Hua and Tang *Journal of Inequalities and Applications* (2015) 2015:272
DOI 10.1186/s13660-015-0784-3

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH

Open Access

A posteriori error estimates of the lowest order Raviart-Thomas mixed finite element methods for convective diffusion optimal control problems

Yuchun Hua and Yuelong Tang*

*Correspondence:
tangyuelonga@163.com
Institute of Computational Mathematics, Department of Mathematics and Computational Science, Hunan University of Science and Engineering, Yongzhou, Hunan 425100, China

Abstract

In this paper, we consider the mixed finite element methods for quadratic optimal control problems governed by convective diffusion equations. The state and the co-state are discretized by the lowest order Raviart-Thomas mixed finite element spaces and the control is approximated by piecewise constant functions. Using some proper duality problems, we derive *a posteriori* $L^2(0,T;L^2(\Omega))$ error estimates for the scalar functions. Such estimates, which are apparently not available in the literature, are an important step toward developing reliable adaptive mixed finite element approximation schemes for the control problem.

MSC: 49J20; 65N30

Keywords: parabolic equations; optimal control problems; *a posteriori* error estimates; mixed finite element methods

1 Introduction

As far as we know, optimal control problems [1] have been extensively utilized in many aspects of the modern life such as social, economic, scientific, and engineering numerical simulation. Thus, they must be solved successfully with efficient numerical methods. Among these numerical methods, finite element method is a good choice. There have been extensive studies in the convergence of finite element approximation of optimal control problems; see [2–6]. A systematic introduction to finite element methods for PDEs and optimal control can be found for example in [7–9].

Recently, the adaptive finite element method has been investigated extensively. It has become one of the most popular methods in the scientific computation and numerical modeling. An adaptive finite element approximation ensures a higher density of nodes in a certain area of the given domain, where the solution is more difficult to approximate, indicated by *a posteriori* error estimators. Hence it is an important approach to boost the accuracy and efficiency of finite element discretizations. There are lots of works concentrating on the adaptivity of various optimal control problems. See, for example, [10–19].

In many control problems, the objective functional contains the gradient of the state variables. Thus, the accuracy of the gradient is important in numerical discretization of the coupled state equations. Mixed finite element methods are appropriate for the state equations in such cases since both the scalar variable and its flux variable can be approximated to the same accuracy by using such methods; see, for example, [20–23].

We shall use the lowest order Raviart-Thomas mixed finite element to discretize the state and the co-state, and use the piecewise constant space to approximate the control variable. Using some proper duality problems, we derive *a posteriori* $L^2(0,T;L^2(\Omega))$ error estimates for the scalar functions. The optimal control problems that we are interested in are as follows:

$$\min_{u\in K\subset U}\left\{\frac{1}{2}\int_0^T\big(\|\mathbf{p}-\mathbf{p}_d\|^2+\|y-y_d\|^2+\|u\|^2\big)\,dt\right\}, \tag{1.1}$$

$$y_t+\operatorname{div}\mathbf{p}+cy=f+u,\quad x\in\Omega,t\in J, \tag{1.2}$$

$$\mathbf{p}=-a(\nabla y+\mathbf{b}y),\quad x\in\Omega,t\in J, \tag{1.3}$$

$$y(x,t)=0,\quad x\in\partial\Omega,t\in J,\qquad y(x,0)=y_0(x),\quad x\in\Omega, \tag{1.4}$$

where the bounded open set $\Omega\subset\mathbf{R}^2$ is a convex polygon with the boundary $\partial\Omega$. $J=[0,T]$. Let K be a closed convex set in the control space $U=L^2(J;L^2(\Omega))$, $\mathbf{p},\mathbf{p}_d\in(L^2(J;H^1(\Omega)))^2$, $u,y,y_d\in L^2(J;H^1(\Omega))$, $f\in L^2(J;L^2(\Omega))$, $y_0(x)\in H_0^1(\Omega)$. Moreover, we assume that $0<a_0\le a\le a^0$, $a(x)\in W^{1,\infty}(\Omega)$, $c(x)\in W^{1,\infty}(\Omega)$, $\mathbf{b}(x)\in(W^{1,\infty}(\Omega))^2$.

We assume that the constraint on the control is an obstacle such that

$$K=\big\{u\in U:u(x,t)\ge 0,\text{a.e. in }\Omega\times J\big\}.$$

In this paper, we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with a norm $\|\cdot\|_{m,p}$ given by $\|v\|_{m,p}^p=\sum_{|\alpha|\le m}\|D^\alpha v\|_{L^p(\Omega)}^p$, a semi-norm $|\cdot|_{m,p}$ given by $|v|_{m,p}^p=\sum_{|\alpha|=m}\|D^\alpha v\|_{L^p(\Omega)}^p$. We set $W_0^{m,p}(\Omega)=\{v\in W^{m,p}(\Omega):v|_{\partial\Omega}=0\}$. For $p=2$, we denote $H^m(\Omega)=W^{m,2}(\Omega)$, $H_0^m(\Omega)=W_0^{m,2}(\Omega)$, and $\|\cdot\|_m=\|\cdot\|_{m,2}$, $\|\cdot\|=\|\cdot\|_{0,2}$.

We denote by $L^s(0,T;W^{m,p}(\Omega))$ the Banach space of all L^s integrable functions from J into $W^{m,p}(\Omega)$ with norm $\|v\|_{L^s(J;W^{m,p}(\Omega))}=(\int_0^T\|v\|_{W^{m,p}(\Omega)}^s\,dt)^{\frac{1}{s}}$ for $s\in[1,\infty)$, and the standard modification for $s=\infty$. Similarly, one can define the spaces $H^l(J;W^{m,p}(\Omega))$ and $C^k(J;W^{m,p}(\Omega))$. In addition C denotes a general positive constant independent of h and Δt, where h is the spatial mesh-size for the control and state discretization and Δt is the time increment.

The plan of this paper is as follows. In next section, we shall give a brief review on the mixed finite element method and the backward Euler discretization, and then we construct the approximation for the optimal control problems (1.1)-(1.4). Then, using two duality problems, we derive *a posteriori* $L^2(0,T;L^2(\Omega))$ error estimates for the scalar functions in Section 3. Finally, we give a conclusion and indicate some possible future work.

2 Mixed methods of parabolic optimal control problems

In this section, we shall study the mixed finite element and the backward Euler discretization approximation of convective diffusion optimal control problems (1.1)-(1.4). For the sake of simplicity, we assume that the domain Ω is a convex polygon. Now, we introduce the co-state parabolic equation

$$-z_t-\operatorname{div}\big(a(\nabla z+\mathbf{p}-\mathbf{p}_d)\big)+\mathbf{b}\cdot(\nabla z+\mathbf{p}-\mathbf{p}_d)+cz=y-y_d,\quad x\in\Omega,t\in J, \tag{2.1}$$

which can be written in the form of the first order system

$$-z_t + \operatorname{div}\mathbf{q} - a^{-1}\mathbf{b}\cdot\mathbf{q} + cz = y - y_d, \quad \mathbf{q} = -a(\nabla z + \mathbf{p} - \mathbf{p}_d), x \in \Omega, t \in J \tag{2.2}$$

and

$$z(x,t) = 0, \quad x \in \partial\Omega, t \in J, \qquad z(x,T) = 0, \quad x \in \Omega. \tag{2.3}$$

To be definite, we shall take the state spaces $\mathbf{L} = L^2(J;\mathbf{V})$ and $Q = H^1(J;W)$, where $\mathbf{V}$ and W are defined as follows:

$$\mathbf{V} = H(\operatorname{div};\Omega) = \{\mathbf{v} \in (L^2(\Omega))^2, \operatorname{div}\mathbf{v} \in L^2(\Omega)\}, \qquad W = L^2(\Omega).$$

The Hilbert space $\mathbf{V}$ is equipped with the following norm:

$$\|\mathbf{v}\|_{H(\operatorname{div};\Omega)} = \left(\|\mathbf{v}\|_{0,\Omega}^2 + \|\operatorname{div}\mathbf{v}\|_{0,\Omega}^2\right)^{1/2}.$$

