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A Hilbert-type operator with a symmetric homogeneous kernel of two parameters and its applications

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Abstract
We introduce a general homogeneous kernel whose degree is given by two parameters to establish the equivalent inequalities with the norm of a new Hilbert-type operator. As applications, we provide new extended Hilbert-type inequalities with the best possible constant factors.

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1 Introduction

Let $\{a_n\}$ and $\{b_m\}$ be two sequences of nonnegative real numbers. The well-known Hilbert's inequality says that if $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \sum_{m=1}^{\infty} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\frac{\pi}{p})}\left(\sum_{m=1}^{\infty} a_m^p\right)^{1/p}\left(\sum_{n=1}^{\infty} b_n^q\right)^{1/q}, \tag{1}$$

where the constant factor $\frac{\pi}{\sin(\frac{\pi}{p})}$ is the best possible [1]. This inequality has been generalized in numerous ways with introducing suitable parameters and weight coefficients. (For example, see [2–13] and the references therein.) In particular, by introducing a Hilbert-type linear operator with a symmetric homogeneous kernel, one can obtain various Hilbert-type inequalities with the best constant factors. For this purpose, let $k(x,y)$ be a nonnegative symmetric function defined on $(0,\infty) \times (0,\infty)$, *i.e.*, $k(x,y) = k(y,x)$. For $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, let ℓ^r $(r = p,q)$ be two normed spaces. If T is a bounded self-adjoint semi-positive definite operator defined by

$$(Ta)(n) := \sum_{m=1}^{\infty} k(m,n)a_m, \quad n \in \mathbb{N}$$

for $a = \{a_m\}_{m=1}^{\infty} \in \ell^p$, or similarly,

$$(Tb)(m) := \sum_{n=1}^{\infty} k(m,n)b_n, \quad m \in \mathbb{N}$$

for $b = \{b_n\}_{n=1}^{\infty} \in \ell^q$. The operator T is called the *Hilbert-type operator* and the function $k(x,y)$ is called the *symmetric kernel* of T. In view of this point, Hilbert's inequality (1) can be expressed by

$$(Ta, b) \le \frac{\pi}{\sin(\frac{\pi}{p})} \|a\|_p \|b\|_q,$$

where the kernel $k(x,y) = \frac{1}{x+y}$ and the formal inner product (Ta,b) between Ta and b is given by $(Ta,b) := \sum_{n=1}^{\infty}(Ta)(n)b_n$. Motivated by this observation, Yang [14] defined a Hilbert-type linear operator $T : \ell^r \to \ell^r$ $(r = p,q)$ with the kernel $k(x,y) = \frac{(xy)^{\frac{\lambda-1}{2}}}{(x+y)^{\lambda}}$ of degree -1. As a consequence, he was able to prove that if $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_m, b_n \ge 0$, $1 - 2\min\{\frac{1}{p}, \frac{1}{q}\} < \lambda < 1 + 2\min\{\frac{1}{p}, \frac{1}{q}\}$, then the following two inequalities are equivalent:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{(mn)^{\frac{\lambda-1}{2}} a_m b_n}{m^{\lambda} + n^{\lambda}} < \frac{1}{\lambda} B\left(\frac{q(\lambda+1)-2}{2q\lambda}, \frac{p(\lambda+1)-2}{2p\lambda}\right) \|a\|_p \|b\|_q,$$

$$\left\{\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} \frac{(mn)^{\frac{\lambda-1}{2}} a_m}{m^{\lambda} + n^{\lambda}}\right)^p\right\}^{\frac{1}{p}} < \frac{1}{\lambda} B\left(\frac{q(\lambda+1)-2}{2q\lambda}, \frac{p(\lambda+1)-2}{2p\lambda}\right) \|a\|_p,$$

where $B(u,v)$ denotes the beta function defined by

$$B(u,v) := \int_0^{\infty} \frac{t^{u-1}}{(1+t)^{u+v}}\, dt = B(u,v) \quad (u, v > 0).$$

Moreover, the constant factor $\frac{1}{\lambda}B(\frac{q(\lambda+1)-2}{2q\lambda}, \frac{p(\lambda+1)-2}{2p\lambda})$ is the best possible. In 2010, Jin and Debnath [15] generalized the Hilbert-type linear operator whose kernel is symmetric and homogeneous of degree -1. In fact, they obtained several extended Hilbert-type inequalities by using the kernel $k(x,y) = \frac{1}{(x^{\frac{1}{\lambda}} + y^{\frac{1}{\lambda}})^{\lambda}}$ $(\lambda > 0)$. For instance, they proved that if $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha, \beta > 0$, $0 < \lambda \le \min\{\frac{q}{\alpha}, \frac{p}{\beta}\}$, then the following two inequalities are equivalent:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{(m^{\alpha} + n^{\beta})^{\lambda}} < \frac{B(\frac{\lambda}{p}, \frac{\lambda}{q})}{\alpha^{\frac{1}{q}} \beta^{\frac{1}{p}}} \left(\sum_{m=1}^{\infty} m^{(p-1)(1-\alpha\lambda)} |a_m|^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} n^{(q-1)(1-\beta\lambda)} |b_n|^q\right)^{\frac{1}{q}},$$

$$\left\{\sum_{n=1}^{\infty} n^{\beta\lambda-1}\left(\sum_{m=1}^{\infty} \frac{a_m}{(m^{\alpha} + n^{\beta})^{\lambda}}\right)^p\right\}^{\frac{1}{p}} < \frac{B(\frac{\lambda}{p}, \frac{\lambda}{q})}{\alpha^{\frac{1}{q}} \beta^{\frac{1}{p}}} \left(\sum_{m=1}^{\infty} m^{(p-1)(1-\alpha\lambda)} |a_m|^p\right)^{\frac{1}{p}},$$

where the constant factor $\frac{B(\frac{\lambda}{p}, \frac{\lambda}{q})}{\alpha^{\frac{1}{q}} \beta^{\frac{1}{p}}}$ is the best possible. See [16–23] for other Hilbert-type operators and the corresponding extended Hilbert-type inequalities with the best factors.

Most of the previous results were, however, obtained by using the Hilbert-type operator with the symmetric homogeneous kernel of $-\lambda$-order, which depends on a parameter $\lambda > 0$. In this paper, we introduce a more general homogeneous kernel whose degree is given by two parameters (Definition 2.3). We establish the equivalent inequalities with the norm of a new Hilbert-type operator (Theorem 3.1). As applications, we provide new extended Hilbert-type inequalities with the best possible constant factors (Corollary 4.1 and Cases 1-3).

2 Hilbert-type operator with a symmetric homogeneous kernel whose degree is given by two parameters

For completeness, we begin with the following definitions and notations.

Definition 2.1 Let $p > 1$, $n_0 \in \mathbb{Z}$, $w(n) \geq 0$ ($n \geq n_0$, $n \in \mathbb{Z}$). Define the normed space ℓ^p_{w,n_0} by

$$\ell^p_{w,n_0} := \left\{ a = \{a_n\}_{n=n_0}^{\infty} : \|a\|_{p,w} := \left(\sum_{n=n_0}^{\infty} w(n)|a_n|^p \right)^{1/p} < \infty \right\}.$$

Definition 2.2 Let $\lambda_1, \lambda_2, \lambda > 0$ satisfying that $\lambda = \lambda_1 + \lambda_2$. Denote by $F_{n_0}(r)$ ($n_0 \in \mathbb{Z}$) the set of all real-valued C^1-functions $\phi(x)$ satisfying the following conditions:

(1) $\phi(x)$ is strictly increasing in $(n_0 - 1, \infty)$ with $\phi((n_0 - 1)+) = 0$, $\phi(\infty) = \infty$.
(2) For $\alpha > 0$, $\frac{\phi'(x)}{\phi(x)^{\alpha+1-\lambda_i}}$ is decreasing in $(n_0 - 1, \infty)$.

Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda = \lambda_1 + \lambda_2$, $\lambda_1, \lambda_2, \lambda > 0$. For $\phi(x) \in F_{m_0}(r)$ and $\psi(y) \in F_{n_9}(s)$, $r, s > 1$, we define the following weight functions:

$$w_1(m) := \frac{\phi(m)^{p(\alpha+1-\lambda_2)-1}}{\phi'(m)^{p-1}}, \qquad w_2(n) := \frac{\psi(n)^{q(\alpha+1-\lambda_1)-1}}{\psi'(n)^{q-1}},$$
$$\widetilde{w}_1(n) := \frac{\psi'(n)}{\psi(n)^{p(\alpha-\lambda_1)+1}}, \qquad \widetilde{w}_2(m) := \frac{\phi'(m)}{\phi(m)^{q(\alpha-\lambda_2)+1}}.$$

Definition 2.3 Let $\lambda_1, \lambda_2, \lambda > 0$ satisfying that $\lambda = \lambda_1 + \lambda_2$. For $\alpha > 0$ and $x, y > 0$, $K_{\alpha,\lambda}(x, y)$ is a continuous real-valued function on $(0, \infty) \times (0, \infty)$ satisfying the following properties:

(1) $K_{\alpha,\lambda}(x, y)$ is a symmetric homogeneous function of degree $2\alpha - \lambda$, that is,

$$K_{\alpha,\lambda}(x, y) = K_{\alpha,\lambda}(y, x),$$
$$K_{\alpha,\lambda}(tx, ty) = t^{2\alpha-\lambda} K_{\alpha,\lambda}(x, y) \quad \text{for any } t > 0.$$

(2) $K_{\alpha,\lambda}(x, y)$ is decreasing with respect to x and y, respectively.
(3) For sufficiently small $\varepsilon \geq 0$, the following integral

$$\widetilde{K}_{\alpha,\lambda}(\lambda_i, \varepsilon) := \int_0^{\infty} K_{\alpha,\lambda}(1, t) t^{-1+\lambda_i-\alpha-\varepsilon} \, dt$$

exists for $i = 1, 2$. Moreover, assume that $\widetilde{K}_{\alpha,\lambda}(\lambda_i, 0) := K_\alpha(\lambda_i) > 0$ and $\widetilde{K}_{\alpha,\lambda}(\lambda_i, \varepsilon) = K_\alpha(\lambda_i) + o(1)$ as $\varepsilon \to 0+$.

(4) Given $p > 1$, $\phi(x) \in F_{m_0}(r)$, and $\psi(y) \in F_{n_0}(s)$ $(r, s > 1)$,

$$\sum_{n=n_0}^{\infty} \frac{\psi'(n)}{\psi(n)^{1+\varepsilon}} \int_0^{\frac{\phi(m_0)}{\psi(n)}} K_{\alpha,\lambda}(1, t) t^{-1+\lambda_i-\alpha-\frac{\varepsilon}{p}} \, dt = O(1)$$

as $\varepsilon \to 0+$.

Lemma 2.4 *Let $\lambda_1, \lambda_2, \lambda > 0$ satisfying that $\lambda = \lambda_1 + \lambda_2$. For any $\alpha > 0$, we have*

$$K_\alpha(\lambda_1) = K_\alpha(\lambda_2).$$

Proof Since

$$K_\alpha(\lambda_1) = \widetilde{K}_{\alpha,\lambda}(\lambda_1, 0) = \int_0^\infty K_{\alpha,\lambda}(1,t)t^{-1+\lambda_1-\alpha}\,dt,$$

letting $t = \frac{1}{s}$ gives

$$K_\alpha(\lambda_1) = \int_0^\infty K_{\alpha,\lambda}(1,s)s^{-1+\lambda_2-\alpha}\,ds = K_\alpha(\lambda_2). \qquad \square$$

In view of Lemma 2.4, we may assume that

$$K_\alpha(\lambda) := K_\alpha(\lambda_1) = K_\alpha(\lambda_2).$$

Lemma 2.5 *Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda_1 + \lambda_2 = \lambda$, $\lambda_1, \lambda_2 > 0$, $\alpha > 0$. For $\phi(x) \in F_{m_0}(r)$ and $\psi(y) \in F_{n_0}(s)$, $r, s > 1$, define the weight coefficients $W_1(m)$ and $W_2(n)$ by*

$$W_1(m) := \sum_{n=n_0}^\infty K_{\alpha,\lambda}\big(\phi(m), \psi(n)\big)\frac{\phi(m)^{\lambda_2-\alpha}}{\psi(n)^{\alpha+1-\lambda_1}}\psi'(n),$$
$$W_2(n) := \sum_{m=m_0}^\infty K_{\alpha,\lambda}\big(\phi(m), \psi(n)\big)\frac{\psi(n)^{\lambda_1-\alpha}}{\phi(m)^{\alpha+1-\lambda_2}}\phi'(m).$$

Then

$$W_1(m) < K_\alpha(\lambda) \quad \text{and} \quad W_2(n) < K_\alpha(\lambda)$$

for any $m \geq m_0$, $n \geq n_0$ $(m, n \in \mathbb{Z})$.

Proof We have

$$W_1(m) = \sum_{n=n_0}^\infty K_{\alpha,\lambda}\left(1, \frac{\psi(n)}{\phi(m)}\right)\frac{\phi(m)^{\alpha-\lambda_1}}{\psi(n)^{\alpha+1-\lambda_1}}\psi'(n)$$
$$< \int_{n_0-1}^\infty K_{\alpha,\lambda}\left(1, \frac{\psi(x)}{\phi(m)}\right)\frac{\psi'(x)}{\psi(x)^{\alpha+1-\lambda_1}}\phi(m)^{\alpha-\lambda_1}\,dx.$$

Setting $t = \frac{\psi(x)}{\phi(m)}$, we get

$$W_1(m) < \int_0^\infty K_{\alpha,\lambda}(1,t)t^{-1+\lambda_1-\alpha}\,dt = K_\alpha(\lambda).$$

Similarly, one can obtain $W_2(n) < K_\alpha(\lambda)$. $\square$

Lemma 2.6 *Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda_1 + \lambda_2 = \lambda$, $\lambda_1, \lambda_2 > 0$. For $a_m, b_n \geq 0$ $(m_0, n_0 \in \mathbb{Z})$, let $a = \{a_m\}_{m=m_0}^\infty \in \ell^p_{w_1,m_0}$ and $b = \{b_n\}_{n=n_0}^\infty \in \ell^q_{w_2,n_0}$. Then, for $\phi(x) \in F_{m_0}(r)$ and $\psi(y) \in F_{n_0}(s)$*

$(r, s > 1)$, *we have*

$$\left\| \sum_{m=m_0}^{\infty} K_{\alpha,\lambda}\big(\phi(m), \psi(n)\big) a_m \right\|_{p,\widetilde{w}_1} \leq K_\alpha(\lambda) \|a\|_{p,w_1} \quad \textit{and}$$

$$\left\| \sum_{n=n_0}^{\infty} K_{\alpha,\lambda}\big(\phi(m), \psi(n)\big) b_n \right\|_{q,\widetilde{w}_2} \leq K_\alpha(\lambda) \|b\|_{q,w_2},$$

and hence

$$\left\{ \sum_{m=m_0}^{\infty} K_{\alpha,\lambda}\big(\phi(m), \psi(n)\big) a_m \right\}_{n=n_0}^{\infty} \in \ell^p_{\tilde{w}_1,n_0} \quad \textit{and}$$

$$\left\{ \sum_{n=n_0}^{\infty} K_{\alpha,\lambda}\big(\phi(m), \psi(n)\big) b_n \right\}_{m=m_0}^{\infty} \in \ell^q_{\tilde{w}_2,m_0}.$$

