^2, f, x).$$

**Theorem 3.1** (Theorem 1.1, (1.5), Corollary 6.2, (6.5) in [5]) *Let  $w \in \mathcal{F}_\lambda(C^3_+)$ ,  $0 < \lambda < 3/2$ , and let  $1 \leq p \leq \infty$ . When  $T^{1/4}wf \in L_p(\mathbb{R})$ , we have, for  $n \geq 1$ ,*

$$\|v_n(f)w\|_{L_p(\mathbb{R})} \leq C \|T^{1/4}wf\|_{L_p(\mathbb{R})},$$

and so

$$\| (f - v_n(f))w \|_{L_p(\mathbb{R})} \leq CE_{p,n}(T^{1/4}w, f).$$

So, equivalently,

$$\| v_n(f)w \|_{L_p(\mathbb{R})} \leq C \| w_{1/4} f \|_{L_p(\mathbb{R})},$$

and so

$$\| (f - v_n(f))w \|_{L_p(\mathbb{R})} \leq CE_{p,n}(w_{1/4}, f). \tag{3.1}$$

When  $w \in \mathcal{F}^*$ , we can replace  $w_{1/4}$  with  $cw$ .

**Lemma 3.2** *Let  $w \in \mathcal{F}(C^2+)$ .*

(1) *(Lemma 3.5(a) in [1]) Let  $L > 0$  be fixed. Then, uniformly for  $t > 0$ ,*

$$a_{Lt} \sim a_t.$$

(2) *(Lemma 3.4, (3.17) in [1]) For  $x > 1$ , we have*

$$|Q'(a_x)| \sim \frac{x\sqrt{T(a_x)}}{a_x} \quad \text{and} \quad |Q(a_x)| \sim \frac{x}{\sqrt{T(a_x)}}.$$

(3) *(Proposition 3 in [6]) If  $T(x)$  is unbounded, then for any  $\eta > 0$  there exists  $C(\eta) > 0$  such that for  $t \geq 1$ ,*

$$a_t \leq C(\eta)t^\eta.$$

To prove the results, we need the following notations. We set

$$\sigma(t) := \inf \left\{ a_u : \frac{a_u}{u} \leq t \right\}, \quad t > 0$$

and

$$\Phi_t(x) := \sqrt{\left| 1 - \frac{|x|}{\sigma(t)} \right|} + T^{-1/2}(\sigma(t)), \quad x \in \mathbb{R}.$$

Define for  $fw \in L_p(\mathbb{R})$ ,  $0 < p \leq \infty$ ,

$$\begin{aligned} \omega_p(f, w, t) := & \sup_{0 < h \leq t} \left\| w(x) \left\{ f\left(x + \frac{h}{2}\Phi_t(x)\right) - f\left(x - \frac{h}{2}\Phi_t(x)\right) \right\} \right\|_{L_p(|x| \leq \sigma(2t))} \\ & + \inf_{c \in \mathbb{R}} \left\| w(x)(f - c)(x) \right\|_{L_p(|x| \geq \sigma(4t))} \end{aligned}$$

(see [7, 8]).

**Proposition 3.3** (cf. Theorem 1.2 in [8], Corollary 1.4 in [7]) *Let  $w \in \mathcal{F}(C^2+)$ . Let  $0 < p \leq \infty$ . Then for  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that  $fw \in L_p(\mathbb{R})$  (where for  $p = \infty$ , we require  $f$  to be continuous, and  $fw$  to vanish at  $\pm\infty$ ), we have, for  $n \geq C_3$ ,*

$$E_{p,n}(w, f) \leq C_1 \omega_p \left( f, w, C_2 \frac{a_n}{n} \right),$$

where  $C_j, j = 1, 2, 3$ , do not depend on  $f$  and  $n$ .

