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On the completeness and the basis property of the modified Frankl problem with a nonlocal oddness condition in the Sobolev space $(W_p^1(0,\pi))$

Naser Abbasi* and Mahmood Shakori

*Correspondence:
naserabbasi_persia@yahoo.com
Department of Mathematics,
Lorestan University, P.O. Box 465,
Khoramabad, Iran

Abstract

In the present paper, we write out the eigenvalues and the corresponding eigenfunctions of the modified Frankl problem with a nonlocal oddness condition of the first kind in the Sobolev space $(W_p^1(0,\pi))$. We analyze the completeness, the basis property, and the minimality of the eigenfunctions in the space $(W_p^1(0,\pi))$.

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Keywords: Frankl problem; Lebesgue integral; Hölder inequality; Bessel equation; Sobolev space

1 Introduction

The classical Frankl problem was considered in [1]. The problem was further developed in [2], pp.339-345, [3], pp.235-252. The modified Frankl problem with a nonlocal boundary condition of the first kind was studied in [4, 5]. The basis property of eigenfunctions of the Frankl problem with nonlocal parity conditions in the Sobolev space was studied in [4]. The coefficients β are found by Theorem 1 in [6], using the results of [6], pp.177-179. In the present paper, we write out the eigenvalues and the corresponding eigenfunctions of the modified Frankl problem with a nonlocal oddness condition of the first kind. We analyze the completeness, the basis property, and the minimality of the eigenfunctions in the space $(W_p^1(0,\pi))$, where $(W_p^1(0,\pi))$ is the space of absolutely continuous functions on $[0,\pi]$. So we can obtain new results by the expansion into cosines that are related to new coefficients which we calculated. This analysis and results may be of interest in itself.

2 Statement of the modified Frankl problem

Definition 2.1 In the domain $D = (D_+ \cup D_{-1} \cup D_{-2})$, we seek a solution of the modified generalized Frankl problem

$$u_{xx} + \operatorname{sgn}(y)u_{yy} + \mu^2 \operatorname{sgn}(x+y)u = 0 \quad \text{in } (D_+ \cup D_{-1} \cup D_{-2}), \tag{1}$$

with the boundary conditions

$$u(1,\theta) = 0, \quad \theta \in \left[0, \frac{\pi}{2}\right], \tag{2}$$

$$\frac{\partial u}{\partial x}(0,y) = 0, \quad y \in (-1,0) \cup (0,1), \tag{3}$$

$$u(0,y) = u(0,-y), \quad y \in [0,1], \tag{4}$$

where $u(x,y)$ is a regular solution in the class

$$u \in C^0(\overline{D_+ \cup D_{-1} \cup D_{-2}}) \cap C^2(\overline{D_{-1}}) \cap C^2(\overline{D_{-2}}),$$

and where

$$\begin{aligned}
&D_+ = \left\{(r,\theta) : 0 < r < 1, 0 < \theta < \frac{\pi}{2}\right\},\\
&D_{-1} = \left\{(x,y) : -y < x < y+1, \frac{-1}{2} < y < 0\right\},\\
&D_{-2} = \left\{(x,y) : x-1 < y < -x, 0 < x < \frac{1}{2}\right\},\\
&\kappa \frac{\partial u}{\partial y}(x,+0) = \frac{\partial u}{\partial y}(x,-0), \quad -\infty < \kappa < \infty, 0 < x < 1.
\end{aligned} \tag{5}$$