Proof Applying Hölder's inequality, we observe

$$\begin{aligned}
&\sum_{m=m_0}^{\infty} K_{\alpha,\lambda}\big(\phi(m), \psi(n)\big) a_m \\
&\quad = \sum_{m=m_0}^{\infty} \left(K_{\alpha,\lambda}\big(\phi(m), \psi(n)\big) \frac{\phi(m)^{\frac{\alpha+1-\lambda_2}{q}}}{\psi(n)^{\frac{\alpha+1-\lambda_1}{p}}} \frac{\psi'(n)^{\frac{1}{p}}}{\phi'(m)^{\frac{1}{q}}} a_m \right) \left(\frac{\psi(n)^{\frac{\alpha+1-\lambda_1}{p}}}{\phi(m)^{\frac{\alpha+1-\lambda_2}{q}}} \frac{\phi'(m)^{\frac{1}{q}}}{\psi'(n)^{\frac{1}{p}}} \right) \\
&\quad \leq \left(\sum_{m=m_0}^{\infty} K_{\alpha,\lambda}\big(\phi(m), \psi(n)\big) \frac{\phi(m)^{(\alpha+1-\lambda_2)(p-1)}}{\psi(n)^{\alpha+1-\lambda_1}} \frac{\psi'(n)}{\phi'(m)^{p-1}} a_m^p \right)^{\frac{1}{p}} \\
&\qquad \times \left(\sum_{m=m_0}^{\infty} K_{\alpha,\lambda}\big(\phi(m), \psi(n)\big) \frac{\psi(n)^{(\alpha+1-\lambda_1)(q-1)}}{\phi(m)^{\alpha+1-\lambda_2}} \frac{\phi'(m)}{\psi'(n)^{q-1}} \right)^{\frac{1}{q}}.
\end{aligned}$$

By Definition 2.2, we get

$$\begin{aligned}
&\sum_{m=m_0}^{\infty} K_{\alpha,\lambda}\big(\phi(m), \psi(n)\big) a_m \\
&\quad \leq \left(\int_0^{\infty} K_{\alpha,\lambda}(1,t) t^{-1+\lambda_2-\alpha}\, dt \right)^{\frac{1}{q}} \left(\frac{\psi(n)^{q(\alpha+1-\lambda_1)-1}}{\psi'(n)^{q-1}} \right)^{\frac{1}{q}} \\
&\qquad \times \left(\sum_{m=m_0}^{\infty} K_{\alpha,\lambda}\big(\phi(m), \psi(n)\big) \frac{\phi(m)^{(\alpha+1-\lambda_2)(p-1)}}{\psi(n)^{\alpha+1-\lambda_1}} \frac{\psi'(n)}{\phi'(m)^{p-1}} a_m^p \right)^{\frac{1}{p}}.
\end{aligned}$$

Therefore, by using Lemma 2.5, we get

$$\begin{aligned}
&\left\| \sum_{m=m_0}^{\infty} K_{\alpha,\lambda}\big(\phi(m), \psi(n)\big) a_m \right\|_{p,\widetilde{w}_1} \\
&\quad = \left\{ \sum_{n=n_0}^{\infty} \frac{\psi'(n)}{\psi(n)^{p(\alpha-\lambda_1)+1}} \left(\sum_{m=m_0}^{\infty} K_{\alpha,\lambda}\big(\phi(m), \psi(n)\big) a_m \right)^p \right\}^{\frac{1}{p}}
\end{aligned}$$

$$\leq K_\alpha(\lambda)^{\frac{1}{q}}\left(\sum_{n=n_0}^{\infty}\sum_{m=m_0}^{\infty}K_{\alpha,\lambda}\big(\phi(m),\psi(n)\big)\frac{\phi(m)^{(\alpha+1-\lambda_2)(p-1)}}{\psi(n)^{\alpha+1-\lambda_1}}\frac{\psi'(n)}{\phi'(m)^{p-1}}a_m^p\right)^{\frac{1}{p}}$$

$$= K_\alpha(\lambda)^{\frac{1}{q}}\left(\sum_{m=m_0}^{\infty}W_1(m)\frac{\phi(m)^{p(\alpha+1-\lambda_2)-1}}{\phi'(m)^{p-1}}a_m^p\right)^{\frac{1}{p}}$$

$$< K_\alpha(\lambda)\|a\|_{p,w_1}.$$

In the same manner, one can obtain

$$\left\|\sum_{n=n_0}^{\infty}K_{\alpha,\lambda}\big(\phi(m),\psi(n)\big)b_n\right\|_{q,\widetilde{w}_2}\leq K_\alpha(\lambda)\|b\|_{q,w_2}.$$ □

In view of Lemma 2.6, we can define a Hilbert-type operator $T:\ell^p_{w_1,m_0}\to\ell^p_{\widetilde{w}_1,n_0}$ by

$$(Ta)(n):=\sum_{m=m_0}^{\infty}K_{\alpha,\lambda}\big(\phi(m),\psi(n)\big)a_m,\quad n\geq n_0, n\in\mathbb{Z}.$$

Similarly, define $T:\ell^q_{w_2,n_0}\to\ell^q_{\widetilde{w}_2,m_0}$ by

$$(Ta)(m):=\sum_{n=n_0}^{\infty}K_{\alpha,\lambda}\big(\phi(m),\psi(n)\big)b_n,\quad m\geq m_0, m\in\mathbb{Z}.$$

It immediately follows from Lemma 2.6 that

$$\|T\|_p:=\sup_{\|a\|_{p,\widetilde{w}_1}=1}\|Ta\|_{p,\widetilde{w}_1}\leq K_\alpha(\lambda)$$

and

$$\|T\|_q:=\sup_{\|a\|_{p,\widetilde{w}_2}=1}\|Tb\|_{q,\widetilde{w}_2}\leq K_\alpha(\lambda).$$

Hence the operator T is bounded. The formal inner product (Ta,b) of Ta and b is defined by

$$(Ta,b):=\sum_{n=n_0}^{\infty}\sum_{m=m_0}^{\infty}K_{\alpha,\lambda}\big(\phi(m),\psi(n)\big)a_m b_n.$$

Lemma 2.7 *Let* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$. *Let* $\widetilde{a}=\{\widetilde{a}_m\}_{m=m_0}^{\infty}$ *and* $\widetilde{b}=\{\widetilde{b}_n\}_{n=n_0}^{\infty}$ *with* $\widetilde{a}_m=\frac{\phi'(m)}{\phi(m)^{\alpha+1-\lambda_2+\frac{\varepsilon}{p}}}$ *and* $\widetilde{b}_n=\frac{\psi'(n)}{\psi(n)^{\alpha+1-\lambda_1+\frac{\varepsilon}{q}}}$ *for* $0<\varepsilon<p\lambda_i$, $i=1,2$. *Then, as* $\varepsilon\to 0+$,

$$K_\alpha(\lambda)\big(1-o(1)\big)\sum_{n=n_0}^{\infty}\frac{\psi'(n)}{\psi(n)^{1+\varepsilon}}<(T\widetilde{a},\widetilde{b})<K_\alpha(\lambda)\big(1+o(1)\big)\sum_{n=n_0}^{\infty}\frac{\psi'(n)}{\psi(n)^{1+\varepsilon}}.$$