*Proof* Damelin and Lubinsky [8] or Damelin [7] have treated a certain class  $\mathcal{E}_1$  of weights containing the ones satisfying conditions (a)-(d) in Definition 1.1 and

$$\frac{yQ'(y)}{xQ'(x)} \leq \left( \frac{Q(y)}{Q(x)} \right)^C, \quad y \geq x > 0, \tag{3.2}$$

where  $C > 0$  is a constant, and they obtain this Proposition for  $w \in \mathcal{E}_1$ . Therefore, we may show  $\mathcal{F}(C^2+) \subset \mathcal{E}_1$ . In fact, from Definition 1.1(d) and (e), we have, for  $y \geq x > 0$ ,

$$\frac{Q'(y)}{Q'(x)} = \exp \left( \int_x^y \frac{Q''(t)}{Q'(t)} dt \right) \leq \exp \left( C_1 \int_x^y \frac{Q'(t)}{Q(t)} dt \right) = \left( \frac{Q(y)}{Q(x)} \right)^{C_1}$$

and

$$\frac{y}{x} = \exp \left( \int_x^y \frac{1}{t} dt \right) \leq \exp \left( \frac{1}{\Lambda} \int_x^y \frac{Q'(t)}{Q(t)} dt \right) = \left( \frac{Q(y)}{Q(x)} \right)^{\frac{1}{\Lambda}}.$$

Therefore, we obtain (3.2) with  $C = C_1 + \frac{1}{\Lambda}$ , that is, we see  $\mathcal{F}(C^2+) \subset \mathcal{E}_1$ . □

**Theorem 3.4** *Let  $w \in \mathcal{F}(C^2+)$ .*

- (1) *If  $f$  is a function having bounded variation on any compact interval and if*

$$\int_{-\infty}^{\infty} w(x) |df(x)| < \infty,$$

*then there exists a constant  $C > 0$  such that, for every  $t > 0$ ,*

$$\omega_1(f, w, t) \leq Ct \int_{-\infty}^{\infty} w(x) |df(x)|,$$

*and so*

$$E_{1,n}(w, f) \leq C \frac{a_n}{n} \int_{-\infty}^{\infty} w(x) |df(x)|.$$

- (2) *If  $f$  is continuous and  $\lim_{|x| \rightarrow \infty} |(\sqrt{T}wf)(x)| = 0$ , then we have*

$$\lim_{t \rightarrow 0} \omega_{\infty}(f, w, t) = 0.$$

To prove this theorem we need the following lemma.

**Lemma 3.5** (Lemma 2.5(b) in [7] and Lemma 7 in [6]) *Let  $w \in \mathcal{F}(C^2+)$ . Uniformly for  $u > 0$  large enough and  $|x|, |y| \leq a_u$  such that*

$$|x - y| \leq t\Phi_t(x), \quad t = a_u/u,$$

then

$$w(x) \sim w(y).$$

*Proof of Theorem 3.4* (1) Let  $g(x) := f(x) - f(0)$ . For  $t > 0$  small enough let  $0 < h \leq t$  and  $|x| \leq \sigma(2t) < \sigma(t)$ . Hence we have  $\Phi_t(x) \leq 2$  for  $|x| \leq \sigma(2t)$ . Then by Lemma 3.5,

$$\begin{aligned} & \int_{|x| \leq \sigma(2t)} w(x) \left| g\left(x + \frac{h}{2}\Phi_t(x)\right) - g\left(x - \frac{h}{2}\Phi_t(x)\right) \right| dx \\ &= \int_{|x| \leq \sigma(2t)} w(x) \left| \int_{x-\frac{h}{2}\Phi_t(x)}^{x+\frac{h}{2}\Phi_t(x)} df(v) \right| dx \leq C \int_{|x| \leq \sigma(2t)} \left| \int_{x-\frac{h}{2}\Phi_t(x)}^{x+\frac{h}{2}\Phi_t(x)} w(v) df(v) \right| dx \\ &\leq \int_{-\infty}^{\infty} \int_{x-h}^{x+h} w(v) |df(v)| dx \leq \int_{-\infty}^{\infty} w(v) \int_{v-h \leq x \leq v+h} dx |df(v)| \\ &\leq 2h \int_{-\infty}^{\infty} w(v) |df(v)|. \end{aligned}$$

Hence we have

$$\int_{|x| \leq \sigma(2t)} w(x) \left| g\left(x + \frac{h}{2}\Phi_t(x)\right) - g\left(x - \frac{h}{2}\Phi_t(x)\right) \right| dx \leq 2t \int_{-\infty}^{\infty} w(x) |df(x)|. \tag{3.3}$$