Theorem 2.2 ([7]) *The eigenvalues and eigenfunctions of problem* (1)-(5) *can be written out in two series. In the first series, the eigenvalues* $\lambda = \mu_{nk}^2$ *are found from the equation*

$$J_{4n}(\mu_{nk}) = 0, \tag{6}$$

where μ_{nk}, $n = 0,1,2,\ldots$, $k = 1,2,\ldots$, *are roots of the Bessel equation* (6), $J_\alpha(z)$ *is the Bessel function* [8], *and the eigenfunctions are given by the formula*

$$u_{nk} = \begin{cases} A_{nk}J_{4n}(\mu_{nk}r)\cos(4n)(\frac{\pi}{2} - \theta) & \text{in } D^+;\\ A_{nk}J_{4n}(\mu_{nk}\rho)\cosh(4n)\psi & \text{in } D_{-1};\\ A_{nk}J_{4n}(\mu_{nk}R)\cosh(4n)\varphi & \text{in } D_{-2}, \end{cases} \tag{7}$$

where $x = r\cos\theta$, $y = r\sin\theta$ *for* $0 \le \theta \le \frac{\pi}{2}$, $0 < r < 1$, *and* $r^2 = x^2 + y^2$ *in* D_+, $x = \rho\cosh\psi$, $y = \rho\sinh\psi$ *for* $0 < \rho < 1$, $-\infty < \psi < 0$, $\rho^2 = x^2 - y^2$ *in* D_{-1} *and* $x = R\sinh\varphi$, $y = -R\cosh\varphi$ *for* $0 < \varphi < +\infty$, $R^2 = y^2 - x^2$ *in* D_{-2}.

In the second series, the eigenvalues $\tilde{\lambda} = \tilde{\mu}_{nk}^2$ *are found from the equation*

$$J_{4(n-\Delta)}(\tilde{\mu}_{nk}) = 0, \tag{8}$$

where $n = 1, 2, \ldots$, *and* $k = 1, 2, \ldots$, *and* $(\tilde{\mu}_{nk})$ *are the roots of the Bessel equation* (8).

$$\tilde{u}_{nk} = \begin{cases} \tilde{A}_{nk} J_{4(n-\Delta)}(\tilde{\mu}_{nk} r) \cos 4(n-\Delta)(\frac{\pi}{2} - \theta) & in\ D^+; \\ \tilde{A}_{nk} J_{4(n-\Delta)}(\tilde{\mu}_{nk} \rho)[\cosh 4(n-\Delta)\varphi \cos 4(n-\Delta)\frac{\pi}{2} & \\ \quad + \kappa \sinh 4(n-\Delta)\psi \cos 4(n-\Delta)] & in\ D_{-1}; \\ \tilde{A}_{nk} J_{4(n-\Delta)}(\tilde{\mu}_{nk} R) \cosh 4(n-\Delta)\varphi [\cos 4(n-\Delta)\frac{\pi}{2} & \\ \quad - \kappa \sin 4(n-\Delta)\frac{\pi}{2}] & in\ D_{-2}, \end{cases} \tag{9}$$

where $\Delta = \frac{1}{\pi} \arcsin \frac{\kappa}{\sqrt{1+\kappa^2}}$, $\Delta \in (0, \frac{1}{2})$, *and*

$$A_{nk}^2 \int_0^1 J_{4n}^2(\mu_{nk} r) r\, dr = 1,$$

$$\tilde{A}_{nk}^2 \int_0^1 J_{4n-1}^2(\tilde{\mu}_{nk} r) r\, dr = 1,$$

$A_{nk} > 0$ *and* $\tilde{A}_{nk} > 0$.

Theorem 2.3 (see [5]) *The function system*

$$\left\{ \cos(4n)\left(\frac{\pi}{2} - \theta\right) \right\}_{n=0}^{\infty}, \qquad \left\{ \cos 4(n-\Delta)\left(\frac{\pi}{2} - \theta\right) \right\}_{n=1}^{\infty} \tag{10}$$

is a Riesz basis in $L_2(0, \frac{\pi}{2})$ *provided that* $\Delta \in (0, \frac{3}{4})$.

3 The completeness, the basis property and minimality of the eigenfunctions

Theorem 3.1 *The system of functions* $\{\cos(n - \frac{\beta}{2})\theta\}_{n=0}^{\infty}$ *is a Riesz basis in* $(W_p^1(0,\pi))$ *if and only if* $\beta \in (-\frac{1}{p}, 2 - \frac{1}{p})$, $\beta \neq 1$.

