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Generalized weighted composition operators from α-Bloch spaces into weighted-type spaces

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Abstract

Some criteria for the boundedness, as well as for the compactness, of the generalized weighted composition operator $D^n_{\varphi,u}$ from α-Bloch spaces into weighted-type spaces are given. Estimates for the norm and the essential norm of the operator are also given. Our results extend and complement some results in the literature.

MSC: 47B33; 30H30

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1 Introduction

Let $\mathbb{D}$ be the unit disk of the complex plane $\mathbb{C}$, $H(\mathbb{D})$ the class of functions analytic on $\mathbb{D}$, and $H^\infty = H^\infty(\mathbb{D})$ the space of bounded analytic functions on $\mathbb{D}$. For $0 < \alpha < \infty$, an $f \in H(\mathbb{D})$ is said to belong to the *α-Bloch space* $\mathcal{B}^\alpha = \mathcal{B}^\alpha(\mathbb{D})$ if

$$b_\alpha(f) = \sup_{z\in\mathbb{D}} \left(1 - |z|^2\right)^\alpha \left|f'(z)\right| < \infty.$$

It is easy to check that $\mathcal{B}^\alpha$ becomes a Banach space with the norm $\|f\|_{\mathcal{B}^\alpha} = |f(0)| + b_\alpha(f)$. The *little α-Bloch space* $\mathcal{B}^\alpha_0 = \mathcal{B}^\alpha_0(\mathbb{D})$, is a subspace of $\mathcal{B}^\alpha$ consisting of all $f \in H(\mathbb{D})$ such that

$$\lim_{|z|\to 1^-} \left(1 - |z|^2\right)^\alpha \left|f'(z)\right| = 0.$$

When $\alpha = 1$, $\mathcal{B}^1 = \mathcal{B}$ is the well-known Bloch space, while $\mathcal{B}^1_0 = \mathcal{B}_0$ is the well-known little Bloch space. For some results on the α-Bloch spaces and the little α-Bloch spaces, see, for example, [1].

A positive continuous function on $\mathbb{D}$ is called a *weight*. Let $\mu(z)$ be a weight. The *weighted-type space* on $\mathbb{D}$ [2, 3], denoted by $H^\infty_\mu = H^\infty_\mu(\mathbb{D})$, consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{H^\infty_\mu} = \sup_{z\in\mathbb{D}} \mu(z)\left|f(z)\right| < \infty.$$

It is obvious that $H_0^\infty = H^\infty$, while for $\mu(z) = (1-|z|^2)^\beta$, $\beta > 0$, is obtained the growth space H_β^∞ [4].

Let $u \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. The weighted composition operator uC_φ, induced by φ and u, is defined by

$$(uC_\varphi f)(z) = u(z) \cdot f\big(\varphi(z)\big), \quad f \in H(\mathbb{D}), z \in \mathbb{D}.$$

When $u(z) \equiv 1$, then the weighted composition operator is reduced to the composition operator, usually denoted by C_φ, while for $\varphi(z) \equiv z$, it is reduced to the multiplication operator, usually denoted by M_u.

A natural generalization of the weighted composition operator is the *generalized weighted composition operator* [5] or the *weighted differentiation composition operator* [6] $D^n_{\varphi,u}$, which is defined as

$$\big(D^n_{\varphi,u} f\big)(z) = u(z) \cdot f^{(n)}\big(\varphi(z)\big), \quad f \in H(\mathbb{D}), z \in \mathbb{D},$$

where $n \in \mathbb{N}_0$, $u \in H(\mathbb{D})$, and φ is an analytic self-map of $\mathbb{D}$. Clearly, when $n = 0$ and $u(z) = 1$, $D^n_{\varphi,u}$ is the composition operator C_φ, if $n = 0$, then $D^n_{\varphi,u}$ is the weighted composition operator uC_φ. If $n = 1$ and $u(z) = \varphi'(z)$, then $D^n_{\varphi,u} = DC_\varphi$, which was studied, for example, in [3, 7–15], while for $u(z) = 1$, $D^n_{\varphi,u} = C_\varphi D^n$, which was studied in [3, 13, 15, 16]. For some other results on the generalized weighted composition operator on various spaces of holomorphic functions, see, for example, [17–22]. A fundamental problem concerning concrete operators is to relate function theoretic properties of their symbols to their operator theoretic properties (see, for example, [3, 5–29]).