Let $\alpha = a^{-1}$ and $\boldsymbol{\beta} = \alpha\mathbf{b}$. We recast (1.1)-(1.4) as the following weak form: find $(\mathbf{p}, y, u) \in \mathbf{L} \times Q \times K$ such that

$$\min_{u \in K \subset U}\left\{\frac{1}{2}\int_0^T \left(\|\mathbf{p} - \mathbf{p}_d\|^2 + \|y - y_d\|^2 + \|u\|^2\right) dt\right\}, \tag{2.4}$$

$$(\alpha\mathbf{p}, \mathbf{v}) - (y, \operatorname{div}\mathbf{v}) + (\boldsymbol{\beta}y, \mathbf{v}) = 0, \quad \forall\mathbf{v} \in \mathbf{V}, t \in J, \tag{2.5}$$

$$(y_t, w) + (\operatorname{div}\mathbf{p}, w) + (cy, w) = (f + u, w), \quad \forall w \in W, t \in J, \tag{2.6}$$

$$y(x,0) = y_0(x), \quad \forall x \in \Omega. \tag{2.7}$$

It follows from [1] and [16] that the optimal control problem (2.4)-(2.7) has a unique solution $(\mathbf{p}, y, u)$, and that a triplet $(\mathbf{p}, y, u)$ is the solution of (2.4)-(2.7) if and only if there is a co-state $(\mathbf{q}, z) \in \mathbf{L} \times Q$ such that $(\mathbf{p}, y, \mathbf{q}, z, u)$ satisfies the following optimality conditions:

$$(\alpha\mathbf{p}, \mathbf{v}) - (y, \operatorname{div}\mathbf{v}) + (\boldsymbol{\beta}y, \mathbf{v}) = 0, \quad \forall\mathbf{v} \in \mathbf{V}, t \in J, \tag{2.8}$$

$$(y_t, w) + (\operatorname{div}\mathbf{p}, w) + (cy, w) = (f + u, w), \quad \forall w \in W, t \in J, \tag{2.9}$$

$$y(x,0) = y_0(x), \quad \forall x \in \Omega, \tag{2.10}$$

$$(\alpha\mathbf{q}, \mathbf{v}) - (z, \operatorname{div}\mathbf{v}) = -(\mathbf{p} - \mathbf{p}_d, \mathbf{v}), \quad \forall\mathbf{v} \in \mathbf{V}, t \in J, \tag{2.11}$$

$$-(z_t, w) + (\operatorname{div}\mathbf{q}, w) - (\boldsymbol{\beta}\cdot\mathbf{q}, w) + (cz, w) = (y - y_d, w), \quad \forall w \in W, t \in J, \tag{2.12}$$

$$z(x,T) = 0, \quad \forall x \in \Omega, \tag{2.13}$$

$$\int_0^T (u + z, \tilde{u} - u)\,dt \geq 0, \quad \forall\tilde{u} \in K, \tag{2.14}$$

where $(\cdot,\cdot)$ is the inner product of $L^2(\Omega)$.

Let $\mathcal{T}_h$ be regular triangulations of Ω. h_τ is the diameter of τ and $h = \max h_\tau$. Let $\mathbf{V}_h \times W_h \subset \mathbf{V} \times W$ denote the lowest order Raviart-Thomas space [24] associated with the triangulations $\mathcal{T}_h$ of Ω. P_k denotes the space of polynomials of total degree of at most k ($k \geq 0$). Let $\mathbf{V}(\tau) = \{\mathbf{v} \in P_0^2(\tau) + x \cdot P_0(\tau)\}$, $W(\tau) = P_0(\tau)$. We define

$$\begin{aligned}
&\mathbf{V}_h := \{\mathbf{v}_h \in \mathbf{V} : \forall \tau \in \mathcal{T}_h, \mathbf{v}_h|_\tau \in \mathbf{V}(\tau)\}, \\
&W_h := \{w_h \in W : \forall \tau \in \mathcal{T}_h, w_h|_\tau \in W(\tau)\}, \\
&K_h := K \cap W_h.
\end{aligned}$$

The mixed finite element discretization of (2.4)-(2.7) is as follows: compute $(\mathbf{p}_h, y_h, u_h) \in L^2(J; \mathbf{V}_h) \times H^1(J; W_h) \times K_h$ such that

$$\min_{u_h(t)\in K_h} \left\{ \frac{1}{2} \int_0^T \left(\|\mathbf{p}_h - \mathbf{p}_d\|^2 + \|y_h - y_d\|^2 + \|u_h\|^2 \right) dt \right\}, \tag{2.15}$$

$$(\alpha \mathbf{p}_h, \mathbf{v}_h) - (y_h, \operatorname{div} \mathbf{v}_h) + (\boldsymbol{\beta} y_h, \mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h, t \in J, \tag{2.16}$$

$$(y_{ht}, w_h) + (\operatorname{div} \mathbf{p}_h, w_h) + (c y_h, w_h) = (f + u_h, w_h), \quad \forall w_h \in W_h, t \in J, \tag{2.17}$$

$$y_h(x, 0) = y_0^h(x), \quad \forall x \in \Omega, \tag{2.18}$$

where $y_0^h(x) \in W_h$ is an approximation of y_0. The optimal control problem (2.15)-(2.18) again has a unique solution $(\mathbf{p}_h, y_h, u_h)$, and that a triplet $(\mathbf{p}_h, y_h, u_h)$ is the solution of (2.15)-(2.18) if and only if there is a co-state $(\mathbf{q}_h, z_h) \in L^2(J; \mathbf{V}_h) \times H^1(J; W_h)$ such that $(\mathbf{p}_h, y_h, \mathbf{q}_h, z_h, u_h)$ satisfies the following optimality conditions:

$$(\alpha \mathbf{p}_h, \mathbf{v}_h) - (y_h, \operatorname{div} \mathbf{v}_h) + (\boldsymbol{\beta} y_h, \mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h, t \in J, \tag{2.19}$$

$$(y_{ht}, w_h) + (\operatorname{div} \mathbf{p}_h, w_h) + (c y_h, w_h) = (f + u_h, w_h), \quad \forall w_h \in W_h, t \in J, \tag{2.20}$$

$$y_h(x, 0) = y_0^h(x), \quad \forall x \in \Omega, \tag{2.21}$$

$$(\alpha \mathbf{q}_h, \mathbf{v}_h) - (z_h, \operatorname{div} \mathbf{v}_h) = -(\mathbf{p}_h - p_d, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_h, t \in J, \tag{2.22}$$

$$\begin{aligned}
&-(z_{ht}, w_h) + (\operatorname{div} \mathbf{q}_h, w_h) - (\boldsymbol{\beta} \cdot \mathbf{q}_h, w_h) + (c z_h, w_h) \\
&\quad = (y_h - y_d, w_h), \quad \forall w_h \in W_h, t \in J,
\end{aligned} \tag{2.23}$$

$$z_h(x, T) = 0, \quad \forall x \in \Omega, \tag{2.24}$$

$$\int_0^T (u_h + z_h, \tilde{u}_h - u_h)\, dt \geq 0, \quad \forall \tilde{u}_h \in K_h. \tag{2.25}$$

We now consider the fully discrete approximation for the above semidiscrete problem. Let $\Delta t > 0$, $N = T/\Delta t \in \mathbb{Z}$, and $t_i = i\Delta t$, $i \in \mathbb{Z}$. Also, let

$$d_t \psi^i = \frac{\psi^i - \psi^{i-1}}{\Delta t}.$$

We address the fully discrete approximation scheme to find $(\mathbf{p}_h^i, y_h^i, u_h^i) \in \mathbf{V}_h \times W_h \times K_h$, $i = 1, 2, \ldots, N$, such that

$$\min_{u_h^i \in K_h} \left\{ \frac{1}{2} \sum_{i=1}^N \Delta t \left(\|\mathbf{p}_h^i - \mathbf{p}_d^i\|^2 + \|y_h^i - y_d^i\|^2 + \|u_h^i\|^2 \right) \right\}, \tag{2.26}$$

$$\big(\alpha \mathbf{p}_h^i, \mathbf{v}_h\big) - \big(y_h^i, \operatorname{div} \mathbf{v}_h\big) + \big(\boldsymbol{\beta} y_h^i, \mathbf{v}_h\big) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{2.27}$$

$$\big(d_t y_h^i, w_h\big) + \big(\operatorname{div} \mathbf{p}_h^i, w_h\big) + \big(c y_h^i, w_h\big) = \big(f^i + u_h^i, w_h\big), \quad \forall w_h \in W_h, \tag{2.28}$$

$$y_h^0(x) = y_0^h(x), \quad \forall x \in \Omega, \tag{2.29}$$

where $f^i = f^i(x) = f(x, t_i)$, $y_d^i = y_d(x, t_i)$, and $\mathbf{p}_d^i = \mathbf{p}_d(x, t_i)$.