Proof Using the formula (20) of [9], we have the relation

$$f(\theta) = \sum_{n=1}^{\infty} B_n \cos\left(n - \frac{\beta}{2}\right)\theta + B_0, \tag{11}$$

where

$$B_n = -\int_0^{\pi} f'(\theta) h_n^{\beta}\, d\theta \left(n - \frac{\beta}{2}\right)^{-1} \quad (n = 1, 2, \ldots). \tag{12}$$

The coefficient B_0 depends on B_n (see [9]). Consider the formally differentiated series (11)

$$\sum_{n=1}^{\infty} B_n \left(n - \frac{\beta}{2}\right) \sin\left(n - \frac{\beta}{2}\right)\theta. \tag{13}$$

Since the coefficient B_n is found by formula (12), using the results of [7], we obtain that series (11) converges to $f'(\theta)$ in the space $L_p(0,\pi)$. Integrating series (11) from 0 to θ, we obtain the relation

$$f(\theta) - f(0) = \sum_{n=1}^{\infty} B_n \cos\left(n - \frac{\beta}{2}\right)\theta - \sum_{n=1}^{\infty} B_n, \tag{14}$$

which has a meaning if the following series converges

$$\sum_{n=1}^{\infty} B_n. \tag{15}$$

By using the results of [9], we obtain that the numerical series (15) converges and relation (11) uniformly converges on $[0,\pi]$, and therefore it converges in the space $L_p(0,\pi)$. Now we assume that

$$B_0 = f(0) - \sum_{n=1}^{\infty} B_n.$$

Then expression (14) coincides with expression (11), and therefore series (11) converges to a function in the space $(W_p^1(0,\pi))$.

Now let us show that the coefficients B_n are uniquely found by using relation (11). Indeed, if series (11) converges in the space $(W_p^1(0,\pi))$, then series (15) converges in the space $L_p(0,\pi)$ (see [9]), this implies that $\lim_{n\to\infty} B_n = 0$. For $\beta \in (-\frac{1}{p}, 2-\frac{1}{p})$. Now let us show that the system $\{\cos(n-\frac{\beta}{2})\theta, 1\}_{n=1}^{\infty}$ does not compose a basis for $\beta \notin (-\frac{1}{p}, 2-\frac{1}{p})$. If $\beta \in (2-\frac{1}{p}, 4-\frac{1}{p})$, then, using the substitution $\beta - 2 = \beta'$ and removing the first cosine, we obtain the cosine system $\{\cos(n-\frac{\beta'}{2})\theta_{n=1}^{\infty}, 1\}$, which, as was proved above, composes a basis in $(W_p^1(0,\pi))$, and therefore the initial cosine system is not minimal in $(W_p^1(0,\pi))$. Analogously, for $\beta \in (-2-\frac{1}{p}, -\frac{1}{p})$, the substitution $\beta + 2 = \beta'$ reduces the initial cosine system to the system with $\beta' \in (-\frac{1}{p}, 2-\frac{1}{p})$, in which there is no function $(\cos(1-\frac{\beta'}{2})\theta)$, and therefore the initial cosine system is not complete. Other ranges of the parameter $\beta \in (-\frac{1}{p}+2k, 2-\frac{1}{p}+2k)$, $k = \pm 1, \pm 2, \ldots$, can be considered analogously. Furthermore, for $\beta = 2-\frac{1}{p}$ in the space $(W_p^1(0,\pi))$, where $\hat{p} > p$, we have $-\frac{1}{\hat{p}} < \beta < 2-\frac{1}{\hat{p}}$, and therefore the cosine system composes a basis in $W_{\hat{p}}^1(0,\pi)$, and hence it is complete in the space $(W_p^1(0,\pi))$.