Moreover, we see

$$\inf_{c \in \mathbb{R}} \|w(x)(f - c)(x)\|_{L_1(|x| \geq \sigma(4t))} \leq \frac{1}{Q'(\sigma(4t))} \|Q'(x)w(x)g(x)\|_{L_1(|x| \geq \sigma(4t))}. \tag{3.4}$$

From Lemma 3.2(2), for  $4t =: \frac{a_u}{u}$ ,

$$Q'(\sigma(4t)) = Q'(a_u) \sim \frac{u\sqrt{T(a_u)}}{a_u} \sim \frac{\sqrt{T(\sigma(4t))}}{t}.$$

On the other hand, we have

$$\begin{aligned} \int_0^{\infty} Q'(x)w(x)|g(x)| dx &= \int_0^{\infty} Q'(x)w(x) \left| \int_0^x dg(u) \right| dx \\ &\leq \int_0^{\infty} Q'(x)w(x) \int_0^x |df(u)| dx \\ &= -w(x) \int_0^x |df(u)| \Big|_0^{\infty} + \int_0^{\infty} w(u) |df(u)|. \end{aligned}$$

Here we see

$$\left| -w(x) \int_0^x |df(u)| \right| \leq \int_0^x w(u) |df(u)|.$$

Therefore, we have

$$\int_0^\infty Q'(x)w(x)|g(x)| dx \leq 2 \int_0^\infty w(u)|df(u)|.$$

Similarly, for  $x < 0$  we see

$$\int_{-\infty}^0 |Q'(x)w(x)g(x)| dx \leq 2 \int_{-\infty}^0 w(x)|df(x)|.$$

Consequently, we have

$$\int_{-\infty}^\infty |Q'(x)w(x)g(x)| dx \leq 2 \int_{-\infty}^\infty w(x)|df(x)|.$$

Hence we have

$$\|Q'wg\|_{L_1(\mathbb{R})} \leq 2 \int_{-\infty}^\infty w(u)|df(u)|. \tag{3.5}$$

Therefore, using (3.4) and (3.5), we have

$$\inf_{c \in \mathbb{R}} \|w(x)(f - c)(x)\|_{L_1(|x| \geq \sigma(4t))} = O(t) \int_{-\infty}^\infty w(x)|df(x)|. \tag{3.6}$$

Consequently, by (3.3) and (3.6) we have

$$\omega_1(f, w, t) \leq Ct \int_{-\infty}^\infty w(x)|df(x)|.$$

Hence, setting  $t = C_2 \frac{a_n}{n}$ , if we use Proposition 3.3, then

$$E_{1,n}(w, f) \leq C \frac{a_n}{n} \int_{-\infty}^\infty w(x)|df(x)|.$$

(2) Given  $\varepsilon > 0$ , and let us take  $L = L(\varepsilon) > 0$  such that

$$\sup_{|x| \geq L} |w(x)f(x)| \leq \sup_{|x| \geq L} |\sqrt{T(x)}w(x)f(x)| < \varepsilon,$$

since  $T(x) > 1$ . Hence, if  $|x| \geq 2L$  and  $0 < t < t_0$ , then

$$\begin{aligned} & \left| w(x) \left\{ f\left(x + \frac{h}{2}\Phi_t(x)\right) - f\left(x - \frac{h}{2}\Phi_t(x)\right) \right\} \right| \\ & \leq C \left[ \left| \sqrt{T\left(x + \frac{h}{2}\Phi_t(x)\right)} w\left(x + \frac{h}{2}\Phi_t(x)\right) f\left(x + \frac{h}{2}\Phi_t(x)\right) \right| \right. \\ & \quad \left. + \left| \sqrt{T\left(x - \frac{h}{2}\Phi_t(x)\right)} w\left(x - \frac{h}{2}\Phi_t(x)\right) f\left(x - \frac{h}{2}\Phi_t(x)\right) \right| \right] \\ & \leq 2C\varepsilon, \end{aligned}$$

where for the first inequality we used Lemma 3.5(2), and for the second inequality we used the fact that  $|x \pm \frac{h}{2}\Phi_t(x)| \geq L$ . On the other hand,

$$\lim_{t \rightarrow 0} \sup_{0 < h \leq t} \left\| w(x) \left\{ f\left(x + \frac{h}{2}\Phi_t(x)\right) - f\left(x - \frac{h}{2}\Phi_t(x)\right) \right\} \right\|_{L_\infty(|x| \leq 2L)} = 0.$$