It follows that the control problem (2.26)-(2.29) has a unique solution $(\mathbf{p}_h^i, y_h^i, u_h^i)$, $i = 1, 2, \dots, N$, and that a triplet $(\mathbf{p}_h^i, y_h^i, u_h^i) \in \mathbf{V}_h \times W_h \times K_h$, $i = 1, 2, \dots, N$, is the solution of (2.26)-(2.29) if and only if there is a co-state $(\mathbf{q}_h^{i-1}, z_h^{i-1}) \in \mathbf{V}_h \times W_h$ such that $(\mathbf{p}_h^i, y_h^i, \mathbf{q}_h^{i-1}, z_h^{i-1}, u_h^i) \in (\mathbf{V}_h \times W_h)^2 \times K_h$ satisfies the following optimality conditions:

$$\big(\alpha \mathbf{p}_h^i, \mathbf{v}_h\big) - \big(y_h^i, \operatorname{div} \mathbf{v}_h\big) + \big(\boldsymbol{\beta} y_h^i, \mathbf{v}_h\big) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{2.30}$$

$$\big(d_t y_h^i, w_h\big) + \big(\operatorname{div} \mathbf{p}_h^i, w_h\big) + \big(c y_h^i, w_h\big) = \big(f^i + u_h^i, w_h\big), \quad \forall w_h \in W_h, \tag{2.31}$$

$$y_h^0(x) = y_0^h(x), \quad \forall x \in \Omega, \tag{2.32}$$

$$\big(\alpha \mathbf{q}_h^{i-1}, \mathbf{v}_h\big) - \big(z_h^{i-1}, \operatorname{div} \mathbf{v}_h\big) = -\big(\mathbf{p}_h^i - \mathbf{p}_d^i, \mathbf{v}_h\big), \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{2.33}$$

$$\begin{aligned} &-\big(d_t z_h^i, w_h\big) + \big(\operatorname{div} \mathbf{q}_h^{i-1}, w_h\big) - \big(\boldsymbol{\beta} \cdot \mathbf{q}_h^{i-1}, w_h\big) + \big(c z_h^{i-1}, w_h\big) \\ &\quad = \big(y_h^i - y_d^i, w_h\big), \quad \forall w_h \in W_h, \end{aligned} \tag{2.34}$$

$$z_h^N(x) = 0, \quad \forall x \in \Omega, \tag{2.35}$$

$$\big(u_h^i + z_h^{i-1}, \tilde{u}_h - u_h^i\big) \geq 0, \quad \forall \tilde{u}_h \in K_h. \tag{2.36}$$

For $i = 0$ and $i = N$, we let

$$\big(\alpha \mathbf{p}_h^0, \mathbf{v}_h\big) - \big(y_h^0, \operatorname{div} \mathbf{v}_h\big) + \big(\boldsymbol{\beta} y_h^0, \mathbf{v}_h\big) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{2.37}$$

$$\big(\alpha \mathbf{q}_h^N, \mathbf{v}_h\big) - \big(z_h^N, \operatorname{div} \mathbf{v}_h\big) = -\big(\mathbf{p}_h^N - \mathbf{p}_d^N, \mathbf{v}_h\big), \quad \forall \mathbf{v}_h \in \mathbf{V}_h. \tag{2.38}$$

For $i = 1, 2, \dots, N$, let

$$\begin{aligned} Y_h|_{(t_{i-1}, t_i]} &= \big((t_i - t) y_h^{i-1} + (t - t_{i-1}) y_h^i\big)/\Delta t, \\ Z_h|_{(t_{i-1}, t_i]} &= \big((t_i - t) z_h^{i-1} + (t - t_{i-1}) z_h^i\big)/\Delta t, \\ P_h|_{(t_{i-1}, t_i]} &= \big((t_i - t) \mathbf{p}_h^{i-1} + (t - t_{i-1}) \mathbf{p}_h^i\big)/\Delta t, \\ Q_h|_{(t_{i-1}, t_i]} &= \big((t_i - t) \mathbf{q}_h^{i-1} + (t - t_{i-1}) \mathbf{q}_h^i\big)/\Delta t, \\ U_h|_{(t_{i-1}, t_i]} &= u_h^i. \end{aligned}$$

For any function $w \in C(J; L^2(\Omega))$, let

$$\hat{w}(x,t)|_{t \in (t_{i-1}, t_i]} = w(x, t_i), \qquad \tilde{w}(x,t)|_{t \in (t_{i-1}, t_i]} = w(x, t_{i-1}).$$

Moreover, we let

$$\begin{aligned} \bar{\mathbf{p}}_d|_{(t_{i-1}, t_i]} &= \big((t_i - t) \mathbf{p}_d^i + (t - t_{i-1}) \mathbf{p}_d^{i+1}\big)/\Delta t, \quad i = 1, 2, \dots, N-1, \qquad \bar{\mathbf{p}}_d|_{(t_{N-1}, t_N]} = \mathbf{p}_d^N, \\ \bar{P}_h|_{(t_{i-1}, t_i]} &= \big((t_i - t) \mathbf{p}_h^i + (t - t_{i-1}) \mathbf{p}_h^{i+1}\big)/\Delta t, \quad i = 1, 2, \dots, N-1, \qquad \bar{P}_h|_{(t_{N-1}, t_N]} = \mathbf{p}_h^N. \end{aligned}$$

Then the optimality conditions (2.30)-(2.36) satisfy

$$(\alpha\hat{P}_h, \mathbf{v}_h) - (\hat{Y}_h, \operatorname{div}\mathbf{v}_h) + (\boldsymbol{\beta}\hat{Y}_h, \mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{2.39}$$

$$(Y_{ht}, w_h) + (\operatorname{div}\hat{P}_h, w_h) + (c\hat{Y}_h, w_h) = (\hat{f} + U_h, w_h), \quad \forall w_h \in W_h, \tag{2.40}$$

$$Y_h(x, 0) = y_0^h(x), \quad \forall x \in \Omega, \tag{2.41}$$

$$(\alpha\tilde{Q}_h, \mathbf{v}_h) - (\tilde{Z}_h, \operatorname{div}\mathbf{v}_h) = -(\hat{P}_h - \hat{\mathbf{p}}_d, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{2.42}$$

$$-(Z_{ht}, w_h) + (\operatorname{div}\tilde{Q}_h, w_h) - (\boldsymbol{\beta}\cdot\tilde{Q}_h, w_h) + (c\tilde{Z}_h, w_h) = (\hat{Y}_h - \hat{y}_d, w_h), \quad \forall w_h \in W_h, \tag{2.43}$$

$$Z_h(x, T) = 0, \quad \forall x \in \Omega, \tag{2.44}$$

$$(U_h + \tilde{Z}_h, \tilde{u}_h - U_h) \geq 0, \quad \forall \tilde{u}_h \in K_h. \tag{2.45}$$

In the rest of the paper, we shall use some intermediate variables. For any control function $U_h \in K_h$, we first define the state solution $(\mathbf{p}(U_h), y(U_h), \mathbf{q}(U_h), z(U_h))$ to satisfy

$$\bigl(\alpha\mathbf{p}(U_h), \mathbf{v}\bigr) - \bigl(y(U_h), \operatorname{div}\mathbf{v}\bigr) + \bigl(\boldsymbol{\beta}y(U_h), \mathbf{v}\bigr) = 0, \quad \forall \mathbf{v} \in \mathbf{V}, \tag{2.46}$$

$$\bigl(y_t(U_h), w\bigr) + \bigl(\operatorname{div}\mathbf{p}(U_h), w\bigr) + \bigl(cy(U_h), w\bigr) = (f + U_h, w), \quad \forall w \in W, \tag{2.47}$$

$$y(U_h)(x, 0) = y_0(x), \quad \forall x \in \Omega, \tag{2.48}$$

$$\bigl(\alpha\mathbf{q}(U_h), \mathbf{v}\bigr) - \bigl(z(U_h), \operatorname{div}\mathbf{v}\bigr) = -\bigl(\mathbf{p}(U_h) - \mathbf{p}_d, \mathbf{v}\bigr), \quad \forall \mathbf{v} \in \mathbf{V}, \tag{2.49}$$

$$\begin{aligned} &-\bigl(z_t(U_h), w\bigr) + \bigl(\operatorname{div}\mathbf{q}(U_h), w\bigr) - \bigl(\boldsymbol{\beta}\cdot\mathbf{q}(U_h), w\bigr) + \bigl(cz(U_h), w\bigr) \\ &\quad = \bigl(y(U_h) - y_d, w\bigr), \quad \forall w \in W, \end{aligned} \tag{2.50}$$

$$z(U_h)(x, T) = 0, \quad \forall x \in \Omega. \tag{2.51}$$

Let $R_h : W \to W_h$ be the orthogonal $L^2(\Omega)$-projection into W_h [25], which satisfies

$$(R_h w - w, \chi) = 0, \quad w \in W, \chi \in W_h, \tag{2.52}$$

$$\|R_h w - w\|_{0,q} \leq Ch\|w\|_{1,q}, \quad \text{if } w \in W \cap W^{1,q}(\Omega). \tag{2.53}$$

Let $\Pi_h : \mathbf{V} \to \mathbf{V}_h$ be the Raviart-Thomas projection operator [26], which satisfies: for any $\mathbf{v} \in \mathbf{V}$,

$$\int_E w_h(\mathbf{v} - \Pi_h\mathbf{v}) \cdot \boldsymbol{\nu}_E \, ds = 0, \quad w_h \in W_h, E \in \mathcal{E}_h, \tag{2.54}$$

$$\int_\tau (\mathbf{v} - \Pi_h\mathbf{v}) \cdot \mathbf{v}_h \, dx\, dy = 0, \quad \mathbf{v}_h \in \mathbf{V}_h, \tau \in \mathcal{T}_h, \tag{2.55}$$

where $\mathcal{E}_h$ denotes the set of element sides in $\mathcal{T}_h$.

