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Approximate moments of extremes

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Abstract

Let $M_{n,i}$ be the i th largest of a random sample of size n from a cumulative distribution function F on $\mathbb{R} = (-\infty, \infty)$. Fix $r \geq 1$ and let $\mathbf{M}_n = (M_{n,1}, \dots, M_{n,r})'$. If there exist b_n and $c_n > 0$ such that as $n \rightarrow \infty$, $(M_{n,1} - b_n)/c_n \xrightarrow{L} Y_1 \sim G$ say, a non-degenerate distribution, then as $n \rightarrow \infty$, $\mathbf{Y}_n = (\mathbf{M}_n - b_n \mathbf{1}_r)/c_n \xrightarrow{L} \mathbf{Y}$, where for $Z_i = -\log G(Y_i)$, $\mathbf{Z} = (Z_1, \dots, Z_r)'$ has joint probability density function $\exp(-z_r)$ on $0 < z_1 < \dots < z_r < \infty$ and $\mathbf{1}_r$ is the r -vector of ones. The moments of \mathbf{Y} are given for the three possible forms of G . First approximations for the moments of \mathbf{M}_n are obtained when these exist.

Keywords: asymptotic distributions; extremes; moments

1 Introduction

For $1 \leq i \leq n$ let $M_{n,i}$ be the i th largest of a random sample of size n from a cumulative distribution function F on \mathbb{R} . Fix $r \geq 1$ and let $\mathbf{M}_n = (M_{n,1}, \dots, M_{n,r})'$ and $\mathbf{Y}_n = (Y_{n,1}, \dots, Y_{n,r})'$, where

$$Y_{n,i} = (M_{n,i} - b_{n,i})/c_{n,i}, \quad (1)$$

and $b_{n,i}, c_{n,i} > 0$ are constants. There are three possible non-degenerate limits in distribution for \mathbf{Y}_n , say $\mathbf{Y} = \mathbf{Y}_I = (Y_{I,1}, \dots, Y_{I,r})'$ for $I = 1, 2, 3$. We take $b_{n,i} \equiv b_{n,1} = b_n$ say and $c_{n,i} \equiv c_{n,1} = c_n$ say. The notation \mathbf{Y}_I conflicts with \mathbf{Y}_n but is simple and clear as dependence on n is always indicated by a subscript n .

The need for approximations for the moments of \mathbf{M}_n based on the moments of \mathbf{Y} arises in many applied areas. For example, in solutions of stochastic traveling salesman problems (Leipala [1]); modeling fire protection and insurance problems (Ramachandran [2]); modeling extreme wind gusts (Revfeim and Hessell [3]); determining failures of jack-ups under environmental loading (van de Graaf *et al.* [4]); and determining the asymptotic cost of algorithms and combinatorial structures such as trie, digital search tree, leader election, adaptive sampling, counting algorithms, trees related to the register function, composition of integers, some structures represented by Markov chains (column-convex polyominoes, Carlitz compositions), runs and number of distinct values of some multiplicity-in-sequences of geometrically distributed random variables (Louchard and Prodinger [5]).

The aim of this note is to provide approximations for the moments of \mathbf{M}_n . Most of the results presented are new to the best of our knowledge. The results are organized as follows. In Section 2, we briefly review the possible forms for the non-degenerate

limit $\mathbf{Y}_I = (Y_{I,1}, \dots, Y_{I,r})'$ for $I = 1, 2, 3$. For $r = 1$ these forms are well known. For example, $P(Y_{1,1} \leq x) = \exp\{-\exp(-x)\}$ on \mathbb{R} . In Section 3, we give the moments $m_j(\mathbf{Y}) = \mathbb{E}[\mathbf{Y}^j]$ and $\mu_j(\mathbf{Y}) = \mathbb{E}\{[\mathbf{Y} - \mathbb{E}(\mathbf{Y})]^j\}$ for each of these forms. Here, $\mathbf{y}^j = \prod_{i=1}^r y_i^{j_i}$. For example, letting $\Gamma(\cdot)$ denote the gamma function and $\psi(z) = (d/dz) \log \Gamma(z)$ the digamma function, we show that

$$\mathbb{E}[Y_{I,r}^j] = \begin{cases} (-1)^j \Gamma^{(j)}(r)/(r-1)!, & I = 1, \\ M_r(-j\alpha^{-1}) \text{ for } r > j\alpha^{-1}, & I = 2, \\ (-1)^j M_r(j\alpha^{-1}), & I = 3, \end{cases}$$