Proof We have

$$(T\widetilde{a},\widetilde{b}) = \sum_{n=n_0}^{\infty}\sum_{m=m_0}^{\infty} K_{\alpha,\lambda}\bigl(\phi(m),\psi(n)\bigr)\frac{\phi'(m)}{\phi(m)^{\alpha+1-\lambda_2+\frac{\varepsilon}{p}}}\frac{\psi'(n)}{\psi(n)^{\alpha+1-\lambda_1+\frac{\varepsilon}{q}}}$$
$$< \sum_{n=n_0}^{\infty}\int_{m_0-1}^{\infty} K_{\alpha,\lambda}\bigl(\phi(x),\psi(n)\bigr)\frac{\phi'(x)}{\phi(x)^{\alpha+1-\lambda_2+\frac{\varepsilon}{p}}}\frac{\psi'(n)}{\psi(n)^{\alpha+1-\lambda_1+\frac{\varepsilon}{q}}}\,dx.$$

Setting $t = \frac{\phi(x)}{\psi(n)}$, we get

$$(T\widetilde{a},\widetilde{b}) < \sum_{n=n_0}^{\infty}\left(\int_0^{\infty} K_{\alpha,\lambda}(1,t)t^{-1+\lambda_2-\alpha-\frac{\varepsilon}{p}}\,dt\right)\frac{\psi'(n)}{\psi(n)^{1+\varepsilon}}$$
$$= K_\alpha(\lambda)\bigl(1+o(1)\bigr)\sum_{n=n_0}^{\infty}\frac{\psi'(n)}{\psi(n)^{1+\varepsilon}}.$$

Moreover,

$$(T\widetilde{a},\widetilde{b}) > \sum_{n=n_0}^{\infty}\left(\int_{\frac{\phi(m_0)}{\psi(n)}}^{\infty} K_{\alpha,\lambda}(1,t)t^{-1+\lambda_2-\alpha-\frac{\varepsilon}{p}}\,dt\right)\frac{\psi'(n)}{\psi(n)^{1+\varepsilon}}$$
$$= \sum_{n=n_0}^{\infty}\frac{\psi'(n)}{\psi(n)^{1+\varepsilon}}\left(\int_0^{\infty} K_{\alpha,\lambda}(1,t)t^{-1+\lambda_2-\alpha-\frac{\varepsilon}{p}}\,dt - \int_0^{\frac{\phi(m_0)}{\psi(n)}} K_{\alpha,\lambda}(1,t)t^{-1+\lambda_2-\alpha-\frac{\varepsilon}{p}}\,dt\right).$$

Note that the definition of $K_{\alpha,\lambda}(x,y)$ implies that

$$\int_0^{\infty} K_{\alpha,\lambda}(1,t)t^{-1+\lambda_2-\alpha-\frac{\varepsilon}{p}}\,dt = K_\alpha(\lambda_2)+o(1)$$

and

$$\sum_{n=n_0}^{\infty}\frac{\psi'(n)}{\psi(n)^{1+\varepsilon}}\int_0^{\frac{\phi(m_0)}{\psi(n)}} K_{\alpha,\lambda}(1,t)t^{-1+\lambda_2-\alpha-\frac{\varepsilon}{p}}\,dt = O(1).$$

Thus, using the fact that for $a > 0$,

$$\sum_{n=n_0}^{\infty}\frac{\psi'(n)}{\psi(n)^{1+\varepsilon}} = \frac{1}{\varepsilon}\bigl(1+o(1)\bigr) \quad\text{and}\quad \sum_{n=n_0}^{\infty}\frac{\psi'(n)}{\psi(n)^{1+a+\frac{\varepsilon}{q}}} = O(1)$$

as $\varepsilon \to 0+$, we obtain

$$(T\widetilde{a},\widetilde{b}) > K_\alpha(\lambda)\bigl(1+o(1)\bigr)\left(\sum_{n=n_0}^{\infty}\frac{\psi'(n)}{\psi(n)^{1+\varepsilon}} - O(1)\right)$$
$$= K_\alpha(\lambda)\left[1+o(1)-O(1)\sum_{n=n_0}^{\infty}\left(\frac{\psi'(n)}{\psi(n)^{1+\varepsilon}}\right)^{-1}\right]\sum_{n=n_0}^{\infty}\frac{\psi'(n)}{\psi(n)^{1+\varepsilon}}$$
$$= K_\alpha(\lambda)\bigl(1-o(1)\bigr)\sum_{n=n_0}^{\infty}\frac{\psi'(n)}{\psi(n)^{1+\varepsilon}},$$

which completes the proof. □

Theorem 2.8 *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda_1 + \lambda_2 = \lambda$, $\lambda_1, \lambda_2 > 0$. *For* $a_m, b_n \geq 0$ $(m_0, n_0 \in \mathbb{Z})$, *let* $a = \{a_m\}_{m=m_0}^{\infty} \in \ell_{w_1,m_0}^p$ *and* $b = \{b_n\}_{n=n_0}^{\infty} \in \ell_{w_2,n_0}^q$. *Then, for* $\phi(x) \in F_{m_0}(r)$ *and* $\psi(y) \in F_{n_0}(s)$ $(r, s > 1)$,

$$\|T\|_p = \|T\|_q = K_\alpha(\lambda).$$

Proof Suppose that $\|T\|_p < K_\alpha(\lambda)$. Consider $\widetilde{a}_m = \phi'(m)\phi(m)^{-1+\lambda_2-\alpha-\frac{\varepsilon}{p}}$ and $\widetilde{b}_n = \phi'(n) \times \psi(n)^{-1+\lambda_1-\alpha-\frac{\varepsilon}{q}}$, where $m \geq m_0$, $n \geq n_0$, $m, n \in \mathbb{Z}$, $0 < \varepsilon < p\lambda_i$, $i = 1, 2$. A simple computation shows that $\widetilde{a} \in \ell_{w_1,m_0}^p$ and $\widetilde{b} \in \ell_{w_2,n_0}^q$ with $\|\widetilde{a}\|_{p,w_1} > 0$ and $\|\widetilde{b}\|_{q,w_2} > 0$. Then

$$\begin{aligned}\|T\widetilde{a}\|_{p,\widetilde{w}_1} &= \left\{\sum_{n=n_0}^{\infty} \psi'(n)\psi(n)^{p(\lambda_1-\alpha)-1}\left(\sum_{m=m_0}^{\infty} K_{\alpha,\lambda}\big(\phi(m), \psi(n)\big)\widetilde{a}_m\right)^p\right\}^{\frac{1}{p}} \\ &\leq \|T\|_p \|\widetilde{a}\|_{p,w_1}.\end{aligned}$$

Moreover, we have

$$\begin{aligned}(T\widetilde{a}, \widetilde{b}) &= \sum_{n=n_0}^{\infty}\sum_{m=m_0}^{\infty} K_{\alpha,\lambda}\big(\phi(m), \psi(n)\big)\widetilde{a}_m\widetilde{b}_n \\ &= \sum_{n=n_0}^{\infty}\left\{\psi'(n)\psi(n)^{p(\lambda_1-\alpha)-1}\left(\sum_{m=m_0}^{\infty} K_{\alpha,\lambda}\big(\phi(m), \psi(n)\big)\widetilde{a}_m\right)^p\right\}^{\frac{1}{p}} \|\widetilde{b}\|_{q,w_2} \\ &\leq \|T\|_p\|\widetilde{a}\|_{p,w_1}\|\widetilde{b}\|_{q,w_2} \\ &= \|T\|_p\left(\sum_{m=m_0}^{\infty}\frac{\phi'(m)}{\phi(m)^{1+\varepsilon}}\right)^{\frac{1}{p}}\left(\sum_{n=n_0}^{\infty}\frac{\psi'(n)}{\psi(n)^{1+\varepsilon}}\right)^{\frac{1}{q}}.\end{aligned} \tag{2}$$

On the other hand, from Lemma 2.7 it follows

$$K_\alpha(\lambda)\big(1 - o(1)\big)\sum_{n=n_0}^{\infty}\frac{\psi'(n)}{\psi(n)^{1+\varepsilon}} < (T\widetilde{a}, \widetilde{b}). \tag{3}$$