We have the commuting diagram property

$$\operatorname{div} \circ \Pi_h = R_h \circ \operatorname{div} : \mathbf{V} \to W_h \quad \text{and} \quad \operatorname{div}(I - \Pi_h)\mathbf{V} \perp W_h, \tag{2.56}$$

where I denotes the identity operator.

Further, the interpolation operator Π_h satisfies a local error estimate:

$$\|\mathbf{v} - \Pi_h\mathbf{v}\|_{0,\Omega} \leq Ch|\mathbf{v}|_{1,\mathcal{T}_h}, \quad \mathbf{v} \in \mathbf{V} \cap H^1(\mathcal{T}_h). \tag{2.57}$$

3 *A posteriori* error estimates

In this section we study *a posteriori* error estimates for the mixed finite element approximation to the parabolic optimal control problems.

For the following analysis, we divide the domain Ω into three parts:

$$\Omega_- = \{x \in \Omega : \tilde{Z}_h(x) \le 0\},$$
$$\Omega_0 = \{x \in \Omega : \tilde{Z}_h(x) > 0, U_h(x) = 0\},$$
$$\Omega_+ = \{x \in \Omega : \tilde{Z}_h(x) > 0, U_h(x) > 0\}.$$

It is easy to see that the partition of the above three subsets is dependent on t. For all t, the three subsets are not intersected each other, and

$$\bar{\Omega} = \bar{\Omega}_- \cup \bar{\Omega}_0 \cup \bar{\Omega}_+.$$

Firstly, let us derive the *a posteriori* error estimates for the control u.

Theorem 3.1 *Let* $(y, \mathbf{p}, z, \mathbf{q}, u)$ *and* $(Y_h, P_h, Z_h, Q_h, U_h)$ *be the solutions of* (2.8)-(2.14) *and* (2.39)-(2.45), *respectively. Then we have*

$$\|u - U_h\|^2_{L^2(J;L^2(\Omega))} \le C\eta_1^2 + \|\tilde{Z}_h - z(U_h)\|^2_{L^2(J;L^2(\Omega))}, \tag{3.1}$$

where

$$\eta_1^2 = \|U_h + \tilde{Z}_h\|^2_{L^2(J;L^2(\Omega_- \cup \Omega_+))}.$$

Proof It follows from (2.14) that

$$\begin{aligned}
&\|u - U_h\|^2_{L^2(J;L^2(\Omega))} \\
&\quad = \int_0^T (u - U_h, u - U_h)\,dt \\
&\quad = \int_0^T (u + z, u - U_h)\,dt + \int_0^T (U_h + \tilde{Z}_h, U_h - u)\,dt \\
&\qquad + \int_0^T \big(\tilde{Z}_h - z(U_h), u - U_h\big)\,dt + \int_0^T \big(z(U_h) - z, u - U_h\big)\,dt \\
&\quad \le \int_0^T (U_h + \tilde{Z}_h, U_h - u)\,dt + \int_0^T \big(\tilde{Z}_h - z(U_h), u - U_h\big)\,dt \\
&\qquad + \int_0^T \big(z(U_h) - z, u - U_h\big)\,dt \\
&\quad =: I_1 + I_2 + I_3.
\end{aligned} \tag{3.2}$$

We first estimate I_1. Note that

$$\begin{aligned}
I_1 &= \int_0^T (U_h + \tilde{Z}_h, U_h - u)\,dt \\
&= \int_0^T \int_{\Omega_- \cup \Omega_+} (U_h + \tilde{Z}_h)(U_h - u)\,dx\,dt + \int_0^T \int_{\Omega_0} (U_h + \tilde{Z}_h)(U_h - u)\,dx\,dt.
\end{aligned} \tag{3.3}$$

It is easy to see that

$$\begin{aligned}
&\int_0^T \int_{\Omega_- \cup \Omega_+} (U_h + \tilde{Z}_h)(U_h - u)\,dx\,dt \\
&\quad \le C(\delta)\|U_h + \tilde{Z}_h\|^2_{L^2(J;L^2(\Omega_- \cup \Omega_+))} + \delta\|u - U_h\|^2_{L^2(J;L^2(\Omega_- \cup \Omega_+))} \\
&\quad = C(\delta)\eta_1^2 + \delta\|u - U_h\|^2_{L^2(J;L^2(\Omega))},
\end{aligned} \tag{3.4}$$

where δ is an arbitrary small positive number, $C(\delta)$ is dependent on δ^{-1}. Furthermore, we have

$$U_h + \tilde{Z}_h \ge \tilde{Z}_h > 0, \qquad U_h - u = 0 - u \le 0 \quad \text{on } \Omega_0.$$

It yields

$$\int_0^T \int_{\Omega_0} (U_h + \tilde{Z}_h)(U_h - u)\,dx\,dt \le 0. \tag{3.5}$$

Then (3.3)-(3.5) imply that

$$I_1 \le C(\delta)\eta_1^2 + \delta\|u - U_h\|^2_{L^2(J;L^2(\Omega))}. \tag{3.6}$$

Moreover, it is clear that

$$\begin{aligned}
I_2 &= \int_0^T \big(\tilde{Z}_h - z(U_h), u - U_h\big)\,dt \\
&\le C(\delta)\big\|\tilde{Z}_h - z(U_h)\big\|^2_{L^2(J;L^2(\Omega))} + \delta\|u - U_h\|^2_{L^2(J;L^2(\Omega))}.
\end{aligned} \tag{3.7}$$

Now we turn to I_3. Note that

$$y(x,0) = y(U_h)(x,0) = y_0(x) \quad \text{and} \quad z(x,T) = z(U_h)(x,T) = 0.$$

Then from (2.8)-(2.13) and (2.46)-(2.51), we have

$$\begin{aligned}
I_3 &= \int_0^T \big(z(U_h) - z, u - U_h\big)\,dt = \int_0^T \big(u - U_h, z(U_h) - z\big)\,dt \\
&= \int_0^T \Big(\big(\big(y - y(U_h)\big)_t, z(U_h) - z\big) + \big(\operatorname{div}\big(\mathbf{p} - \mathbf{p}(U_h)\big), z(U_h) - z\big)\Big)\,dt \\
&\quad + \int_0^T \Big(\big(c\big(y - y(U_h)\big), z(U_h) - z\big) - \big(\boldsymbol{\beta}\big(y - y(U_h)\big), \mathbf{q}(U_h) - \mathbf{q}\big)\Big)\,dt \\
&\quad - \int_0^T \Big(\big(\alpha\big(\mathbf{p} - \mathbf{p}(U_h)\big), \mathbf{q}(U_h) - \mathbf{q}\big) - \big(y - y(U_h), \operatorname{div}\big(\mathbf{q}(U_h) - \mathbf{q}\big)\big)\Big)\,dt \\
&= \int_0^T \Big(-\big(\big(z(U_h) - z\big)_t, y - y(U_h)\big) + \big(\operatorname{div}\big(\mathbf{q}(U_h) - \mathbf{q}\big), y - y(U_h)\big)\Big)\,dt \\
&\quad + \int_0^T \Big(\big(c\big(z(U_h) - z\big), y - y(U_h)\big) - \big(\boldsymbol{\beta} \cdot \big(\mathbf{q}(U_h) - \mathbf{q}\big), y - y(U_h)\big)\Big)\,dt
\end{aligned}$$

$$
\begin{aligned}
&-\int_0^T \Big(\big(\alpha\big(\mathbf{q}(U_h)-\mathbf{q}\big), \mathbf{p}-\mathbf{p}(U_h)\big) - \big(z(U_h)-z, \operatorname{div}\big(\mathbf{p}-\mathbf{p}(U_h)\big)\big)\Big)\,dt \\
&= \int_0^T \Big(\big(y(U_h)-y, y-y(U_h)\big) + \big(\mathbf{p}(U_h)-\mathbf{p}, \mathbf{p}-\mathbf{p}(U_h)\big)\Big)\,dt \le 0. \qquad (3.8)
\end{aligned}
$$