where

$$M_r(t) = \Gamma(r+t)/(r-1)! \tag{2}$$

for $r + \Re(e(t)) > 0$, $\Re(z)$ denoting the real part of z , and $\omega^{(j)}(\cdot)$ denoting $\omega^{(j)}(t) = d^j \omega(t)/dt^j$. We also show that for $1 \leq s \leq r$, the covariance between $Y_{I,s}$ and $Y_{I,r}$ is

$$\text{covar}(Y_{I,s}, Y_{I,r}) = \begin{cases} \psi^{(1)}(r), & I = 1, \\ c_{s,r}(\alpha) \text{ for } r > 2\alpha^{-1}, s > \alpha^{-1}, & I = 2, \\ c_{s,r}(-\alpha), & I = 3, \end{cases}$$

where $c_{s,r}(\alpha) = \Gamma(r - 2\alpha^{-1}) / \{(s-1)!(s-\alpha^{-1})_{r-s}\} - M_r(-\alpha^{-1})M_s(-\alpha^{-1})$, $\langle x \rangle_j = x(x+1) \cdots (x+j-1)$, and that the j .th order cumulant of \mathbf{Y}_1 is

$$\kappa_j(\mathbf{Y}_1) = (-1)^j \psi^{(j-1)}(r) \tag{3}$$

for $j_r \neq 0$, where $\mathbf{j} = \sum_{i=1}^r j_i$. This is a remarkable result as (3) does not depend on \mathbf{j} except via j_r . These results appear to be all new except for $\mathbb{E}[Y_{1,r}]$ and the moments of $Y_{2,1}, Y_{3,1}$.

In Section 4, we give $b_{n,1}, c_{n,1}$, and first approximations for \mathbf{M}_n and its moments when these exist, for the most important classes of tail behavior for F . These classes cover all the examples we have come across. For $r = 1$ these results extend those of McCord [6] and verify a conjecture of his concerning $\mu_2(M_{n,1})$.

As well as the notation above, we use $n_1 = \log n, n_2 = \log n_1, \langle x \rangle_j = x(x-1) \cdots (x-j+1)$, $\text{var}(Z)$ = variance of Z, J = Jacobian, \mathbb{C} = the set of all complex numbers, $\text{corr}(Z_1, Z_2)$ = the correlation between Z_1 and $Z_2, \rho_{s,r}$ = the correlation between the s th and r th components of a vector of variables, and \sup_n = supremum over all possible values of n . Also $\alpha(x) \approx \beta(x)$ means that $\alpha(x)$ and $\beta(x)$ are approximately equal.

2 A brief review

Fisher and Tippett [7] showed that if $Y_{n,1}$ of (1) has a non-degenerate limit in distribution, Y_1 , then for some b and $c > 0, (Y_1 - b)/c \sim G(y)$ has only three possible forms, known as EV1, EV2, and EV3: $Y_{1,1} \sim G_1(y) = \exp\{-\exp(-y)\}$ on \mathbb{R} or $Y_{2,1} \sim G_2(y) = G_2(y, \alpha) = \exp(-y^\alpha)$ on $(0, \infty)$, where $\alpha > 0$, or $Y_{3,1} \sim G_3(y) = G_3(y, \alpha) = \exp\{-(-y)^\alpha\}$ on $(-\infty, 0)$, where $\alpha > 0$. Note the representation

$$Y_{2,1} = \exp(Y_{1,1}/\alpha), \quad Y_{3,1} = -Y_{2,1}^{-1} = -\exp(\alpha/Y_{1,1}). \tag{4}$$

Since we may take $c = 1$ and $b = 0$ without loss of generality, this gives the representation (with $b_n = b_{n,1}$, $c_n = c_{n,1}$) $M_{n,1} = b_n + c_n(Y_1 + o_p(1))$. So, for example, by Theorem 5.4 of Billingsley [8], $\mathbb{E}[M_{n,1}] = b_n + c_n\{\mathbb{E}[Y_1] + o(1)\}$ when $\mathbb{E}[Y_{n,1}^2]$ is bounded and $\mu_j(M_{n,1}) = c_n^j\{\mu_j(Y_1) + o(1)\}$ when $\mathbb{E}[|Y_{n,1}|^{j+\varepsilon}]$ is bounded for some $\varepsilon > 0$. Fisher and Tippett [7] also essentially showed that

$$\mathbb{E}[\exp(tY_{1,1})] = \Gamma(1 - t) \tag{5}$$

for $\Re(e(t)) < 1$ so that $\mathbb{E}[Y_{1,1}^j] = (-1)^j\Gamma^{(j)}(1)$ for $j \geq 0$, $\mathbb{E}[Y_{1,1}] = -\psi(1) = \gamma$, Euler's constant, and for $j \geq 2$, $\kappa_j(Y_{1,1}) = (j-1)!S_j$, where $S_j = \sum_{i=1}^{\infty} i^{-j}$ since $\log \Gamma(1 - t) = \gamma t + \sum_{j=2}^{\infty} S_j t^j/j$. For example, $S_2 = \pi^2/6 = \psi^{(1)}(1) = 1.6449 \dots$.