Therefore, combining these inequalities (2) and (3),

$$K_\alpha(\lambda)\big(1 - o(1)\big)\left(\sum_{n=n_0}^{\infty}\frac{\psi'(n)}{\psi(n)^{1+\varepsilon}}\right)^{\frac{1}{p}} \leq \|T\|_p\left(\sum_{m=m_0}^{\infty}\frac{\phi'(m)}{\phi(m)^{1+\varepsilon}}\right)^{\frac{1}{p}}.$$

Since

$$\sum_{n=n_0}^{\infty}\frac{\psi'(n)}{\psi(n)^{1+\varepsilon}} = \frac{1}{\varepsilon}\big(1 + o(1)\big) \quad \text{and} \quad \sum_{m=m_0}^{\infty}\frac{\phi'(m)}{\phi(m)^{1+\varepsilon}} = \frac{1}{\varepsilon}\big(1 + o(1)\big)$$

as $\varepsilon \to 0+$, we obtain that $K_\alpha(\lambda) \leq \|T\|_p$, which is a contradiction. Thus we conclude that $\|T\|_p = K_\alpha(\lambda)$. Applying the same argument, we have $\|T\|_q = K_\alpha(\lambda)$, which completes the proof. □

3 Two equivalent inequalities for the Hilbert-type operator

Equipped with the Hilbert-type operator defined as above, we have the following theorem.

Theorem 3.1 *Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda_1 + \lambda_2 = \lambda$, $\lambda_1, \lambda_2 > 0$. For $a_m, b_n \geq 0$ $(m_0, n_0 \in \mathbb{Z})$, let $a = \{a_m\}_{m=m_0}^{\infty} \in \ell^p_{w_1,m_0}$, $b = \{b_n\}_{n=n_0}^{\infty} \in \ell^q_{w_2,n_0}$ $\|a\|_{p,w_1} > 0$, $\|b\|_{q,w_2} > 0$. Then, for $\phi(x) \in F_{m_0}(r)$ and $\psi(y) \in F_{n_0}(s)$ $(r, s > 1)$, we have the following equivalent inequalities*:

$$(Ta, b) = \sum_{n=n_0}^{\infty} \sum_{m=m_0}^{\infty} K_{\alpha,\lambda}\big(\phi(m), \psi(n)\big) a_m b_n < K_\alpha(\lambda) \|a\|_{p,w_1} \|b\|_{q,w_2}, \tag{4}$$

$$\|Ta\|_{p,\widetilde{w}_1} < K_\alpha(\lambda) \|a\|_{p,w_1}. \tag{5}$$

Furthermore, the constant factor $K_\alpha(\lambda)$ is the best possible.

Proof It follows from Hölder's inequality that

$$\begin{aligned}
(Ta, b) &= \sum_{n=n_0}^{\infty} \sum_{m=m_0}^{\infty} K_{\alpha,\lambda}\big(\phi(m), \psi(n)\big) \left(\frac{\phi(m)^{\frac{\alpha+1-\lambda_2}{q}}}{\psi(n)^{\frac{\alpha+1-\lambda_1}{p}}} \frac{\psi'(n)^{\frac{1}{p}}}{\phi'(m)^{\frac{1}{q}}} a_m \right) \\
&\quad \times \left(\frac{\psi(n)^{\frac{\alpha+1-\lambda_1}{p}}}{\phi(m)^{\frac{\alpha+1-\lambda_2}{q}}} \frac{\phi'(m)^{\frac{1}{q}}}{\psi'(n)^{\frac{1}{p}}} b_n \right) \\
&\leq \left(\sum_{n=n_0}^{\infty} \sum_{m=m_0}^{\infty} K_{\alpha,\lambda}\big(\phi(m), \psi(n)\big) \frac{\phi(m)^{(\alpha+1-\lambda_2)(p-1)}}{\psi(n)^{\alpha+1-\lambda_1}} \frac{\psi'(n)}{\phi'(m)^{p-1}} a_m^p \right)^{\frac{1}{p}} \\
&\quad \times \left(\sum_{n=n_0}^{\infty} \sum_{m=m_0}^{\infty} K_{\alpha,\lambda}\big(\phi(m), \psi(n)\big) \frac{\psi(n)^{(\alpha+1-\lambda_1)(q-1)}}{\phi(m)^{\alpha+1-\lambda_2}} \frac{\phi'(m)}{\psi'(n)^{q-1}} b_n^q \right)^{\frac{1}{q}} \\
&= \left(\sum_{m=m_0}^{\infty} W_1(m) \frac{\phi(m)^{p(\alpha+1-\lambda_2)-1}}{\phi'(m)^{p-1}} a_m^p \right)^{\frac{1}{p}} \left(\sum_{n=n_0}^{\infty} W_2(n) \frac{\psi(n)^{q(\alpha+1-\lambda_1)-1}}{\psi'(n)^{q-1}} b_n^q \right)^{\frac{1}{q}}.
\end{aligned}$$

Applying Lemma 2.5, we see that

$$(Ta, b) < K_\alpha(\lambda) \|a\|_{p,w_1} \|b\|_{q,w_2}.$$

In order to prove that inequality (4) implies inequality (5), we define $\widetilde{b}$ as follows:

$$\widetilde{b}_n := \frac{\psi'(n)}{\psi(n)^{p(\alpha-\lambda_1)+1}} \left(\sum_{m=m_0}^{\infty} K_{\alpha,\lambda}\big(\phi(m), \psi(n)\big) \right)^{p-1}$$

for $n \geq n_0$, $n \in \mathbb{Z}$. Then we see that $\widetilde{b} \in \ell^q_{w_2,n_0}$ and $\|\widetilde{b}\|_{q,w_2} > 0$ as before. Thus using inequality (4) shows that

$$\begin{aligned}
\|\widetilde{b}\|_{q,w_2}^q &= \sum_{n=n_0}^{\infty} \frac{\psi'(n)}{\psi(n)^{p(\alpha-\lambda_1)+1}} \left(\sum_{m=m_0}^{\infty} K_{\alpha,\lambda}\big(\phi(m), \psi(n)\big) a_m \right)^p \\
&= \sum_{n=n_0}^{\infty} \sum_{m=m_0}^{\infty} K_{\alpha,\lambda}\big(\phi(m), \psi(n)\big) a_m \widetilde{b}_n < K_\alpha(\lambda) \|a\|_{p,w_1} \|\widetilde{b}\|_{q,w_2},
\end{aligned}$$

which gives $\|Ta\|_{p,\widetilde{w}_1} = \|\widetilde{b}\|_{q,w_2}^{q-1} < K_\alpha(\lambda)\|a\|_{p,w_1}$. Hence inequality (4) implies inequality (5).