Thus, we obtain from (3.2) and (3.6)-(3.8)

$$
\|u-U_h\|^2_{L^2(J;L^2(\Omega))} \le C\eta_1^2 + \big\|\tilde{Z}_h - z(U_h)\big\|^2_{L^2(J;L^2(\Omega))}, \qquad (3.9)
$$

which proves (3.1). □

In order to estimate the error $\|\tilde{Z}_h - z(U_h)\|^2_{L^2(J;L^2(\Omega))}$, we need the following well-known stability results (see [27, 28] for the details) for the following dual equations:

$$
\begin{cases}
\phi_t - \operatorname{div}(a\nabla\phi + \mathbf{b}\phi) + c\phi = F, & x\in\Omega, t\in J,\\
\phi|_{\partial\Omega} = 0, & t\in J,\\
\phi(x,0) = 0, & x\in\Omega
\end{cases} \qquad (3.10)
$$

and

$$
\begin{cases}
-\psi_t - \operatorname{div}(a\nabla\psi) + \mathbf{b}\cdot\nabla\psi + c\psi = F, & x\in\Omega, t\in J,\\
\psi|_{\partial\Omega} = 0, & t\in J,\\
\psi(x,T) = 0, & x\in\Omega.
\end{cases} \qquad (3.11)
$$

Lemma 3.1 [28] *Let ϕ and ψ be the solutions of* (3.10) *and* (3.11), *respectively. Let Ω be a convex domain. Then, for $\varphi = \phi$ or $\varphi = \psi$,*

$$
\begin{aligned}
&\int_\Omega \big|\varphi(x,t)\big|^2\,dx \le C\|F\|^2_{L^2(J;L^2(\Omega))}, \quad \forall t\in J,\\
&\int_0^T\int_\Omega |\nabla\varphi|^2\,dx\,dt \le C\|F\|^2_{L^2(J;L^2(\Omega))},\\
&\int_0^T\int_\Omega \big|D^2\varphi\big|^2\,dx\,dt \le C\|F\|^2_{L^2(J;L^2(\Omega))},\\
&\int_0^T\int_\Omega |\varphi_t|^2\,dx\,dt \le C\|F\|^2_{L^2(J;L^2(\Omega))},
\end{aligned}
$$

where $|D^2\varphi| = \max\{|\partial^2\varphi/\partial x_i\,\partial x_j|, 1\le i,j\le 2\}$.

We also need the following Gronwall lemma.

Lemma 3.2 [29] *Let f and g be piecewise continuous nonnegative functions defined on $0\le t\le T$, g being non-decreasing. If, for each $t\in J$,*

$$
f(t) \le g(t) + \int_0^t f(s)\,ds, \qquad (3.12)
$$

then $f(t)\le e^t g(t)$.

In the following two theorems, we shall estimate the error $\|\tilde{Z}_h - z(U_h)\|_{L^2(J;L^2(\Omega))}$.

Theorem 3.2 *Let* $(Y_h, P_h, Z_h, Q_h, U_h)$ *and* $(y(U_h), \mathbf{p}(U_h), z(U_h), \mathbf{q}(U_h), U_h)$ *be the solutions of* (2.39)-(2.45) *and* (2.46)-(2.51), *respectively. Then we have*

$$\|Y_h - y(U_h)\|^2_{L^2(J;L^2(\Omega))} \le C\sum_{i=2}^{7}\eta_i^2, \tag{3.13}$$

where

$$\eta_2^2 = \int_0^T \sum_\tau h_\tau^2 \int_\tau (Y_{ht} + \operatorname{div}\hat{P}_h + c\hat{Y}_h - \hat{f} - U_h)^2\,dx\,dt;$$

$$\eta_3^2 = \int_0^T \sum_\tau h_\tau^2 \int_\tau (\alpha P_h + \boldsymbol{\beta} Y_h)^2\,dx\,dt; \qquad \eta_4^2 = \|\hat{P}_h - P_h\|^2_{L^2(J;L^2(\Omega))};$$

$$\eta_5^2 = \|\hat{f} - f\|^2_{L^2(J;L^2(\Omega))}; \qquad \eta_6^2 = \|\hat{Y}_h - Y_h\|^2_{L^2(J;L^2(\Omega))}; \qquad \eta_7^2 = \|y_0^h(x) - y_0(x)\|^2_{L^2(\Omega)}.$$

Proof From (2.30) and (2.37), we get the equality

$$(\alpha P_h, \mathbf{v}_h) - (Y_h, \operatorname{div}\mathbf{v}_h) + (\boldsymbol{\beta} Y_h, \mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h. \tag{3.14}$$

Let ψ be the solution of (3.11) with $F = Y_h - y(U_h)$, using (2.39)-(2.41), (2.46)-(2.48), and (2.54)-(2.56), we infer that

$$\begin{aligned}
&\|Y_h - y(U_h)\|^2_{L^2(J;L^2(\Omega))}\\
&\quad= \int_0^T \big(Y_h - y(U_h), F\big)\,dt\\
&\quad= \int_0^T \big(Y_h - y(U_h), -\psi_t - \operatorname{div}(a\nabla\psi) + \mathbf{b}\cdot\nabla\psi + c\psi\big)\,dt\\
&\quad= \int_0^T \Big(\big(\big(Y_h - y(U_h)\big)_t, \psi\big) - \big(Y_h, \operatorname{div}\big(\Pi_h(a\nabla\psi)\big)\big) + \big(\mathbf{p}(U_h), \nabla\psi\big)\Big)\,dt\\
&\qquad+ \int_0^T \Big((\mathbf{b}Y_h, \nabla\psi) + \big(c\big(Y_h - y(U_h)\big), \psi\big)\Big)\,dt + \Big(\big(Y_h - y(U_h)\big)(x,0), \psi(x,0)\Big)\\
&\quad= \int_0^T \Big(\big(\big(Y_h - y(U_h)\big)_t, \psi\big) - \big(\alpha P_h, \Pi_h(a\nabla\psi)\big)\\
&\qquad- \big(\boldsymbol{\beta} Y_h, \Pi_h(a\nabla\psi)\big) - \big(\operatorname{div}\mathbf{p}(U_h), \psi\big)\Big)\,dt\\
&\qquad+ \int_0^T \Big((\boldsymbol{\beta} Y_h, a\nabla\psi) + \big(c\big(Y_h - y(U_h)\big), \psi\big)\Big)\,dt + \big(y_0^h(x) - y_0(x), \psi(x,0)\big)\\
&\quad= \int_0^T \Big((Y_{ht}, \psi) + \big(\alpha P_h, a\nabla\psi - \Pi_h(a\nabla\psi)\big) - (\hat{P}_h - P_h, \nabla\psi) - (\operatorname{div}\hat{P}_h, \psi)\Big)\,dt\\
&\qquad+ \int_0^T \Big(\big(\boldsymbol{\beta} Y_h, a\nabla\psi - \Pi_h(a\nabla\psi)\big) + (cY_h - f - U_h, \psi)\Big)\,dt + \big(y_0^h(x) - y_0(x), \psi(x,0)\big)\\
&\quad= \int_0^T (Y_{ht} + \operatorname{div}\hat{P}_h + c\hat{Y}_h - \hat{f} - U_h, \psi)\,dt + \int_0^T \big(\alpha P_h + \boldsymbol{\beta} Y_h, a\nabla\psi - \Pi_h(a\nabla\psi)\big)\,dt\\
&\qquad+ \int_0^T \Big((\hat{f} - f, \psi) + \big(c(Y_h - \hat{Y}_h), \psi\big) + (\hat{P}_h - P_h, \nabla\psi)\Big)\,dt + \big(y_0^h(x) - y_0(x), \psi(x,0)\big)\\
&\quad=: L_1 + L_2 + L_3 + L_4.
\end{aligned} \tag{3.15}$$