For $\beta = -\frac{1}{p}$, the cosine system is minimal since, as was proved above, the coefficients B_n are found by concrete formulas in the form of an integral. Let us show that for $\beta = 2-\frac{1}{p}$, the cosine system is not minimal. By using the results of [7], we obtain that for $\beta = 2-\frac{1}{p}$, the cosine system is complete but not minimal, and hence, for $\beta = -\frac{1}{p}$, the cosine system is complete (since it is minimal in this case). Now let us prove that for $\beta = -\frac{1}{p}$, the cosine system does not compose a basis. Let $f(\theta) = \theta$, then $f(\theta) \in (W_p^1(0,\pi))$, $f'(\theta) = 1$, and the coefficients B_n can be calculated by using formula (12) exactly in the same way as in [7], where it was shown that a series converges to a function not belonging to $L_p(0,\pi)$, thus Theorem 3.1 is proved. □

Theorem 3.2 *The cosine system* $\{\cos(n-\frac{\beta}{2})\theta\}_{n=0}^{\infty}$ *forms a basis in the space* $(W_p^1(0,\pi))$ *if and only if* $\beta \in (-\frac{1}{p}, 2-\frac{1}{p})$, $\beta \neq 1$. *The expansion into cosines has the form*

$$f(\theta) = \sum_{n=0}^{\infty} D_n \cos\left(n - \frac{\beta}{2}\right)\theta,$$

where the coefficients D_n *are calculated according to the formulas*

$$D_0 = \int_0^{\pi} f(\theta) H_0^{\beta}(\theta)\, d(\theta) \tag{16}$$

for $\beta < 1$ *and*

$$D_0 = \frac{8(1-\beta)}{\pi\beta(2-\beta)}\int_0^\pi \frac{\sin(\theta)\sin(\frac{\beta\theta}{2})}{(2\cos\frac{\theta}{2})^\beta}\,d(\theta)$$
$$= \int_0^\pi f(\theta)H_0^\beta(\theta)\,d(\theta) + \int_0^\pi \frac{f'(\theta)h_1^\beta}{1-\frac{\beta}{2}}\,d(\theta) \tag{17}$$

for $\beta > 1$ *and for all* $n \in N$, D_n *is given by*

$$D_n = -\int_0^\pi \left(f' + D_0\left(\frac{\beta}{2}\right)\sin\left(\frac{\beta\theta}{2}\right)\right)h_n^\beta\,d(\theta)\left(n-\frac{\beta}{2}\right)^{-1}, \tag{18}$$

where H_n^β *and* $h_n^\beta(\theta)$ *were studied in* [10].

Proof Analogously to the proof of relation (14), we obtain the relation

$$f(\theta) - f(0) = \sum_{n=0}^\infty D_n\cos\left(n-\frac{\beta}{2}\right)\theta - \sum_{n=1}^\infty D_n. \tag{19}$$

The convergence of numerical series $\sum_{n=0}^\infty D_n$ is proved analogously to the proof of the convergence of series $\sum_{n=1}^\infty B_n$. This implies the uniform convergence of series (19).

First let $\beta < 1$, then multiply series (19) by H_0^β. Integrating over the closed interval $[0,\pi]$ and taking into account relations (6) of [9] and (16) or (17), we have the relation

$$f(0) = \sum_{n=0}^\infty D_n.$$

Therefore, instead of the relation, we can write

$$f(\theta) = \sum_{n=0}^\infty D_n\cos\left(n-\frac{\beta}{2}\right)\theta. \tag{20}$$

For $\beta > 0$, we multiply series (19) by $H_0^{\beta-2}(\theta)$ and integrate the obtained relation over the closed interval $[0,\pi]$. Using relation (9) of [9], we obtain

$$\int_0^\pi f(\theta)H_0^{\beta-2}(\theta)\,d(\theta) = D_0\int_0^\pi \cos\frac{\beta\theta}{2}H_0^{\beta-2}(\theta)\,d(\theta) + D_1$$
$$+ \left(f(0) - \sum_{n=0}^\infty D_n\right)\int_0^\pi H_0^{\beta-2}(\theta)\,d(\theta).$$