Finally, we will show

$$\inf_{c \in \mathbb{R}} \|w(f - c)\|_{L_\infty(|x| \geq \sigma(4t))} \rightarrow 0, \quad t \rightarrow 0. \tag{3.7}$$

If we let  $4t := \frac{a_n}{n}$ , then we see  $n \rightarrow \infty$  and  $\sigma(4t) = a_n \rightarrow \infty$  as  $t \rightarrow 0$ . Hence using  $\lim_{|x| \rightarrow \infty} |(\sqrt{T}wf)(x)| = 0$ , we have for  $|x| \geq \sigma(4t)$ ,

$$a_n < x \rightarrow \infty \Rightarrow |f(x)w(x)| \leq |T^{1/2}(x)f(x)w(x)| \rightarrow 0$$

and  $|cw(x)| \leq cw(a_n) \rightarrow 0$  as  $t \rightarrow 0$ . Therefore, (3.7) is proved. Consequently, we have the result.  $\square$

**Lemma 3.6** (cf. Lemma 4.4 in [9]) *Let  $g$  be a real valued function on  $\mathbb{R}$  satisfying  $\|gw\|_{L_\infty(\mathbb{R})} < \infty$  and, for some  $n \geq 1$ ,*

$$\int_{-\infty}^{\infty} gPw^2 dt = 0, \quad P \in \mathcal{P}_n. \tag{3.8}$$

Then we have

$$\left\| w(x) \int_0^x g(t) dt \right\|_{L_\infty(\mathbb{R})} \leq C \frac{a_n}{n} \|gw\|_{L_\infty(\mathbb{R})}. \tag{3.9}$$

*Epecially, if  $w \in \mathcal{F}_\lambda(C^3+)$ ,  $0 < \lambda < 3/2$  and  $T^{1/4}wf' \in L_\infty(\mathbb{R})$ , then we have*

$$\left\| w(x) \int_0^x (f'(t) - v_n(f')(t)) dt \right\|_{L_\infty(\mathbb{R})} \leq C \frac{a_n}{n} E_n(w_{1/4}, f'). \tag{3.10}$$

When  $w \in \mathcal{F}^*$ , we also have (3.10) replacing  $w_{1/4}$  with  $cw$ .

*Proof* We let

$$\phi_x(t) = \begin{cases} w^{-2}(t), & 0 \leq t \leq x; \\ 0, & \text{otherwise,} \end{cases} \tag{3.11}$$

then we have, for arbitrary  $P_n \in \mathcal{P}_n$ ,

$$\begin{aligned} \left| \int_0^x g(t) dt \right| &= \left| \int_{-\infty}^{\infty} g(t)\phi_x(t)w^2(t) dt \right| \\ &= \left| \int_{-\infty}^{\infty} g(t)(\phi_x(t) - P_n(t))w^2(t) dt \right|. \end{aligned} \tag{3.12}$$

Therefore, we have

$$\begin{aligned} \left| \int_0^x g(t) dt \right| &\leq \|gw\|_{L_\infty(\mathbb{R})} \inf_{P_n \in \mathcal{P}_n} \int_{-\infty}^\infty |\phi_x(t) - P_n(t)| w(t) dt \\ &= \|gw\|_{L_\infty(\mathbb{R})} E_{1,n}(w, \phi_x). \end{aligned}$$

Here, from Theorem 3.4 we see that

$$\begin{aligned} E_{1,n}(w, \phi_x) &\leq C \frac{a_n}{n} \int_{-\infty}^\infty w(t) |d\phi_x(t)| \\ &\leq C \frac{a_n}{n} \int_0^x w(t) |Q'(t)| w^{-2}(t) dt \\ &= C \frac{a_n}{n} \int_0^x Q'(t) w^{-1}(t) dt \\ &\leq C \frac{a_n}{n} w^{-1}(x). \end{aligned}$$

So, we have

$$\begin{aligned} \left| w(x) \int_0^x g(t) dt \right| &\leq \|gw\|_{L_\infty(\mathbb{R})} w(x) E_{1,n}(w, \phi_x) \\ &\leq C \frac{a_n}{n} \|gw\|_{L_\infty(\mathbb{R})}. \end{aligned}$$