Using (2.52), (2.40), the Cauchy inequality, and Lemma 3.1, we have

$$\begin{aligned} L_1 &= \int_0^T (Y_{ht} + \operatorname{div} \hat{P}_h + c\hat{Y}_h - \hat{f} - U_h, \psi - P_h\psi)\,dt \\ &\le C(\delta)\eta_2^2 + \delta\|\psi\|^2_{L^2(J;H^1(\Omega))} \\ &\le C\eta_2^2 + \frac{1}{5}\big\|Y_h - y(U_h)\big\|^2_{L^2(J;L^2(\Omega))}. \end{aligned} \tag{3.16}$$

Similarly, using the Cauchy inequality and Lemma 3.1, we have

$$L_2 \le C\eta_3^2 + \frac{1}{5}\big\|Y_h - y(U_h)\big\|^2_{L^2(J;L^2(\Omega))}, \tag{3.17}$$

$$L_3 \le C\big(\eta_4^2 + \eta_5^2 + \eta_6^2\big) + \frac{1}{5}\big\|Y_h - y(U_h)\big\|^2_{L^2(J;L^2(\Omega))}, \tag{3.18}$$

$$L_4 \le C\eta_7^2 + \frac{1}{5}\big\|Y_h - y(U_h)\big\|^2_{L^2(J;L^2(\Omega))}. \tag{3.19}$$

Hence, using (3.15)-(3.19), we get

$$\big\|Y_h - y(U_h)\big\|^2_{L^2(J;L^2(\Omega))} \le C\sum_{i=2}^{7}\eta_i^2. \tag{3.20}$$

This proves (3.13). □

Theorem 3.3 *Let* $(y,\mathbf{p},z,\mathbf{q},u)$ *and* (Y_h,P_h,Z_h,Q_h,U_h) *be the solutions of* (2.8)-(2.14) *and* (2.39)-(2.45), *respectively. Let* $(y(U_h),\mathbf{p}(U_h),z(U_h),\mathbf{q}(U_h),U_h)$ *be defined as in* (2.46)-(2.51). *Then we have the following error estimate*:

$$\big\|\tilde{Z}_h - z(U_h)\big\|^2_{L^2(J;L^2(\Omega))} \le C\sum_{i=3,6,8-14}\eta_i^2 + C\big\|Y_h - y(U_h)\big\|^2_{L^2(J;L^2(\Omega))}, \tag{3.21}$$

where

$$\eta_8^2 = \int_0^T \sum_\tau h_\tau^2 \int_\tau \big(-Z_{ht} + \operatorname{div}\tilde{Q}_h - \boldsymbol{\beta}\cdot\tilde{Q}_h + c\tilde{Z}_h - \hat{Y}_h + \hat{y}_d\big)^2\,dx\,dt;$$

$$\eta_9^2 = \int_0^T \sum_\tau h_\tau^2 \int_\tau (\alpha Q_h + \bar{P}_h - \bar{\mathbf{p}}_d)^2\,dx\,dt; \qquad \eta_{10}^2 = \|\tilde{Q}_h - Q_h\|^2_{L^2(J;L^2(\Omega))};$$

$$\eta_{11}^2 = \|\bar{P}_h - P_h\|^2_{L^2(J;L^2(\Omega))}; \qquad \eta_{12}^2 = \|\tilde{Z}_h - Z_h\|^2_{L^2(J;L^2(\Omega))};$$

$$\eta_{13}^2 = \|\bar{\mathbf{p}}_d - \mathbf{p}_d\|^2_{L^2(J;L^2(\Omega))}; \qquad \eta_{14}^2 = \|\hat{y}_d - y_d\|^2_{L^2(J;L^2(\Omega))},$$

η_3 *and* η_6 *are defined in Theorem* 3.2.

Proof Similar to (3.14), using (2.33), (2.38), and the definitions of Z_h, Q_h, $\bar{P}_h$, and $\bar{\mathbf{p}}_d$, we get

$$(\alpha Q_h, \mathbf{v}_h) - (Z_h, \operatorname{div}\mathbf{v}_h) = -(\bar{P}_h - \bar{\mathbf{p}}_d, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_h. \tag{3.22}$$

Let ϕ be the solution of (3.10) with $F = Z_h - z(U_h)$. Then it follows from (2.42)-(2.44), (2.49)-(2.51), and (2.54)-(2.56) that

$$\begin{aligned}
&\|Z_h - z(U_h)\|^2_{L^2(J;L^2(\Omega))} \\
&\quad = \int_0^T \big(Z_h - z(U_h), F\big)\,dt \\
&\quad = \int_0^T \big(Z_h - z(U_h), \phi_t - \operatorname{div}(a\nabla\phi + \mathbf{b}\phi) + c\phi\big)\,dt \\
&\quad = \int_0^T \big(\big(-\big(Z_h - z(U_h)\big)_t, \phi\big) - \big(Z_h, \operatorname{div}\big(\Pi_h(a\nabla\phi + \mathbf{b}\phi)\big)\big)\big)\,dt \\
&\qquad + \int_0^T \big(\alpha\mathbf{q}(U_h) + \mathbf{p}(U_h) - \mathbf{p}_d, a\nabla\phi + \mathbf{b}\phi\big)\,dt + \int_0^T \big(c\big(Z_h - z(U_h)\big), \phi\big)\,dt \\
&\quad = \int_0^T \big(\big(-\big(Z_h - z(U_h)\big)_t, \phi\big) - \big(\alpha Q_h + \bar{P}_h - \bar{\mathbf{p}}_d, \Pi_h(a\nabla\phi + \mathbf{b}\phi)\big)\big)\,dt \\
&\qquad + \int_0^T \big(\big(\mathbf{p}(U_h) - \mathbf{p}_d, a\nabla\phi + \mathbf{b}\phi\big) - \big(\operatorname{div}\mathbf{q}(U_h), \phi\big) + \big(\boldsymbol{\beta}\cdot\mathbf{q}(U_h), \phi\big)\big)\,dt \\
&\qquad + \int_0^T \big(c\big(Z_h - z(U_h)\big), \phi\big)\,dt \\
&\quad = \int_0^T \big(\big(-\big(Z_h - z(U_h)\big)_t, \phi\big) + \big(\alpha Q_h + \bar{P}_h - \bar{\mathbf{p}}_d, a\nabla\phi + \mathbf{b}\phi - \Pi_h(a\nabla\phi + \mathbf{b}\phi)\big)\big)\,dt \\
&\qquad + \int_0^T \big(\big(\alpha(\tilde{Q}_h - Q_h) - \alpha\tilde{Q}_h, a\nabla\phi + \mathbf{b}\phi\big) - \big(\operatorname{div}\mathbf{q}(U_h), \phi\big) + \big(\boldsymbol{\beta}\cdot\mathbf{q}(U_h), \phi\big)\big)\,dt \\
&\qquad + \int_0^T \big(c\big(Z_h - z(U_h), \phi\big)\big)\,dt + \int_0^T \big(\mathbf{p}(U_h) - \bar{P}_h + \bar{\mathbf{p}}_d - \mathbf{p}_d, a\nabla\phi + \mathbf{b}\phi\big)\,dt \\
&\quad = \int_0^T (-Z_{ht} + \operatorname{div}\tilde{Q}_h - \boldsymbol{\beta}\cdot\tilde{Q}_h + c\tilde{Z}_h - \hat{Y}_h + \hat{y}_d, \phi)\,dt + \int_0^T \big(c(Z_h - \tilde{Z}_h), \phi\big)\,dt \\
&\qquad + \int_0^T \big(\alpha Q_h + \bar{P}_h - \bar{\mathbf{p}}_d, a\nabla\phi + \mathbf{b}\phi - \Pi_h(a\nabla\phi + \mathbf{b}\phi)\big)\,dt \\
&\qquad + \int_0^T \big(\alpha(\tilde{Q}_h - Q_h), a\nabla\phi + \mathbf{b}\phi\big)\,dt + \int_0^T \big(y_d - \hat{y}_d + \hat{Y}_h - y(U_h), \phi\big)\,dt \\
&\qquad + \int_0^T \big(\mathbf{p}(U_h) - \bar{P}_h + \bar{\mathbf{p}}_d - \mathbf{p}_d, a\nabla\phi + \mathbf{b}\phi\big)\,dt \\
&\quad =: J_1 + J_2 + \cdots + J_6.
\end{aligned} \tag{3.23}$$