It is well known that the composition operator is bounded on the Bloch space $\mathcal{B}$. See, for example, [26, 28, 29] for the compactness and essential norm of the composition operator on $\mathcal{B}$. In [28], it was shown that C_φ is compact on $\mathcal{B}$ if and only if

$$\|C_\varphi p_j\|_{\mathcal{B}} = \big\|\varphi^j\big\|_{\mathcal{B}} \to 0 \quad \text{as } j \to \infty,$$

where $p_j(z) = z^j$, $j \in \mathbb{N}_0$.

Motivated by this result, in [22], the author proved that $D^n_{\varphi,u} : \mathcal{B} \to H^\infty_\beta$ is compact if and only if it is bounded and

$$\lim_{j\to\infty} \big\|D^n_{\varphi,u}(p_j)\big\|_{H^\infty_\beta} = 0.$$

Following the line of the above mentioned investigations, in this work, we consider the operators $D^n_{\varphi,u} : \mathcal{B}^\alpha$ (or $\mathcal{B}^\alpha_0$) $\to H^\infty_\mu$, and show that $D^n_{\varphi,u} : \mathcal{B}^\alpha$ (or $\mathcal{B}^\alpha_0$) $\to H^\infty_\mu$ is bounded (respectively, compact) if and only if the sequence $(j^{\alpha-1}\|D^n_{\varphi,u}(p_j)\|_{H^\infty_\mu})_{j=n}^\infty$ is bounded (respectively, convergent to 0 as $j \to \infty$). Moreover, we give some estimates for the norm, as well as for the essential norm of the operator $D^n_{\varphi,u} : \mathcal{B}^\alpha$ (or $\mathcal{B}^\alpha_0$) $\to H^\infty_\mu$. Recall that the essential norm of the operator $T : X \to Y$ is its distance to the set of compact operators K mapping X to Y, that is,

$$\|T\|_{e,X\to Y} = \inf\big\{\|T-K\|_{X\to Y} : K \text{ is compact}\big\},$$

where X and Y are Banach spaces and $\|\cdot\|_{X\to Y}$ is the operator norm. Consequently, $\|T\|_{e,X\to Y} = 0$ if and only if T is compact.

Throughout the paper, we denote by C a positive constant which may differ from one occurrence to the next. We write $P \preceq Q$ if there exists a positive constant C independent of the quantities P and Q such that $P \leq CQ$. The symbol $P \approx Q$ means that $P \preceq Q \preceq P$.

2 Boundedness of $D^n_{\varphi,u} : \mathcal{B}^\alpha$ (or $\mathcal{B}^\alpha_0$) $\to H^\infty_\mu$

For $w \in \mathbb{D}$, set

$$f_w(z) = \frac{1-|w|^2}{(1-\overline{w}z)^\alpha}, \quad z \in \mathbb{D}.$$

Note that

$$f_w^{(n)}(z) = \frac{(1-|w|^2)\overline{w}^n}{(1-\overline{w}z)^{\alpha+n}} \prod_{j=0}^{n-1}(\alpha+j), \quad z \in \mathbb{D}, n \in \mathbb{N}. \tag{1}$$

In this section, we will use this family of functions, as well as the sequence of functions $(j^{\alpha-1}p_j)_{j\in\mathbb{N}}$ to characterize the boundedness and compactness of $D^n_{\varphi,u} : \mathcal{B}^\alpha$ (or $\mathcal{B}^\alpha_0$) $\to H^\infty_\mu$.