Gumbel [9], p.174 tabled μ_j and κ_j for $Y_{1,1}$ for $j \leq 6$. From (4), (5) we obtain

$$\mathbb{E}[Y_{2,1}^t] = \Gamma(1 - t/\alpha)$$

for $\Re(e(t)) < \alpha$ and

$$\mathbb{E}[(-Y_{3,1})^t] = \Gamma(1 + t/\alpha)$$

for $\Re(e(t)) > -\alpha$, as noted on p.264 and 281 of Gumbel [9]. For example, $\mathbb{E}[Y_{2,1}] = \Gamma(1 - \alpha^{-1})$ if $\alpha > 1$, $\text{var}[Y_{2,1}] = \Gamma(1 - 2\alpha^{-1}) - \Gamma(1 - \alpha^{-1})^2$ if $\alpha > 2$, $\mathbb{E}[Y_{3,1}] = -\Gamma(1 + \alpha^{-1})$ and $\text{var}[Y_{3,1}] = \Gamma(1 + 2\alpha^{-1}) - \Gamma(1 + \alpha^{-1})^2$.

If we have N observations distributed as $Y_{n,1}$ we can estimate c_n , b_n , α by assuming a choice of I and using maximum likelihood or moment estimates on

$$P(M_{n,1} \leq x) \approx G((x - b_n)/c_n), \tag{6}$$

where $G = G_I$. Gumbel [9] does this in Sections 6.2.5-6.2.6 for $I = 1$ for the maximum likelihood method, and in Sections 7.3.1-7.3.2 for $I = 3$ for the moment method. See also Kotz and Johnson [10], p.609. A better approach is to avoid assuming a choice of I (but still assuming that a non-degenerate limit exists) by replacing G in (6) by the generalized extreme value cumulative distribution function:

$$G(y) = \exp[-\{1 - k(y - \xi)/\sigma\}^{1/k}], \tag{7}$$

where $\sigma > 0$, $y < \xi + \sigma/k$ if $k > 0$, $y > \xi + \sigma/k$ if $k < 0$, σ denotes the scale parameter, ξ denotes the location parameter and k denotes the shape parameter. Furthermore, $G(y) = G_1(y)$ as $k \rightarrow 0$ if $\xi = 0$, $\sigma = 1$; $G(y) = G_2(y)$ if $\xi = 1$, $\sigma = 1/\alpha$, $k = -1/\alpha$; $G(y) = G_3(y)$ if $\xi = -1$, $\sigma = k = 1/\alpha$. For $n \leq 100$, an estimation of ξ , σ , k using L -moments is more efficient than using maximum likelihood estimates: see Hosking *et al.* [11].

Gnedenko [12] gave necessary and sufficient conditions on F that $\mathcal{L}Y_{n,1} \rightarrow G_1$ for $I = 1, 2$ or 3 . This is Theorem 1.6.2 of Leadbetter *et al.* [13]; appropriate choices of b_n and $a_n = c_n^{-1}$ in each is given by their Corollary 1.6.3. A choice of b_n , c_n can be found by setting $u_n = b_n + c_n x$ in their result (p.13) so that

$$n[1 - F(u_n)] \rightarrow t \in [0, \infty] \quad \text{if and only if} \quad P(M_{n,1} \leq u_n) \rightarrow \exp(t). \tag{8}$$

Table 1 Some examples

$X \sim F(x)$	Support	i	α	b_n	c_n
exponential	$(0, \infty)$	1		n_1	1
logistic	\mathbb{R}	1		n_1	1
normal	\mathbb{R}	1		B_n	$(2n_1)^{1/2}$
log-normal	$(0, \infty)$	1		B'_n	$(2n_1)^{-1/2} B'_n$
$1 - \exp(1/x)$	$(-\infty, 1)$	1		$-n_1^{-1}$	n_1^{-2}
Pareto $1 - (a/x)^\alpha$	(a, ∞)	2	α	0	$an^{1/\alpha}$
Cauchy	\mathbb{R}	2	1	0	n^{-1}
uniform	$(0,1)$	3	1	1	n^{-1}
$1 - (x_0 - x)^\alpha$	$(x_0 - 1, x_0)$	3	α	x_0	$n^{-1/\alpha}$
$\{1 - \exp(-x)\}/\{1 - \exp(-x_0)\}$	$(0, x_0)$	3	1	x_0	$\{\exp(x_0) - 1\}/n$
G_1	\mathbb{R}	1		1	1
G_2	$(0, \infty)$	2	α	0	$n^{1/\alpha}$
G_3	$(-\infty, 0)$	3	α	0	$n^{-1/\alpha}$