First, using the same estimates as (3.16)-(3.19), we have

$$J_1 \le C\eta_8^2 + \frac{1}{8}\|Z_h - z(U_h)\|^2_{L^2(J;L^2(\Omega))}, \tag{3.24}$$

$$J_2 \le C\eta_{12}^2 + \frac{1}{8}\|Z_h - z(U_h)\|^2_{L^2(J;L^2(\Omega))}, \tag{3.25}$$

$$J_3 \le C\eta_9^2 + \frac{1}{8}\|Z_h - z(U_h)\|^2_{L^2(J;L^2(\Omega))}, \tag{3.26}$$

$$J_4 \le C\eta_{10}^2 + \frac{1}{8}\|Z_h - z(U_h)\|^2_{L^2(J;L^2(\Omega))}. \tag{3.27}$$

For J_5, using the Cauchy inequality and Lemma 3.1, we have

$$\begin{aligned} J_5 &= \int_0^T \big(\hat{Y}_h - Y_h + Y_h - y(U_h) + y_d - \hat{y}_d, \phi\big)\,dt \\ &\le C\big(\eta_6^2 + \eta_{14}^2\big) + C\big\|Y_h - y(U_h)\big\|^2_{L^2(J;L^2(\Omega))} + \frac{1}{8}\big\|Z_h - z(U_h)\big\|^2_{L^2(J;L^2(\Omega))}. \end{aligned} \tag{3.28}$$

Finally, for J_6, using (2.39), (2.46), the Cauchy inequality, and Lemma 3.1, we derive

$$\begin{aligned} J_6 &= \int_0^T \big(\mathbf{p}(U_h) - P_h + P_h - \bar{P}_h + \bar{\mathbf{p}}_d - \mathbf{p}_d, a\nabla\phi + \mathbf{b}\phi\big)\,dt \\ &= \int_0^T \big(\alpha\big(\mathbf{p}(U_h) - P_h\big), a^2\nabla\phi + a\mathbf{b}\phi\big)\,dt \\ &\quad + \int_0^T (P_h - \bar{P}_h + \bar{\mathbf{p}}_d - \mathbf{p}_d, a\nabla\phi + \mathbf{b}\phi)\,dt \\ &= \int_0^T \Big(\big(y(U_h), \operatorname{div}\big(a^2\nabla\phi + a\mathbf{b}\phi\big)\big) - \big(\boldsymbol{\beta} y(U_h), a^2\nabla\phi + a\mathbf{b}\phi\big)\Big)\,dt \\ &\quad + \int_0^T \big(\alpha P_h, \Pi_h\big(a^2\nabla\phi + a\mathbf{b}\phi\big) - a^2\nabla\phi - a\mathbf{b}\phi\big)\,dt \\ &\quad + \int_0^T \Big(\big(\boldsymbol{\beta} Y_h, \Pi_h\big(a^2\nabla\phi + a\mathbf{b}\phi\big)\big) - \big(Y_h, \operatorname{div}\big(\Pi_h\big(a^2\nabla\phi + a\mathbf{b}\phi\big)\big)\big)\Big)\,dt \\ &\quad + \int_0^T (P_h - \bar{P}_h + \bar{\mathbf{p}}_d - \mathbf{p}_d, a\nabla\phi + \mathbf{b}\phi)\,dt \\ &= \int_0^T \Big(\big(y(U_h) - Y_h, \operatorname{div}\big(a^2\nabla\phi + a\mathbf{b}\phi\big)\big) + \big(\boldsymbol{\beta}\big(Y_h - y(U_h)\big), a^2\nabla\phi + a\mathbf{b}\phi\big)\Big)\,dt \\ &\quad + \int_0^T \big(\alpha P_h + \boldsymbol{\beta} Y_h, \Pi_h\big(a^2\nabla\phi + a\mathbf{b}\phi\big) - a^2\nabla\phi - a\mathbf{b}\phi\big)\,dt \\ &\quad + \int_0^T (P_h - \bar{P}_h + \bar{\mathbf{p}}_d - \mathbf{p}_d, a\nabla\phi + \mathbf{b}\phi)\,dt \\ &\le C\big(\eta_3^2 + \eta_{11}^2 + \eta_{13}^2\big) + C\big\|Y_h - y(U_h)\big\|^2_{L^2(J;L^2(\Omega))} + \frac{1}{8}\big\|Z_h - z(U_h)\big\|^2_{L^2(J;L^2(\Omega))}. \end{aligned} \tag{3.29}$$

Therefore, it follows from the above estimates that

$$\big\|Z_h - z(U_h)\big\|^2_{L^2(J;L^2(\Omega))} \le C \sum_{i=3,6,8-14} \eta_i^2 + C\big\|Y_h - y(U_h)\big\|^2_{L^2(J;L^2(\Omega))}. \tag{3.30}$$

The triangle inequality and (3.30) yield (3.21). □

Remark 3.1 If we use the higher order RT mixed finite elements to approximate the state variables and the co-state variables, then the estimators η_2^2, η_3^2, η_8^2, and η_9^2 in Theorem 3.2 and Theorem 3.3 can be improved by

$$\eta_2^2 = \int_0^T \sum_\tau h_\tau^4 \int_\tau (Y_{ht} + \operatorname{div}\hat{P}_h + c\hat{Y}_h - \hat{f} - U_h)^2\,dx\,dt;$$

$$\eta_3^2 = \int_0^T \sum_\tau h_\tau^2 \int_\tau (\alpha P_h + \nabla_h Y_h + \boldsymbol{\beta} Y_h)^2\,dx\,dt;$$

$$\eta_8^2 = \int_0^T \sum_\tau h_\tau^4 \int_\tau (-Z_{ht} + \operatorname{div} \tilde{Q}_h - \boldsymbol{\beta} \cdot \tilde{Q}_h + c\tilde{Z}_h - \hat{Y}_h + \hat{y}_d)^2 \, dx \, dt;$$

$$\eta_9^2 = \int_0^T \sum_\tau h_\tau^2 \int_\tau (\alpha Q_h + \nabla_h Z_h + \bar{P}_h - \bar{\mathbf{p}}_d)^2 \, dx \, dt,$$

where $\nabla_h \chi|_\tau = \nabla(\chi|_\tau)$.

Let $(\mathbf{p}, y, \mathbf{q}, z, u)$ and $(P_h, Y_h, Q_h, Z_h, U_h)$ be the solutions of (2.8)-(2.14) and (2.39)-(2.45), respectively. We decompose the errors as follows:

$$\mathbf{p} - P_h = \mathbf{p} - \mathbf{p}(U_h) + \mathbf{p}(U_h) - P_h := \epsilon_1 + \varepsilon_1,$$

$$y - Y_h = y - y(U_h) + y(U_h) - Y_h := r_1 + e_1,$$

$$\mathbf{q} - Q_h = \mathbf{q} - \mathbf{q}(U_h) + \mathbf{q}(U_h) - Q_h := \epsilon_2 + \varepsilon_2,$$

$$z - Z_h = z - z(U_h) + z(U_h) - Z_h := r_2 + e_2.$$

From (2.8)-(2.13) and (2.46)-(2.51), we derive the error equations:

$$(\alpha\epsilon_1, \mathbf{v}) - (r_1, \operatorname{div} \mathbf{v}) + (\boldsymbol{\beta} r_1, \mathbf{v}) = 0, \quad \forall \mathbf{v} \in \mathbf{V}, \tag{3.31}$$

$$(r_{1t}, w) + (\operatorname{div} \epsilon_1, w) + (cr_1, w) = (u - U_h, w), \quad \forall w \in W, \tag{3.32}$$

$$(\alpha\epsilon_2, \mathbf{v}) - (r_2, \operatorname{div} \mathbf{v}) = -(\epsilon_1, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}, \tag{3.33}$$

$$-(r_{2t}, w) + (\operatorname{div} \epsilon_2, w) - (\boldsymbol{\beta} \cdot \epsilon_2, w) + (cr_2, w) = (r_1, w), \quad \forall w \in W. \tag{3.34}$$

Theorem 3.4 *There is a constant $C > 0$, independent of h, such that*

$$\|\epsilon_1\|_{L^2(J;L^2(\Omega))} + \|r_1\|_{L^2(J;L^2(\Omega))} \leq C\|u - U_h\|_{L^2(J;L^2(\Omega))}, \tag{3.35}$$

$$\|\epsilon_2\|_{L^2(J;L^2(\Omega))} + \|r_2\|_{L^2(J;L^2(\Omega))} \leq C\|u - U_h\|_{L^2(J;L^2(\Omega))}. \tag{3.36}$$

Proof Choosing $\mathbf{v} = \epsilon_1$ and $w = r_1$ as the test functions and add the two relations of (3.31)-(3.32), we have

$$(\alpha\epsilon_1, \epsilon_1) + (r_{1t}, r_1) = (u - U_h, r_1) - (\boldsymbol{\beta} r_1, \epsilon_1) - (cr_1, r_1). \tag{3.37}$$