Now suppose that inequality (5) holds for any $a \in \ell^p_{w_1,m_0}$.

$$\begin{aligned}
(Ta,b) &= \sum_{n=n_0}^{\infty}\sum_{m=m_0}^{\infty} K_{\alpha,\lambda}\big(\phi(m),\psi(n)\big)a_m b_n \\
&= \sum_{n=n_0}^{\infty}\left(\frac{\psi'(n)^{\frac{1}{p}}}{\psi(n)^{\alpha-\lambda_1+\frac{1}{p}}}\sum_{m=m_0}^{\infty} K_{\alpha,\lambda}\big(\phi(m),\psi(n)\big)a_m\right)\left(\frac{\psi(n)^{\alpha-\lambda_1+\frac{1}{p}}}{\psi'(n)^{\frac{1}{p}}}b_n\right) \\
&\le \left\{\sum_{n=n_0}^{\infty}\frac{\psi'(n)}{\psi(n)^{p(\alpha-\lambda_1)+1}}\left(\sum_{m=m_0}^{\infty} K_{\alpha,\lambda}\big(\phi(m),\psi(n)\big)a_m\right)^p\right\}^{\frac{1}{p}}\|b\|_{q,w_2} \\
&< K_\alpha(\lambda)\|a\|_{p,w_1}\|b\|_{q,w_2},
\end{aligned}$$

which means that inequality (5) implies inequality (4). Therefore inequality (4) is equivalent to inequality (5). Furthermore, Theorem 2.8 implies that the constant factor $K_\alpha(\lambda)$ in inequalities (4) and (5) is the best possible, which completes the proof. □

4 Applications to various Hilbert-type inequalities

In this section, we apply our previous theorems to obtain several Hilbert-type inequalities. Recall that the beta function $B(u,v)$ is defined by

$$B(u,v) := \int_0^\infty \frac{t^{u-1}}{(1+t)^{u+v}}\,dt = B(u,v) \quad (u,v>0).$$

Define the function $K_{\alpha,\lambda}(x,y)$ by

$$K_{\alpha,\lambda}(x,y) := \frac{(xy)^\alpha}{(x+y)^\lambda}$$

for $\lambda > \alpha \ge 0$. Then $K_{\alpha,\lambda}(x,y)$ is a symmetric homogeneous function of degree $2\alpha-\lambda$ and is decreasing with respect to x and y, respectively. Moreover,

$$\sum_{n=n_0}^{\infty}\frac{\psi'(n)}{\psi(n)^{1+\varepsilon}}\int_0^{\frac{\phi(m_0)}{\psi(n)}} K_{\alpha,\lambda}(1,t)t^{-1+\lambda_2-\alpha-\frac{\varepsilon}{p}}\,dt = O(1).$$

To see this, for $0 < \varepsilon < p\lambda_2$,

$$\begin{aligned}
\sum_{n=n_0}^{\infty}\frac{\psi'(n)}{\psi(n)^{1+\varepsilon}}\int_0^{\frac{\phi(m_0)}{\psi(n)}}\frac{t^{-1+\lambda_2-\frac{\varepsilon}{p}}}{(1+t)^\lambda}\,dt &\le \sum_{n=n_0}^{\infty}\frac{\psi'(n)}{\psi(n)^{1+\varepsilon}}\int_0^{\frac{\phi(m_0)}{\psi(n)}} t^{-1+\lambda_2-\frac{\varepsilon}{p}}\,dt \\
&= \sum_{n=n_0}^{\infty}\frac{\psi'(n)}{\psi(n)^{1+\varepsilon}}\frac{1}{\lambda_2-\frac{\varepsilon}{p}}\left(\frac{\phi(m_0)}{\psi(n)}\right)^{\lambda_2-\frac{\varepsilon}{p}} \\
&= \frac{\phi(m_0)^{\lambda_2-\frac{\varepsilon}{p}}}{\lambda_2-\frac{\varepsilon}{p}}\sum_{n=n_0}^{\infty}\frac{\psi'(n)}{\psi(n)^{1+\lambda_2+\frac{\varepsilon}{q}}} \\
&= O(1).
\end{aligned}$$

Note that since

$$\widetilde{K}_{\alpha,\lambda}(\lambda_i,\varepsilon) := \int_0^\infty K_{\alpha,\lambda}(1,t)t^{-1+\lambda_i-\alpha-\varepsilon}\,dt = \int_0^\infty \frac{t^{-1+\lambda_i-\varepsilon}}{(1+t)^\lambda}\,dt,$$

we see that

$$\widetilde{K}_{\alpha,\lambda}(\lambda_i,\varepsilon) \to \int_0^\infty \frac{t^{\lambda_i-1}}{(1+t)^\lambda}\,dt = B(\lambda_1,\lambda_2) = K_\alpha(\lambda_i) = K_\alpha(\lambda)$$

as $\varepsilon \to 0+$. Therefore from Theorem 3.1 we observe the following.

Corollary 4.1 *Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda_1 + \lambda_2 = \lambda$, $\lambda_1, \lambda_2 > 0$, $\lambda > \alpha \geq 0$. For $a_m, b_n \geq 0$ ($m_0, n_0 \in \mathbb{Z}$), let $a = \{a_m\}_{m=m_0}^\infty \in \ell^p_{w_1,m_0}$, $b = \{b_n\}_{n=n_0}^\infty \in \ell^q_{w_2,n_0}$ and $\|a\|_{p,w_1} > 0$, $\|b\|_{q,w_2} > 0$. Then, for $\phi(x) \in F_{m_0}(r)$ and $\psi(y) \in F_{n_0}(s)$ $(r, s > 1)$, we have the following equivalent inequalities*:

$$\sum_{n=n_0}^\infty \sum_{m=m_0}^\infty \frac{\phi(m)^\alpha \psi(n)^\alpha a_m b_n}{(\phi(m)+\psi(n))^\lambda} < B(\lambda_1,\lambda_2)\|a\|_{p,w_1}\|b\|_{q,w_2},$$

$$\left\{\sum_{n=n_0}^\infty \psi'(n)\psi(n)^{p(\lambda_1-\alpha)-1}\left(\sum_{m=m_0}^\infty \frac{\phi(m)^\alpha\psi(n)^\alpha a_m}{(\phi(m)+\psi(n))^\lambda}\right)^p\right\}^{\frac{1}{p}} < B(\lambda_1,\lambda_2)\|a\|_{p,w_1}.$$

Furthermore, the constant factor $B(\lambda_1,\lambda_2)$ is the best possible.

As applications, we have the following.

Case 1. Let $\phi(x) = x^\beta$ and $\psi(x) = x^\gamma$ $(\beta,\gamma > 0)$ for $m_0 = n_0 = 1$. For $0 < \lambda_i < \alpha + \min\{\frac{1}{\beta}, \frac{1}{\gamma}\}$ and $0 \leq \alpha < \lambda$, one has the following equivalent inequalities:

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{(m^\beta n^\gamma)^\alpha}{(m^\beta+n^\gamma)^\lambda} a_m b_n < \frac{B(\lambda_1,\lambda_2)}{\beta^{\frac{1}{q}}\gamma^{\frac{1}{p}}}\|a\|_{p,w_1}\|b\|_{q,w_2},$$

$$\left\{\sum_{n=1}^\infty n^{\gamma p(\lambda_1-\alpha)-1}\left(\sum_{m=1}^\infty \frac{(m^\beta n^\gamma)^\alpha}{(m^\beta+n^\gamma)^\lambda} a_m\right)^p\right\}^{\frac{1}{p}} < \frac{B(\lambda_1,\lambda_2)}{\beta^{\frac{1}{q}}\gamma^{\frac{1}{p}}}\|a\|_{p,w_1},$$

where $w_1(m) = m^{p(1-\lambda_2\beta+\alpha\beta)-1}$ and $w_2(n) = n^{q(1-\lambda_1\gamma+\alpha\gamma)-1}$.

(I) For $\lambda_1 = \frac{\lambda}{p}$ and $\lambda_2 = \frac{\lambda}{q}$ with $0 < \lambda_i < \alpha + \min\{\frac{1}{\beta}, \frac{1}{\gamma}\}$ and $0 \leq \alpha < \lambda$, one has the following equivalent inequalities:

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{(m^\beta n^\gamma)^\alpha}{(m^\beta+n^\gamma)^\lambda} a_m b_n < \frac{B(\frac{\lambda}{p},\frac{\lambda}{q})}{\beta^{\frac{1}{q}}\gamma^{\frac{1}{p}}}\|a\|_{p,w_1}\|b\|_{q,w_2},$$

$$\left\{\sum_{n=1}^\infty n^{\gamma(\lambda-p\alpha)-1}\left(\sum_{m=1}^\infty \frac{(m^\beta n^\gamma)^\alpha}{(m^\beta+n^\gamma)^\lambda} a_m\right)^p\right\}^{\frac{1}{p}} < \frac{B(\frac{\lambda}{p},\frac{\lambda}{q})}{\beta^{\frac{1}{q}}\gamma^{\frac{1}{p}}}\|a\|_{p,w_1},$$

where $w_1(m) = m^{(p-1)(1-\lambda\beta)+p\alpha\beta}$ and $w_2(n) = n^{(q-1)(1-\lambda\gamma)+q\alpha\gamma}$.