They gave the examples in Table 1. (The logistic is from Stuart and Ord [14].) Recall $n_1 = \log n$ and $n_2 = \log n_1$. In Table 1,

$$B_n = (2n_1)^{1/2} \{1 - (n_2 + \log 4\pi)/(4n_1)\} \tag{9}$$

and $B'_n = \exp(B_n)$. Leadbetter *et al.* [13] also noted that no non-degenerate limit exists for an F Poisson or a geometric cumulative distribution function. The last three results in Table 1 are exact, not asymptotic, since if $F = G_I$, then

$$F(x)^n = \begin{cases} G_1(x + n_1), & \text{if } I = 1, \\ G_2(n^{-1/\alpha} x), & \text{if } I = 2, \\ G_3(n^{1/\alpha} x), & \text{if } I = 3. \end{cases} \tag{10}$$

For the convergence rate of $Y_{n,1}$ to Y_1 , see Balkema and de Haan [15] and references therein. For an estimate of α , see Dekkers and de Haan [16] and Dekkers *et al.* [17] and references therein.

We now consider extensions to $M_{n,i}$, where $i \geq 1$. Leadbetter *et al.* [13], p.33 showed that (8) implies $P(M_{n,i} \leq u_n) \rightarrow \exp(-t) \sum_{j=0}^{i-1} t^j/j!$, so that if $Y_{n,1} \xrightarrow{\mathcal{L}} Y \sim G(x)$ then for $i \geq 1$ with $b_{n,i} \equiv b_{n,1}$, $c_{n,i} \equiv c_{n,1}$

$$Y_{n,i} \xrightarrow{\mathcal{L}} Y_{i,G} \sim G(x) \sum_{j=0}^{i-1} (-\log G(x))^j/j!,$$

that is,

$$Z_{n,i} = -\log G(Y_{n,i}) \xrightarrow{\mathcal{L}} \text{gamma}(i), \tag{11}$$

where $\text{gamma}(\gamma)$ has probability density function $z^{\gamma-1} \exp(-z)/\Gamma(\gamma)$ on $(0, \infty)$. By Theorem 2.3.1 in Leadbetter *et al.* [13], if $Y_{n,1} \xrightarrow{\mathcal{L}} Y_1 \sim G$ then $\mathbf{Z}_n = (Z_{n,1}, \dots, Z_{n,r})$ of (11) satisfies

$$P(\mathbf{Z}_n \geq \mathbf{z}) \rightarrow \sum p(\mathbf{z}, \mathbf{k}) \tag{12}$$

summing over k_1 to k_i , $0 \leq k_i, 1 \leq i \leq r$, where

$$p(\mathbf{z}, \mathbf{k}) = \exp(-z_r) \prod_{i=1}^r (z_i - z_{i-1})^{k_i} / k_i!$$

where $z_0 = 0$, and the support is $0 < z_1 < z_2 < \dots < z_r < \infty$. So,

$$\mathbf{Z}_n \xrightarrow{\mathcal{L}} \mathbf{Z} \text{ with joint probability density function } \exp(-z_r) \tag{13}$$

on $0 < z_1 < \dots < z_r < \infty$, $(G(Y_{n,1}), \dots, G(Y_{n,r})) \xrightarrow{\mathcal{L}} \mathbf{U}$ say with joint probability density function $(u_1 \dots u_{r-1})^{-1}$ on $0 < u_r < \dots < u_1 < 1$, and

$$\mathbf{Y}_n \xrightarrow{\mathcal{L}} \mathbf{Y}_G \tag{14}$$

with joint probability density function $G(y_r)J$ on $-\infty < y_r < \dots < y_1 < \infty$, where $J = |\prod_{i=1}^r \partial z_i / \partial y_i|$ for $z_i = -\log G(y_i)$. If G is the generalized extreme value cumulative distribution function of (7) this joint probability density function reduces to (5) of Tawn [18] and (2.2) of Smith [19]. For convergence rates in (14), see Omey and Rachev [20]. The results (12)-(14) are not given, although (14) is easily proven for $r = 2$ from their Theorem 2.3.2. From (13), $\{Z_{r_i} - Z_{r_{i-1}}\}$ are independently distributed as $\{\text{gamma}(r_i - r_{i-1})\}$. If $F = G_I$ for $I = 1, 2$ or 3 then $\xrightarrow{\mathcal{L}}$ in (14) can be replaced by $\stackrel{\mathcal{L}}{=}$ with $G = G_I$. For $r = 1$ this is just (10).