Substituting the expression for D_1 from (18) in the latter relation, we obtain

$$\int_0^\pi f(\theta)H_0^{\beta-2}(\theta)\,d(\theta) - D_0\int_0^\pi \cos\frac{\beta\theta}{2}H_0^{\beta-2}(\theta)\,d(\theta)$$
$$+ \int_0^\pi f'(\theta)h_1^\beta(\theta)\,d(\theta)\frac{1}{1-\frac{\beta}{2}}$$

$$
\begin{aligned}
&+ D_0 \int_0^\pi \sin\frac{\beta\theta}{2} h_1^\beta(\theta)\,d(\theta) \frac{\beta}{2(1-\frac{\beta}{2})} \\
&= \left(f(0) - \sum_{n=0}^{\infty} D_n\right) \int_0^\pi H_0^{\beta-2}(\theta)\,d(\theta). \qquad (21)
\end{aligned}
$$

Now let us show that the left-hand side of relation (21) vanishes, this will imply

$$
f(0) = \sum_{n=0}^{\infty} D_n.
$$

Indeed, integrating relation (9) of [9] by parts, we obtain the relation

$$
\frac{\beta}{2(1-\frac{\beta}{2})} \int_0^\pi H_0^{\beta-2}(\theta) \cos\frac{\beta\theta}{2}\,d(\theta) = \left(1 - \frac{\beta}{2}\right)\frac{2}{\beta} \int_0^\pi \sin\frac{\beta\theta}{2} h_1^\beta(\theta)\,d(\theta).
$$

Furthermore, substituting this formula in (21), we immediately see that

$$
\begin{aligned}
&\int_0^\pi \left(f(\theta) H_0^{\beta-2}(\theta) + \frac{f'(\theta) h_1^\beta(\theta)}{1-\frac{\beta}{2}}\right) d(\theta) \\
&\quad + D_0 \int_0^\pi \sin\frac{\beta\theta}{2} h_1^\beta(\theta)\,d(\theta) \left(\frac{2}{2-\beta} - \frac{2-\beta}{\beta}\right) \\
&= \int_0^\pi \left(f(\theta) H_0^{\beta-2}(\theta) + \frac{f'(\theta) h_1^\beta(\theta)}{1-\frac{\beta}{2}}\right) d(\theta) \\
&\quad + \left(\frac{4D_0(\beta-1)}{\beta(2-\beta)}\right) \int_0^\pi \sin\frac{\beta\theta}{2} h_1^\beta(\theta)\,d(\theta).
\end{aligned}
$$

By using relations (16) (or (17)) and (9) of [9], we annihilate the latter relation, *i.e.*, we obtain relation (20) for $\beta > 1$. The remaining part of Theorem 3.2 is proved analogously to Theorem 3.1. □

Remark 3.3 In case $\kappa > 0$. The system of functions (10) is a Riesz basis in $(W_p^1(0,\pi))$ if $\Delta \in (\frac{-1}{4}, 0) \cup (0, \frac{3}{4})$.

If $\Delta \geq \frac{3}{4}$, $\Delta \neq 1, 2, 3, \ldots$, then system (10) is complete but is not minimal in $(W_p^1(0,\pi))$.

If $\Delta = \frac{-1}{4}$, then system (10) is complete and minimal but is not basis in $(W_p^1(0,\pi))$.

If $\Delta < \frac{-1}{4}$, $\Delta \neq 1, 2, 3, \ldots$, then system (10) is not complete but is minimal in $(W_p^1(0,\pi))$.

Proof The proof of Remark 3.3 reproduces that of Theorem 3.1 and Theorem 3.2. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The research and writing of this manuscript was a collaborative effort of all the authors. All authors read and approved the final manuscript.

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