(II) For $\lambda_1 = \frac{\lambda}{q}$ and $\lambda_2 = \frac{\lambda}{p}$ with $0 < \lambda_i < \alpha + \min\{\frac{1}{\beta}, \frac{1}{\gamma}\}$ and $0 \le \alpha < \lambda$, one has the following equivalent inequalities:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(m^\beta n^\gamma)^\alpha}{(m^\beta + n^\gamma)^\lambda}a_m b_n < \frac{B(\frac{\lambda}{p}, \frac{\lambda}{q})}{\beta^{\frac{1}{q}}\gamma^{\frac{1}{p}}}\|a\|_{p,w_1}\|b\|_{q,w_2},$$

$$\left\{\sum_{n=1}^{\infty}n^{\gamma\lambda(p-1)-p\alpha\gamma-1}\left(\sum_{m=1}^{\infty}\frac{(m^\beta n^\gamma)^\alpha}{(m^\beta + n^\gamma)^\lambda}a_m\right)^p\right\}^{\frac{1}{p}} < \frac{B(\frac{\lambda}{p}, \frac{\lambda}{q})}{\beta^{\frac{1}{q}}\gamma^{\frac{1}{p}}}\|a\|_{p,w_1},$$

where $w_1(m) = m^{p-1-\beta\lambda+p\alpha\beta}$ and $w_2(n) = n^{q-1-\gamma\lambda+q\alpha\gamma}$.

(III) Let $\lambda_1 = \frac{p+\lambda-2}{p}$, $\lambda_2 = \frac{q+\lambda-2}{q}$, $\lambda > \max\{2-p, 2-q\}$, $0 < \beta < \frac{p}{p+\lambda-2-p\alpha}$, $0 < \gamma < \frac{q}{q+\lambda-2-q\alpha}$, $0 \le \alpha < \min\{\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\}$. Then one has the following equivalent inequalities:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(m^\beta n^\gamma)^\alpha}{(m^\beta + n^\gamma)^\lambda}a_m b_n < \frac{B(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q})}{\beta^{\frac{1}{q}}\gamma^{\frac{1}{p}}}\|a\|_{p,w_1}\|b\|_{q,w_2},$$

$$\left\{\sum_{n=1}^{\infty}n^{\gamma(p+\lambda-2)-p\alpha\gamma-1}\left(\sum_{m=1}^{\infty}\frac{(m^\beta n^\gamma)^\alpha}{(m^\beta + n^\gamma)^\lambda}a_m\right)^p\right\}^{\frac{1}{p}} < \frac{B(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q})}{\beta^{\frac{1}{q}}\gamma^{\frac{1}{p}}}\|a\|_{p,w_1},$$

where $w_1(m) = m^{(p-1)(1-\beta(q+\lambda-2))+p\alpha\beta}$ and $w_2(n) = n^{(q-1)(1-\gamma(p+\lambda-2))+q\alpha\gamma}$.

(IV) Let $\lambda_1 = \frac{q+\lambda-2}{q}$, $\lambda_2 = \frac{p+\lambda-2}{p}$, $\lambda > \max\{2-p, 2-q\}$, $0 < \beta < \frac{q}{q+\lambda-2-q\alpha}$, $0 < \gamma < \frac{p}{p+\lambda-2-p\alpha}$, $0 \le \alpha < \min\{\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\}$. Then one has the following equivalent inequalities:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(m^\beta n^\gamma)^\alpha}{(m^\beta + n^\gamma)^\lambda}a_m b_n < \frac{B(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q})}{\beta^{\frac{1}{q}}\gamma^{\frac{1}{p}}}\|a\|_{p,w_1}\|b\|_{q,w_2},$$

$$\left\{\sum_{n=1}^{\infty}n^{\gamma(p-1)(q+\lambda-2)-p\alpha\gamma-1}\left(\sum_{m=1}^{\infty}\frac{(m^\beta n^\gamma)^\alpha}{(m^\beta + n^\gamma)^\lambda}a_m\right)^p\right\}^{\frac{1}{p}} < \frac{B(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q})}{\beta^{\frac{1}{q}}\gamma^{\frac{1}{p}}}\|a\|_{p,w_1},$$

where $w_1(m) = m^{p-1-\beta(p+\lambda-2)+p\alpha\beta}$ and $w_2(n) = n^{q-1-\gamma(q+\lambda-2)+q\alpha\gamma}$.

Case 2. For $A, B > 0$, let $\phi(x) = A(\ln x)^\beta$ and $\psi(x) = B(\ln x)^\gamma$ $(\beta, \gamma > 0)$, $m_0 = n_0 = 2$. For $0 < \lambda_i < \alpha + \min\{\frac{1}{\beta}, \frac{1}{\gamma}\}$ and $0 \le \alpha < \lambda$, one has the following equivalent inequalities:

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{((\ln m)^\beta(\ln n)^\gamma)^\alpha}{(A(\ln m)^\beta + B(\ln n)^\gamma)^\lambda}a_m b_n < \frac{B(\lambda_1, \lambda_2)}{A^{\lambda_2}B^{\lambda_1}\beta^{\frac{1}{q}}\gamma^{\frac{1}{p}}}\|a\|_{p,w_1}\|b\|_{q,w_2},$$

$$\left\{\sum_{n=2}^{\infty}\frac{1}{n}(\ln n)^{p\gamma(\lambda_1-\alpha)-1}\left(\sum_{m=2}^{\infty}\frac{((\ln m)^\beta(\ln n)^\gamma)^\alpha}{(A(\ln m)^\beta + B(\ln n)^\gamma)^\lambda}a_m\right)^p\right\}^{\frac{1}{p}} < \frac{B(\lambda_1, \lambda_2)}{A^{\lambda_2}B^{\lambda_1}\beta^{\frac{1}{q}}\gamma^{\frac{1}{p}}}\|a\|_{p,w_1},$$

where $w_1(m) = m^{p-1}(\ln m)^{p(1-\lambda_2\beta+\alpha\beta)-1}$ and $w_2(n) = n^{q-1}(\ln n)^{q(1-\lambda_1\gamma+\alpha\gamma)-1}$.

(I) For $\lambda_1 = \frac{\lambda}{p}$ and $\lambda_2 = \frac{\lambda}{q}$ with $0 < \lambda_i < \alpha + \min\{\frac{1}{\beta}, \frac{1}{\gamma}\}$ and $0 \le \alpha < \lambda$, one has the following equivalent inequalities:

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{((\ln m)^\beta(\ln n)^\gamma)^\alpha}{(A(\ln m)^\beta + B(\ln n)^\gamma)^\lambda}a_m b_n < \frac{B(\frac{\lambda}{p}, \frac{\lambda}{q})}{A^{\lambda_2}B^{\lambda_1}\beta^{\frac{1}{q}}\gamma^{\frac{1}{p}}}\|a\|_{p,w_1}\|b\|_{q,w_2},$$

$$\left\{\sum_{n=2}^{\infty}\frac{1}{n}(\ln n)^{\gamma-p\alpha\gamma-1}\left(\sum_{m=2}^{\infty}\frac{((\ln m)^{\beta}(\ln n)^{\gamma})^{\alpha}}{(A(\ln m)^{\beta}+B(\ln n)^{\gamma})^{\lambda}}a_m\right)^p\right\}^{\frac{1}{p}} < \frac{B(\frac{\lambda}{p},\frac{\lambda}{q})}{A^{\lambda_2}B^{\lambda_1}\beta^{\frac{1}{q}}\gamma^{\frac{1}{p}}}\|a\|_{p,w_1},$$

where $w_1(m)=m^{p-1}(\ln m)^{(p-1)(1-\lambda\beta)+p\alpha\beta}$ and $w_2(n)=n^{q-1}(\ln n)^{(q-1)(1-\lambda\gamma)+q\alpha\gamma}$.