For $1 \leq I \leq 3$, set

$$\mathbf{Y}_I = (Y_{1,1}, \dots, Y_{I,r})' = \mathbf{Y}_G \tag{15}$$

for $G = G_I$. By Section 14.20 of Stuart and Ord [14], for $r \geq 1$ and M_r of (2), $\mathbb{E}[\exp(Y_{i,r}t)] = M_r(-t)$, so $\mathbb{E}[Y_{1,r}^j] = (-1)^j \Gamma^{(j)}(r) / (r-1)!$ and $\kappa_j(Y_r) = (-1)^j \psi^{(j-1)}(r)$, which they table for $j = 1, 2$ and $r = 1, 3, 5, 10$.

3 Moments for limits

From (13) or (14) we have the following lemma.

Lemma 3.1 *Note that $\mathbf{Y}_1, \mathbf{Y}_2$, and \mathbf{Y}_3 of (15) can be represented in terms of each other by $Y_{2,i} = \exp(Y_{1,i}/\alpha)$ and $Y_{3,i} = -Y_{2,i}^{-1} = -\exp(-Y_{1,i}/\alpha)$, $1 \leq i \leq r$.*

Theorem 3.1 *Set $Z_i = -\log G_1(Y_{1,i})$. For \mathbf{t} in \mathbb{C}^r ,*

$$\mathbb{E}[\exp(\mathbf{t}'\mathbf{Y}_1)] = \mathbb{E} \left[\prod_{i=1}^r Z_i^{-t_i} \right] = B_r(\mathbf{t}), \tag{16}$$

where $B_r(\mathbf{t}) = \Gamma(r - T_r) / \prod_{i=1}^{r-1} (i - T_i)$ and $T_i = \sum_{j=1}^i t_j$ for $i - \Re(T_i) > 0, 1 \leq i \leq r$.

Proof By (13), \mathbf{Z} has joint probability density function $\exp(-z_r)$ on $0 < z_1 < \dots < z_r < \infty$. Now use induction. □

Applying $(\partial/\partial t)^j$ to (16) and setting $t = 0$ gives an expression for $m_j(\mathbf{Y}_1) = \mathbb{E}[\prod_{i=1}^r Y_{1,i}^j]$. However, these are more easily found from the cumulants: taking logs of (16) yields the following remarkable result.

Corollary 3.1 For $j_r > 0$ and $\mathbf{j} = \sum_{i=1}^r j_i$,

$$\kappa_j(\mathbf{Y}_1) = (-1)^j \psi^{(j-1)}(r). \tag{17}$$

For example, for $1 \leq s \leq r$, $\text{covar}(Y_{s,1}, Y_{r,1}) = \psi^{(1)}(r)$, so $0 \leq \text{var}[Y_{s,1} - Y_{r,1}] = \text{var}[Y_{s,1}] - \text{var}[Y_{r,1}]$ and $\text{corr}(Y_{s,1}, Y_{r,1}) = \{\psi^{(1)}(r)/\psi^{(1)}(s)\}^{1/2}$. By (6.3.6) of Abramowitz and Stegun [21], the right hand side of (17) is equal to $-\sum_{i=1}^{r-1} i^{-j} + \gamma_j$, where $\gamma_j = (-1)^j \psi^{(j-1)}(1)$. Also $\gamma_1 = -\psi(1) = \gamma = 0.57721 \dots$ and $\gamma_2 = \pi^2/6$.

Lemma 3.1 and Theorem 3.1 imply the following result.

Theorem 3.2 We have $m_j(\mathbf{Y}_2) = B_r(\mathbf{j}/\alpha)$ and $m_j(-\mathbf{Y}_3) = B_r(-\mathbf{j}/\alpha)$. In fact, this holds with \mathbf{j} replaced by $\mathbf{t} \in \mathbb{C}^r$. So, for $M_r(t)$ of (2) and $V_r(t) = M_r(2t) - M_r(t)^2$,

$$\mathbb{E}[Y_{2,r}^t] = M_r(-t/\alpha), \quad \mathbb{E}[(-Y_{3,r})^t] = M_r(t/\alpha),$$

and

$$\text{var}[Y_{2,r}] = V_r(-\alpha^{-1}), \quad \text{var}[Y_{3,r}] = V_r(\alpha^{-1}).$$

Similarly, for $1 \leq s \leq r$,

$$\mathbb{E}[Y_{2,s}^{j_s} Y_{2,r}^{j_r}] = C(j_s/\alpha, j_r/\alpha), \quad \mathbb{E}[(-Y_{3,s})^{j_s} (-Y_{3,r})^{j_r}] = C(-j_s/\alpha, -j_r/\alpha),$$

where

$$C(t_s, t_r) = \Gamma(r - t_s - t_r) / \left\{ (s-1)! \prod_{j=s}^{r-1} (j - t_s) \right\}$$

and $\text{covar}(Y_{1,s}, Y_{1,r})$ is given by (3) for $I = 2, 3$.