Therefore, we have (3.9). Next we show (3.10). Since

$$v_n(f')(t) = \frac{1}{n} \sum_{j=n+1}^{2n} s_j(f', t),$$

and, for any  $P \in \mathcal{P}_n, j \geq n + 1$ ,

$$\int_{-\infty}^\infty (f'(t) - s_j(f'; t)) P(t) w^2(t) dt = 0,$$

we have

$$\int_{-\infty}^\infty (f'(t) - v_n(f')(t)) P(t) w^2(t) dt = 0. \tag{3.13}$$

Using (3.9) and (3.1), we have (3.10). □

**Lemma 3.7** *Let  $w = \exp(-Q) \in \mathcal{F}_\lambda(C^3+), 0 < \lambda < 3/2$ . Let  $\|w_{1/4} f'\|_{L_\infty(\mathbb{R})} < \infty$ , and let  $q_{n-1} \in \mathcal{P}_{n-1} (n \geq 1)$  be the best approximation of  $f'$  with respect to the weight  $w$ , that is,*

$$\|(f' - q_{n-1})w\|_{L_\infty(\mathbb{R})} = E_{n-1}(w, f').$$

Now we set

$$F(x) := f(x) - \int_0^x q_{n-1}(t) dt,$$

then there exists  $S_{2n} \in \mathcal{P}_{2n}$  such that

$$\|w(F - S_{2n})\|_{L_\infty(\mathbb{R})} \leq C \frac{a_n}{n} E_n(w_{1/4}, f')$$

and

$$\|wS'_{2n}\|_{L_\infty(\mathbb{R})} \leq CE_{n-1}(w_{1/4}, f').$$

When  $w \in \mathcal{F}^*$ , we have the same results replacing  $w_{1/4}$  with  $cw$ .

*Proof* Let

$$S_{2n}(x) = f(0) + \int_0^x v_n(f' - q_{n-1})(t) dt, \tag{3.14}$$

then, by Lemma 3.6 and (3.10),

$$\begin{aligned} & \|w(F - S_{2n})\|_{L_\infty(\mathbb{R})} \\ &= \left\| w \left( f - \int_0^x q_{n-1}(t) dt - f(0) - \int_0^x v_n(f' - q_{n-1})(t) dt \right) \right\|_{L_\infty(\mathbb{R})} \\ &= \left\| w \left( \int_0^x [f'(t) - v_n(f')(t)] dt \right) \right\|_{L_\infty(\mathbb{R})} \leq C \frac{a_n}{n} E_n(w_{1/4}, f'). \end{aligned}$$

Now by Theorem 3.1, (3.1),

$$\begin{aligned} \|wS'_{2n}\|_{L_\infty(\mathbb{R})} &= \|w(v_n(f' - q_{n-1}))\|_{L_\infty(\mathbb{R})} \\ &\leq \|(f' - v_n(f'))w\|_{L_\infty(\mathbb{R})} + \|(f' - q_{n-1})w\|_{L_\infty(\mathbb{R})} \\ &\leq E_n(w_{1/4}, f') + E_{n-1}(w, f') \leq 2E_{n-1}(w_{1/4}, f'). \end{aligned} \quad \square$$

To prove Theorem 2.3 we need the following theorems with  $p = \infty$ .

**Theorem 3.8** (Corollary 3.4 in [6]) *Let  $w \in \mathcal{F}(C^{2+})$ , and let  $r \geq 0$  be an integer. Let  $1 \leq p \leq \infty$ , and let  $wf^{(r)} \in L_p(\mathbb{R})$ . Then we have, for  $n \geq r$ ,*

$$E_{p,n}(f, w) \leq C \left(\frac{a_n}{n}\right)^k \|f^{(k)}w\|_{L_p(\mathbb{R})}, \quad k = 1, 2, \dots, r,$$

and equivalently,

$$E_{p,n}(w, f) \leq C \left(\frac{a_n}{n}\right)^k E_{p,n-k}(w, f^{(k)}).$$

**Theorem 3.9** (Corollary 6.2 in [4]) *Let  $r \geq 1$  be an integer and  $w \in \mathcal{F}_\lambda(C^{r+2+})$ ,  $0 < \lambda < (r + 2)/(r + 1)$ , and let  $1 \leq p \leq \infty$ . Then there exists a constant  $C > 0$  such that, for any  $1 \leq k \leq r$ , any integer  $n \geq 1$ , and any polynomial  $P \in \mathcal{P}_n$ ,*

$$\|P^{(k)}w\|_{L_p(\mathbb{R})} \leq C \left(\frac{n}{a_n}\right)^k \|T^{k/2}Pw\|_{L_p(\mathbb{R})}.$$