(II) Let $\lambda_1=\frac{p+\lambda-2}{p}$, $\lambda_2=\frac{q+\lambda-2}{q}$, $\lambda>\max\{2-p,2-q\}$, $0<\beta<\frac{p}{p+\lambda-2-p\alpha}$, $0<\gamma<\frac{q}{q+\lambda-2-q\alpha}$, $0\le\alpha<\min\{\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\}$. Then one has the following equivalent inequalities:

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{((\ln m)^{\beta}(\ln n)^{\gamma})^{\alpha}}{(A(\ln m)^{\beta}+B(\ln n)^{\gamma})^{\lambda}}a_m b_n < \frac{B(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q})}{A^{\lambda_2}B^{\lambda_1}\beta^{\frac{1}{q}}\gamma^{\frac{1}{p}}}\|a\|_{p,w_1}\|b\|_{q,w_2},$$

$$\left\{\sum_{n=2}^{\infty}\frac{1}{n}(\ln n)^{\gamma(p+\lambda-2)-p\alpha\gamma-1}\left(\sum_{m=2}^{\infty}\frac{((\ln m)^{\beta}(\ln n)^{\gamma})^{\alpha}}{(A(\ln m)^{\beta}+B(\ln n)^{\gamma})^{\lambda}}a_m\right)^p\right\}^{\frac{1}{p}} < \frac{B(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q})}{A^{\lambda_2}B^{\lambda_1}\beta^{\frac{1}{q}}\gamma^{\frac{1}{p}}}\|a\|_{p,w_1},$$

where $w_1(m)=m^{p-1}(\ln m)^{(p-1)(1-\beta(q+\lambda-2))+p\alpha\beta}$ and $w_2(n)=n^{q-1}(\ln n)^{(q-1)(1-\gamma(p+\lambda-2))+q\alpha\gamma}$.

Case 3. For $A,B>0$, let $\phi(x)=A(\ln x)^{\beta}$ and $\psi(x)=Bx^{\gamma}$ $(\beta,\gamma>0)$, $m_0=2$, $n_0=1$. For $0<\lambda_i<\alpha+\min\{\frac{1}{\beta},\frac{1}{\gamma}\}$ and $0\le\alpha<\lambda$, one has the following equivalent inequalities:

$$\sum_{n=1}^{\infty}\sum_{m=2}^{\infty}\frac{((\ln m)^{\beta}n^{\gamma})^{\alpha}}{(A(\ln m)^{\beta}+Bn^{\gamma})^{\lambda}}a_m b_n < \frac{B(\lambda_1,\lambda_2)}{A^{\lambda_2}B^{\lambda_1}\beta^{\frac{1}{q}}\gamma^{\frac{1}{p}}}\|a\|_{p,w_1}\|b\|_{q,w_2},$$

$$\left\{\sum_{n=1}^{\infty}n^{p\gamma(\lambda_1-\alpha)-1}\left(\sum_{m=2}^{\infty}\frac{((\ln m)^{\beta}n^{\gamma})^{\alpha}}{(A(\ln m)^{\beta}+Bn^{\gamma})^{\lambda}}a_m\right)^p\right\}^{\frac{1}{p}} < \frac{B(\lambda_1,\lambda_2)}{A^{\lambda_2}B^{\lambda_1}\beta^{\frac{1}{q}}\gamma^{\frac{1}{p}}}\|a\|_{p,w_1},$$

where $w_1(m)=m^{p-1}(\ln m)^{p(1-\lambda_2\beta+\alpha\beta)-1}$ and $w_2(n)=n^{q(1-\lambda_1\gamma+\alpha\gamma)-1}$.

(I) For $\lambda_1=\frac{\lambda}{p}$ and $\lambda_2=\frac{\lambda}{q}$ with $0<\lambda_i<\alpha+\min\{\frac{1}{\beta},\frac{1}{\gamma}\}$ and $0\le\alpha<\lambda$, one has the following equivalent inequalities:

$$\sum_{n=1}^{\infty}\sum_{m=2}^{\infty}\frac{((\ln m)^{\beta}n^{\gamma})^{\alpha}}{(A(\ln m)^{\beta}+Bn^{\gamma})^{\lambda}}a_m b_n < \frac{B(\frac{\lambda}{p},\frac{\lambda}{q})}{A^{\lambda_2}B^{\lambda_1}\beta^{\frac{1}{q}}\gamma^{\frac{1}{p}}}\|a\|_{p,w_1}\|b\|_{q,w_2},$$

$$\left\{\sum_{n=1}^{\infty}n^{\gamma(1-p\alpha)-1}\left(\sum_{m=2}^{\infty}\frac{((\ln m)^{\beta}n^{\gamma})^{\alpha}}{(A(\ln m)^{\beta}+Bn^{\gamma})^{\lambda}}a_m\right)^p\right\}^{\frac{1}{p}} < \frac{B(\frac{\lambda}{p},\frac{\lambda}{q})}{A^{\lambda_2}B^{\lambda_1}\beta^{\frac{1}{q}}\gamma^{\frac{1}{p}}}\|a\|_{p,w_1},$$

where $w_1(m)=m^{p-1}(\ln m)^{(p-1)(1-\lambda\beta)+p\alpha\beta}$ and $w_2(n)=n^{(q-1)(1-\lambda\gamma)+q\alpha\gamma}$.

(II) Let $\lambda_1=\frac{p+\lambda-2}{p}$, $\lambda_2=\frac{q+\lambda-2}{q}$, $\lambda>\max\{2-p,2-q\}$, $0<\beta<\frac{p}{p+\lambda-2-p\alpha}$, $0<\gamma<\frac{q}{q+\lambda-2-q\alpha}$, $0\le\alpha<\min\{\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\}$. Then one has the following equivalent inequalities:

$$\sum_{n=1}^{\infty}\sum_{m=2}^{\infty}\frac{((\ln m)^{\beta}n^{\gamma})^{\alpha}}{(A(\ln m)^{\beta}+Bn^{\gamma})^{\lambda}}a_m b_n < \frac{B(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q})}{A^{\lambda_2}B^{\lambda_1}\beta^{\frac{1}{q}}\gamma^{\frac{1}{p}}}\|a\|_{p,w_1}\|b\|_{q,w_2},$$

$$\left\{\sum_{n=1}^{\infty} n^{\gamma(p+\lambda-2)-p\alpha\gamma-1}\left(\sum_{m=2}^{\infty}\frac{((\ln m)^{\beta}n^{\gamma})^{\alpha}}{(A(\ln m)^{\beta}+Bn^{\gamma})^{\lambda}}a_m\right)^p\right\}^{\frac{1}{p}} < \frac{B(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q})}{A^{\lambda_2}B^{\lambda_1}\beta^{\frac{1}{q}}\gamma^{\frac{1}{p}}}\|a\|_{p,w_1},$$

where $w_1(m) = m^{p-1}(\ln m)^{(p-1)(1-\beta(q+\lambda-2))+p\alpha\beta}$ and $w_2(n) = n^{(q-1)(1-\gamma(p+\lambda-2))+q\alpha\gamma}$.

Competing interests
The author declares that he has no competing interests.

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