Example 3.1 Set $\alpha = 1$. Then in the notation of (3), $\mathbb{E}[Y_{2,r}^j] = 1/\langle r-1 \rangle_j$, $\mathbb{E}[(-Y_{3,r})^j] = (r)_j$, $\text{var}[Y_{2,r}] = (r-1)^{-2}(r-2)^{-1}$ for $3 \leq r$, $\text{var}[Y_{3,r}] = r$, $\text{covar}(Y_{2,s}, Y_{2,r}) = c_{s,r}(1) = \{(s-1)(r-1)(r-2)\}^{-1}$ for $2 \leq s \leq r$, $3 \leq r$ and $\text{covar}(Y_{3,s}, Y_{3,r}) = c_{s,r}(-1) = s$ for $1 \leq s \leq r$, with corresponding correlations $\rho_{s,r}(1) = \{(s-2)/(r-2)\}^{1/2}$ and $\rho_{s,r}(-1) = (s/r)^{1/2}$.

Example 3.2 Set $\alpha = 1/2$. Then $\mathbb{E}[Y_{2,r}] = 1/\langle r-1 \rangle_2$ for $r \geq 2$, $\mathbb{E}[Y_{3,r}] = -(r)_2$ for $r \geq 0$, $\text{var}[Y_{2,r}] = 2\langle r-1 \rangle_4^{-1}(2r-5)$ for $r \geq 4$, and $\text{var}[Y_{3,r}] = 2(r)_2(2r+5)$ for $r \geq 0$.

By (6.1.47) of Abramowitz and Stegun [21], as $r \rightarrow \infty$,

$$M_r(t) = r^t \{ 1 + \langle t \rangle_2 r^{-1}/2 + \langle t \rangle_3 (3t-1)r^{-2}/24 + O(r^{-3}) \},$$

so $V_r(t) = 2r^{2t-1}\{t^2 + O(r^{-1})\}$, $\mathbb{E}[Y_{2,r}] = r^{-1/\alpha}\{1 + (\alpha^{-1})_2 + O(r^{-2})\}$, $\mathbb{E}[Y_{3,r}] = -r^{1/\alpha}\{1 + (\alpha^{-1})_2 r^{-1}/2 + O(r^{-2})\}$, $\text{var}[Y_{2,r}] = 2r^{2/\alpha-1}\{\alpha^{-2} + O(r^{-1})\}$ and similarly for $\mathbb{E}[Y_{2,r}^j]$ and $\mathbb{E}[Y_{3,r}^j]$.

4 Approximations for M_n

Suppose

$$Y_n = (M_n - b_n \mathbf{1}_r) c_n \xrightarrow{\mathcal{L}} Y \tag{18}$$

as $n \rightarrow \infty$. Then we can represent Y_n as $Y + o_p(1)$, that is,

$$M_n = b_n \mathbf{1}_r + c_n(Y + o_p(1)). \tag{19}$$

By Theorem 5.4 of Billingsley [8], as $n \rightarrow \infty$, $\mathbb{E}[g(Y_n)] \xrightarrow{\mathcal{L}} \mathbb{E}[g(Y)]$ if $g(Y_n)$ is uniformly integrable, for example, if $\sup_n \mathbb{E}[|g(Y_n)|^{1+\epsilon}] < \infty$ for some $\epsilon > 0$. So, if $\sup_n \mathbb{E}[|Y_n^{\mathbf{k}}|] < \infty$ for some $\mathbf{k} > \mathbf{j}$ then

$$\mathbb{E}[Y_n^{\mathbf{j}}] = c_n^{-\mathbf{j}} \mathbb{E}[(M_n - b_n \mathbf{1}_r)^{\mathbf{j}}] \rightarrow \mathbb{E}[Y^{\mathbf{j}}]$$

and

$$\mu_{\mathbf{j}}(Y_n) = c_n^{-\mathbf{j}} \mu_{\mathbf{j}}(M_n) \rightarrow \mu_{\mathbf{j}}(Y). \tag{20}$$