*Proof of Theorem 2.3* We show that for  $k = 0, 1, \dots, r$ ,

$$|f^{(k)}(x) - P_{n,f,w}^{(k)}(x)| \leq CT^{k/2}(x)E_{n-k}(w_{1/4}, f^{(k)}). \tag{3.15}$$

If  $r = 0$ , then (3.15) is trivial. For some  $r \geq 0$  we suppose that (3.15) holds, and let  $f \in C^{(r+1)}(\mathbb{R})$  be satisfying

$$\lim_{|x| \rightarrow \infty} T^{1/4}(x)f^{(r+1)}(x)w(x) = 0.$$

Then  $f' \in C^{(r)}(\mathbb{R})$ , and

$$\lim_{|x| \rightarrow \infty} T^{1/4}(x)(f')^{(r)}(x)w(x) = 0.$$

So we may apply the induction assumption to  $f'$ , for  $0 \leq k \leq r$ . Let  $q_{n-1} \in \mathcal{P}_{n-1}$  be the polynomial of best approximation of  $f'$  with respect to the weight  $w$ . Then from our assumption we have, for  $0 \leq k \leq r$ ,

$$|f^{(k+1)}(x) - q_{n-1}^{(k)}(x)| \leq CT^{k/2}(x)E_{n-1-k}(w_{1/4}, f^{(k+1)}),$$

that is, for  $1 \leq k \leq r + 1$ ,

$$|f^{(k)}(x) - q_{n-1}^{(k-1)}(x)| \leq CT^{\frac{k-1}{2}}(x)E_{n-k}(w_{1/4}, f^{(k)}). \tag{3.16}$$

Let

$$F(x) := f(x) - \int_0^x q_{n-1}(t) dt = f(x) - Q_n(x), \tag{3.17}$$

then

$$|F'(x)w(x)| \leq CE_{n-1}(w, f').$$

As (3.14) we set  $S_{2n} = \int_0^x (v_n(f')(t) - q_{n-1}(t)) dt + f(0)$ , then from Lemma 3.7

$$\|(F - S_{2n})w\|_{L_\infty(\mathbb{R})} \leq C \frac{a_n}{n} E_n(w_{1/4}, f') \tag{3.18}$$

and

$$\|S'_{2n}w\|_{L_\infty(\mathbb{R})} \leq CE_{n-1}(w_{1/4}, f').$$

Here we apply Theorem 3.9 with the weight  $w_{-(k-1)/2}$ . In fact, by Theorem 2.2 we have  $w_{-(k-1)/2} \in \mathcal{F}_\lambda(C^{r+2}, +)$ . Then, noting  $a_{2n} \sim a_n$  from Lemma 3.2(1), we see

$$\begin{aligned} |S_{2n}^{(k)}(x)w_{-(k-1)/2}(x)| &\leq C \left(\frac{n}{a_n}\right)^{k-1} \|S'_{2n}w\|_{L_\infty(\mathbb{R})} \\ &\leq C \left(\frac{n}{a_n}\right)^{k-1} E_{n-1}(w_{1/4}, f'), \end{aligned}$$

that is,

$$|S_{2n}^{(k)}(x)w(x)| \leq C \left( \frac{n\sqrt{T(x)}}{a_n} \right)^{k-1} E_{n-1}(w_{1/4}, f'), \quad 1 \leq k \leq r + 1. \tag{3.19}$$