Then, using the ϵ-Cauchy inequality, we can find an estimate as follows:

$$(a\epsilon_1, \epsilon_1) + (r_{1t}, r_1) \leq C\bigl(\|r_1\|_{L^2(\Omega)}^2 + \|u - U_h\|_{L^2(\Omega)}^2\bigr) + \frac{1}{2}(a\epsilon_1, \epsilon_1). \tag{3.38}$$

Note that

$$(r_{1t}, r_1) = \frac{1}{2}\frac{\partial}{\partial t}\|r_1\|_{L^2(\Omega)}^2,$$

then, using the assumption on a, we can obtain

$$\frac{1}{2}a_0\|\epsilon_1\|_{L^2(\Omega)}^2 + \frac{1}{2}\frac{\partial}{\partial t}\|r_1\|_{L^2(\Omega)}^2 \leq C\bigl(\|r_1\|_{L^2(\Omega)}^2 + \|u - U_h\|_{L^2(\Omega)}^2\bigr). \tag{3.39}$$

Integrating (3.39) in time, and since $r_1(0) = 0$, using Lemma 3.2 to get

$$\|\epsilon_1\|^2_{L^2(J;L^2(\Omega))} + \|r_1\|^2_{L^\infty(J;L^2(\Omega))} \leq C\|u - U_h\|^2_{L^2(J;L^2(\Omega))}, \tag{3.40}$$

implies (3.35).

Similarly, we can obtain

$$\|\epsilon_2\|^2_{L^2(J;L^2(\Omega))} + \|r_2\|^2_{L^\infty(J;L^2(\Omega))} \leq C\big(\|\epsilon_1\|^2_{L^2(J;L^2(\Omega))} + \|r_1\|^2_{L^2(J;L^2(\Omega))}\big). \tag{3.41}$$

Using (3.41) and (3.35), we complete the proof of Theorem 3.4. □

Collecting Theorems 3.1-3.4, we can derive the following results.

Theorem 3.5 *Let* $(\mathbf{p}, y, \mathbf{q}, z, u)$ *and* $(P_h, Y_h, Q_h, Z_h, U_h)$ *be the solutions of* (2.8)-(2.14) *and* (2.39)-(2.45), *respectively. Then we have*

$$\|u - U_h\|^2_{L^2(J;L^2(\Omega))} + \|y - Y_h\|^2_{L^2(J;L^2(\Omega))} + \|z - Z_h\|^2_{L^2(J;L^2(\Omega))} \leq C\sum_{i=1}^{14}\eta_i^2, \tag{3.42}$$

where η_1 *is defined in Theorem* 3.1, $\eta_2, \ldots, \eta_7$ *are defined in Theorem* 3.2, *and* $\eta_8, \ldots, \eta_{14}$ *are defined in Theorems* 3.3, *respectively.*

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The first author carried out the molecular genetic studies, participated in the sequence alignment, and drafted the manuscript. The second author conceived of the study, and participated in its design and coordination and helped to draft the manuscript. All authors read and approved the final manuscript.

Acknowledgements

The first author is supported by the scientific research program in Hunan University of Science and Engineering (2015). The second author is supported by the National Natural Science Foundation of China (11401201), the Foundation of Hunan Educational Committee (13C338), and the construct program of the key discipline in Hunan University of Science and Engineering.

Received: 7 February 2015 Accepted: 13 August 2015 Published online: 16 September 2015

References

1. Lions, JL: Optimal Control of Systems Governed by Partial Differential Equations. Springer, Berlin (1971)
2. Tröltzsch, F: Semidiscrete Ritz-Galerkin approximation of nonlinear parabolic boundary control problems-strong convergence of optimal control. Appl. Math. Optim. **29**, 309-329 (1994)
3. Hou, L, Turner, JC: Analysis and finite element approximation of an optimal control problem in electrochemistry with current density controls. Numer. Math. **71**, 289-315 (1995)
4. Knowles, G: Finite element approximation of parabolic time optimal control problems. SIAM J. Control Optim. **20**, 414-427 (1982)
5. Mcknight, R, Bosarge, W Jr.: The Ritz-Galerkin procedure for parabolic control problems. SIAM J. Control Optim. **11**, 510-524 (1973)
6. Arada, N, Casas, E, Tröltzsch, F: Error estimates for the numerical approximation of a semilinear elliptic control problem. Comput. Optim. Appl. **23**, 201-229 (2002)
7. Tiba, D: Lectures on the Optimal Control of Elliptic Problems. University of Jyvaskyla Press, Jyvaskyla (1995)
8. Neittaanmaki, P, Tiba, D: Optimal Control of Nonlinear Parabolic Systems: Theory, Algorithms and Applications. Dekker, New York (1994)
9. Haslinger, J, Neittaanmaki, P: Finite Element Approximation for Optimal Shape Design. Wiley, Chichester (1989)
10. Becker, R, Kapp, H, Rannacher, R: Adaptive finite element methods for optimal control of partial differential equations: basic concept. SIAM J. Control Optim. **39**, 113-132 (2000)
11. Brunner, H, Yan, N: Finite element methods for optimal control problems governed by integral equations and integro-differential equations. Numer. Math. **101**, 1-27 (2005)
12. Li, R, Liu, W, Ma, H, Tang, T: Adaptive finite element approximation of elliptic control problems. SIAM J. Control Optim. **41**, 1321-1349 (2002)

13. Liu, W, Ma, H, Tang, T, Yan, N: *A posteriori* error estimates for discontinuous Galerkin time-stepping method for optimal control problems governed by parabolic equations. SIAM J. Numer. Anal. **42**, 1032-1061 (2004)
14. Liu, W, Yan, N: *A posteriori* error analysis for convex distributed optimal control problems. Adv. Comput. Math. **15**, 285-309 (2001)
15. Liu, W, Yan, N: *A posteriori* error estimates for optimal control problems governed by Stokes equations. SIAM J. Numer. Anal. **40**, 1850-1869 (2003)
16. Liu, W, Yan, N: *A posteriori* error estimates for optimal control problems governed by parabolic equations. Numer. Math. **93**, 497-521 (2003)
17. Hoppe, RHW, Iliash, Y, Iyyunni, C, Sweilam, NH: *A posteriori* error estimates for adaptive finite element discretizations of boundary control problems. J. Numer. Math. **14**, 57-82 (2006)
18. Liu, W, Yan, N: *A posteriori* error estimates for convex boundary control problems. SIAM J. Numer. Anal. **39**, 73-99 (2001)
19. Gong, W, Yan, N: *A posteriori* error estimate for boundary control problems governed by the parabolic partial differential equations. J. Comput. Math. **27**, 68-88 (2009)
20. Chen, Y: Superconvergence of quadratic optimal control problems by triangular mixed finite elements. Int. J. Numer. Methods Eng. **75**, 881-898 (2008)
21. Chen, Y, Huang, Y, Liu, WB, Yan, NN: Error estimates and superconvergence of mixed finite element methods for convex optimal control problems. J. Sci. Comput. **42**, 382-403 (2009)
22. Chen, Y, Liu, WB: *A posteriori* error estimates for mixed finite element solutions of convex optimal control problems. J. Comput. Appl. Math. **211**, 76-89 (2008)
23. Hou, T: *A posteriori* $L^\infty(L^2)$-error estimates of semidiscrete mixed finite element methods for hyperbolic optimal control problems. Bull. Korean Math. Soc. **50**(1), 321-341 (2013)
24. Brezzi, F, Fortin, M: Mixed and Hybrid Finite Element Methods. Springer Series in Computational Mathematics, vol. 15, pp. 65-187. Springer, Berlin (1991)
25. Babuska, I, Strouboulis, T: The Finite Element Method and Its Reliability. Oxford University Press, Oxford (2001)
26. Carstensen, C: *A posteriori* error estimate for the mixed finite element method. Math. Comput. **66**, 465-476 (1997)
27. Houston, P, Süli, E: *A posteriori* error analysis for linear convection-diffusion problems under weak mesh regularity assumptions. Technical report NA97/03, Oxford University Computing Laboratory, Wolfson Building, Parks Road, Oxford OX1 3QD (1997)
28. Houston, P, Süli, E: Adaptive Lagrange-Galerkin methods for unsteady convection-diffusion problems. Math. Comput. **70**, 77-106 (2000)
29. Thomée, V: Galerkin Finite Element Methods for Parabolic Problems. Springer, Berlin (1997)