Here, $\mathbf{k} > \mathbf{j}$ means $k_i > j_i$ for $1 \leq i \leq r$. If

$$b_n = B_n(1 + \delta_n), \quad \text{where } \delta_n \rightarrow 0 \text{ and } C_n = c_n/B_n \rightarrow 0 \tag{21}$$

then (19) implies $M_n^{\mathbf{j}} = b_{n,\mathbf{j}} + c_{n,\mathbf{j}}(\mathbf{j}'Y + o_p(1))$, where $b_{n,\mathbf{j}} = B_n^{\mathbf{j}}(1 + \mathbf{j}.\delta_n)$ and $c_{n,\mathbf{j}} = B_n^{\mathbf{j}}C_n$, so

$$\mathbb{E}[M_n^{\mathbf{j}}] = b_{n,\mathbf{j}} + c_{n,\mathbf{j}}(\mathbf{j}'\mathbb{E}[Y] + o(1))$$

provided $(M_n^{\mathbf{j}} - b_{n,\mathbf{j}})/c_{n,\mathbf{j}}$ is uniformly integrable. McCord [6] provided methods of proving uniform integrability of such functions of Y_n for the case $r = 1$. We shall not do so here. It is easy to show that there exists $c_{n,r} < \infty$ such that

$$\mathbb{E}[|M_n^{\mathbf{j}}|] \leq c_{n,r} \prod_{i=1}^r \mathbb{E}[|X|^{j_i}],$$

where $X \sim F(x)$. For example, the reader can try this on (14.2) of Stuart and Ord [14]. It follows that $\mathbb{E}[M_n^{\mathbf{j}}]$ exists if and only if $\mathbb{E}[X^{j_0}]$ exists, that is,

$$\mathbb{E}[|X|^{j_0}] < \infty, \tag{22}$$

where $j_0 = \max\{j_i : 1 \leq i \leq r\}$. We conjecture that this condition is sufficient for $Y_n^{\mathbf{j}}$ to be uniformly integrable. We shall see that McCord [6] proved this conjecture when $r = 1$ for several large classes of F . Table 1 gave Y_1, b_n, c_n for a number of choices of $F(x)$. In fact, each of these $F(x)$ may be replaced by the class of $F(x)$ with the same asymptotic tail behavior as $x \rightarrow \infty$. We now illustrate this and expand these classes for some important cases. Set $b_n(X) = b_n, c_n(X) = c_n, M_n(X) = M_n$, and $Y_n(X) = Y_n$.

Theorem 4.1 *Suppose*

$$1 - F(x) \approx (a/x)^\alpha \tag{23}$$

as $x \rightarrow \infty$, where $a > 0$ and $\alpha > 0$. Then (18) holds with $b_n = 0$, $c_n = an^{1/\alpha}$, and $Y = Y_2$.

Proof Note that (8) holds with $u_n = c_n x$, $t = x^{-\alpha}$, so (14) gives (18). □

Note that (23) holds if

$$F \text{ has probability density function } \approx (a/x)^{\alpha+1} a^{-1} \tag{24}$$

as $x \rightarrow \infty$. This includes the Pareto, Cauchy and G_2 distributions.

For $r = 1$, when (24) holds, Theorem 5.2 of McCord [6] proved

$$Y_n^j \text{ is uniformly integrable if (22) holds.} \tag{25}$$

We conjecture this is true for all r .

Theorem 4.2 *Suppose $X \sim F$ on $(-\infty, x_0)$ with $x_0 < \infty$ and*

$$1 - F(x) \approx c(x_0 - x)^\alpha \tag{26}$$

as $x \uparrow x_0$, where $c > 0$ and $\alpha > 0$. Then (18) holds with $Y = Y_3$, $b_n = x_0$, and $c_n = (cn)^{-1/\alpha}$.

Proof Note that $u_n = b_n + c_n x$ satisfies (8) with $t = (-x)^\alpha$, so (14) gives (18). □

Note that (26) holds if F has probability density function

$$f(x) \approx \alpha c(x_0 - x)^{\alpha-1} \tag{27}$$

as $x \uparrow x_0$.

For $r = 1$ Theorem 1 in McCord [6] proved (25) when (27) holds. We conjecture it is true for all r .

Note that (26) holds for $F = G_3$ with $c = 1$ and $x_0 = 0$.

Theorem 4.3 *Suppose as $x \rightarrow \infty$,*

$$1 - F(x) \approx kx^d \exp(-x). \tag{28}$$

Then (18) holds with $Y = Y_1$, $c_n = 1$, and $b_n = n_1 + dn_2 + k_1$, where $k_1 = \log k$.