Let  $R_n \in \mathcal{P}_n$  denote the polynomial of best approximation of  $F$  with  $w$ . By Theorem 3.9 with  $w_{-\frac{k}{2}}$  again, for  $0 \leq k \leq r + 1$ , we have

$$\begin{aligned} |(R_n^{(k)} - S_{2n}^{(k)}(x))w_{-\frac{k}{2}}(x)| &\leq C \left( \frac{n}{a_n} \right)^k \|(R_n - S_{2n})w_{-\frac{k}{2}}(x)T^{k/2}(x)\|_{L_\infty(\mathbb{R})} \\ &\leq C \left( \frac{n}{a_n} \right)^k \|(R_n - S_{2n})w\|_{L_\infty(\mathbb{R})} \end{aligned} \tag{3.20}$$

and by (3.18)

$$\begin{aligned} \|(R_n - S_{2n})w\|_{L_\infty(\mathbb{R})} &\leq C [\|(F - R_n)w\|_{L_\infty(\mathbb{R})} + \|(F - S_{2n})w\|_{L_\infty(\mathbb{R})}] \\ &\leq C \left[ E_n(w, F) + \frac{a_n}{n} E_n(w_{1/4}, f') \right] \\ &\leq C \left[ \frac{a_n}{n} E_{n-1}(w, f') + \frac{a_n}{n} E_{n-1}(w_{1/4}, f') \right] \\ &\leq C \frac{a_n}{n} E_{n-1}(w_{1/4}, f'). \end{aligned} \tag{3.21}$$

Hence, from (3.20) and (3.21) we have, for  $0 \leq k \leq r + 1$ ,

$$\begin{aligned} |(R_n^{(k)} - S_{2n}^{(k)}(x))w(x)| &\leq C |T^{k/2}(x)| |(R_n^{(k)} - S_{2n}^{(k)}(x))w_{-\frac{k}{2}}(x)| \\ &\leq C \left( \frac{n\sqrt{T(x)}}{a_n} \right)^k \frac{a_n}{n} E_{n-1}(w_{1/4}, f'). \end{aligned} \tag{3.22}$$

Therefore by (3.19), (3.22), and Theorem 3.8,

$$\begin{aligned} |R_n^{(k)}(x)w(x)| &\leq C T^{k/2}(x) \left( \frac{n}{a_n} \right)^{k-1} E_{n-1}(w_{1/4}, f') \\ &\leq C T^{k/2}(x) E_{n-k}(w_{1/4}, f^{(k)}). \end{aligned} \tag{3.23}$$

Since  $E_n(w, F) = E_n(w, f)$  and

$$E_n(w, F) = \|w(F - R_n)\|_{L_\infty(\mathbb{R})} = \|w(f - Q_n - R_n)\|_{L_\infty(\mathbb{R})} \tag{3.24}$$

(see (3.17)), we know that  $P_{n;f,w} := Q_n + R_n$  is the polynomial of best approximation of  $f$  with  $w$ . Now, from (3.16), (3.17), and (3.23) we have, for  $1 \leq k \leq r + 1$ ,

$$\begin{aligned} |(f^{(k)}(x) - P_{n;f,w}^{(k)}(x))w(x)| &= |(f^{(k)}(x) - Q_n^{(k)}(x) - R_n^{(k)}(x))w(x)| \\ &\leq |(f^{(k)}(x) - q_{n-1}^{(k-1)}(x))w(x)| + |R_n^{(k)}(x)w(x)| \\ &\leq C T^{k/2}(x) E_{n-k}(w_{1/4}, f^{(k)}). \end{aligned}$$

For  $k = 0$  it is trivial. Consequently, we have (3.15) for all  $r \geq 0$ . Moreover, using Theorem 3.8, we conclude Theorem 2.3. □

*Proof of Corollary 2.4* It follows from Theorem 2.3. □

*Proof of Corollary 2.5* Applying Theorem 2.3 with  $w_{k/2}$ , we have, for  $0 \leq j \leq r$ ,

$$\| (f^{(j)} - P_{n;f,w_{k/2}}^{(j)}) w \|_{L_\infty(\mathbb{R})} \leq CE_{n-k}(w_{(2k+1)/4}, f^{(j)}).$$

Especially, when  $j = k$ , we obtain

$$\| (f^{(k)} - P_{n;f,w_{k/2}}^{(k)}) w \|_{L_\infty(\mathbb{R})} \leq CE_{n-k}(w_{(2k+1)/4}, f^{(k)}). \quad \square$$

**Competing interests**

The authors declare that they have no competing interests.

**Authors' contributions**

All authors conceived of the study, participated in its design and coordination, drafted the manuscript and participated in the sequence alignment. All authors read and approved the final manuscript.

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