Proof Note that (8) holds with $u_n = b_n + x$ and $t = \exp(-x)$, so (14) gives (18). □

Note that (28) holds if F has probability density function $f(x) \approx kx^d \exp(-x)$ as $x \rightarrow \infty$. So, with $k = 1$ this covers the exponential ($d = 0$), logistic ($d = 0$) and gamma (γ) ($d = \gamma - 1$) distributions.

Theorem 4.4 *Suppose as $z \rightarrow \infty$,*

$$1 - F(x) \approx kz^d \exp(-z) \tag{29}$$

for $z = \{(x - b)/c\}^a$. So, either $a > 0$ and $c > 0$, or $c < 0$ and a is a negative odd integer. Then (18) holds with $Y = Y_1$, $c_n = ca^{-1}n_1^{1/a-1}$, and $b_n = bI(a < 0) + cn_1^{1/a}\{1 + a^{-1}n_1^{-1}(dn_2 + k_1)\}$, where $I(A) = 1$ if A is true, or $I(A) = 0$ if A is false.

Proof The cumulative distribution function of $Z = \{(X - b)/c\}^a$ satisfies (28). So, $M_{n,i} = b + cM_{n,i}(Z)^{1/a}$. Now apply Theorem 4.3. □

If $a > 0$ and $c > 0$, then $z \rightarrow \infty$ if and only if $x \rightarrow \infty$. But if $a < 0$ and $c < 0$, then $X < b$ with probability 1 and $z \rightarrow \infty$ if and only if $x \uparrow b$.

Note that (29) holds if F has probability density function

$$f(x) \approx kz^d \exp(-z) \partial z / \partial x = kac^{-1}z^{d+1-1/a} \exp(-z) \tag{30}$$

as $z \rightarrow \infty$. For $F = \Phi$, the unit normal cumulative distribution function, this holds with $a = 2, b = 0, c = 2^{1/2}, d = -1/2, k + (4\pi)^{-1/2}$, giving b_n, c_n of (9). Another example is $1 - F(x) \approx \exp(1/x)$: in this case $k = 1, d = b = 0$, and $a = c = -1$.

For $r = 1, d = 0, a > 0$, and (30), Theorem 3 in McCord [6] proved (25). We conjecture it is true for all r . For this case, he conjectured (20) for $j = 2$. So, his conjecture is true if $\mathbb{E}[X^2] < \infty$.

If $a > 0$ or $b = 0$ then (21) holds with

$$B_n = cn_1^{1/a} \quad \text{and} \quad \delta_n = a^{-1}n_1^{-1}(dn_2 + k_1), \tag{31}$$

so $b_{n,j} = c^j n_1^{j/a} (1 + j \cdot \delta_n)$ and $c_{n,j} = a^{-1}c^j n_1^{j/a-1}$.

Theorem 4.5 *Suppose $Z = \exp(X)$ for X of Theorem 4.2. If $a > 1$, (18) holds for $Y_n(Z)$ with $Y = Y_1, b_n(Z) = \exp(B_n)(1 + B_n \delta_n)$ and $c_n(Z) = \exp(B_n)c_n$ for B_n, δ_n of (31). If $a = 1$, (18) holds for $Y_n(Z)$ with $Y = Y_2$ at $\alpha = c^{-1}4, b_n(Z) = 0, c_n(Z) = \exp(b_n)$, and $b_n = c(n_1 + dn_2 + k_1)$.*

Proof Note that $M_{n,i}(Z) = \exp(M_{n,i}) = \exp(b_n + c_n Y_{n,i})$. By (21), (31), if $a > 1, B_n \delta_n \rightarrow 0$ and $c_n \rightarrow 0$, so $M_{n,i}(Z) = \exp(B_n)(1 + B_n \delta_n + c_n Y_{1,i} + o_p(c_n))$. If $a = 1, c_n = c$, so $\exp(c_n Y_{n,1}) = \exp(cY_1)(1 + o_p(1)) = Y_2 + o_p(1)$ at $\alpha = c^{-1}$. □

For the log-normal distribution this gives $Y = Y_1, B_n = c_n^{-1} = (2n_1)^{1/2}, \delta_n = -(n_2 + \log 4\pi)/(4n_1)$, so $b_n(Z) = \exp(B_n)\{1 - (2n_1)^{-1/2}(n_2 + \log 4\pi)/2\}$ and $c_n(Z) = \exp(B_n)/\sqrt{2n_1}$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

CW derived the results for Sections 1, 2 and 3. SN derived the results for Section 4. All authors read and approved the final manuscript.

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