Theorem 2.1 *Let n be a positive integer, $\alpha > 0$, μ a weight, $u \in H(\mathbb{D})$, and φ be an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(a) *The operator $D^n_{\varphi,u} : \mathcal{B}^\alpha \to H^\infty_\mu$ is bounded.*
(b) *The operator $D^n_{\varphi,u} : \mathcal{B}^\alpha_0 \to H^\infty_\mu$ is bounded.*
(c) $M_1 := \sup_{j\geq n} j^{\alpha-1}\|D^n_{\varphi,u}(p_j)\|_{H^\infty_\mu} < \infty$.
(d) $M_2 := \sup_{w\in\mathbb{D}} \|D^n_{\varphi,u}f_{\varphi(w)}\|_{H^\infty_\mu} < \infty$ *and* $u \in H^\infty_\mu$.
(e) $M_3 := \sup_{z\in\mathbb{D}} \frac{\mu(z)|u(z)|}{(1-|\varphi(z)|^2)^{n+\alpha-1}} < \infty$ *and* $u \in H^\infty_\mu$.

Moreover, if the operator $D^n_{\varphi,u} : \mathcal{B}^\alpha \to H^\infty_\mu$ is bounded, then the following asymptotic relations hold:

$$\|D^n_{\varphi,u}\|_{\mathcal{B}^\alpha\to H^\infty_\mu} \approx \|D^n_{\varphi,u}\|_{\mathcal{B}^\alpha_0\to H^\infty_\mu} \approx M_1 \approx \max\{M_2, \|u\|_{H^\infty_\mu}\} \approx M_3. \tag{2}$$

Proof (a) $\Rightarrow$ (b) Since $\mathcal{B}^\alpha_0 \subset \mathcal{B}^\alpha$, this implication, as well as the inequality

$$\|D^n_{\varphi,u}\|_{\mathcal{B}^\alpha_0\to H^\infty_\mu} \leq \|D^n_{\varphi,u}\|_{\mathcal{B}^\alpha\to H^\infty_\mu}, \tag{3}$$

is obvious.

(b) $\Rightarrow$ (c) It is easy to see that the sequence $(j^{\alpha-1}p_j)_{j\in\mathbb{N}}$ is bounded in $\mathcal{B}^\alpha_0$ and

$$\|p_j\|_{\mathcal{B}^\alpha} = j\left(\frac{2\alpha}{j-1+2\alpha}\right)^\alpha\left(\frac{j-1}{j-1+2\alpha}\right)^{\frac{j-1}{2}}, \quad \text{for } j \in \mathbb{N},$$

which implies that $\|j^{\alpha-1}p_j\|_{\mathcal{B}^\alpha} \approx 1$. Notice that $(D^n_{\varphi,u}p_n)(z) = u(z)n!$, $z \in \mathbb{D}$, while for $j < n$, $D^n_{\varphi,u}(p_j) = 0$. Therefore, by the boundedness of $D^n_{\varphi,u} : \mathcal{B}^\alpha_0 \to H^\infty_\mu$, we get

$$j^{\alpha-1}\|D^n_{\varphi,u}(p_j)\|_{H^\infty_\mu} = \|D^n_{\varphi,u}(j^{\alpha-1}p_j)\|_{H^\infty_\mu} \leq C\|D^n_{\varphi,u}\|_{\mathcal{B}^\alpha_0\to H^\infty_\mu} < \infty,$$

for every $j \in \mathbb{N}$, proving (c), as well as the asymptotic relation

$$M_1 \preceq \|D^n_{\varphi,u}\|_{\mathcal{B}^\alpha_0 \to H^\infty_\mu}. \tag{4}$$

(c) ⇒ (a) If $\|\varphi\|_\infty = \sup_{z\in\mathbb{D}} |\varphi(z)| < 1$, then by Proposition 8 in [1], we have

$$\|D^n_{\varphi,u} f\|_{H^\infty_\mu} \preceq \frac{\|u\|_{H^\infty_\mu} \|f\|_{\mathcal{B}^\alpha}}{(1-\|\varphi\|^2_\infty)^{n+\alpha-1}},$$

from which the boundedness of $D^n_{\varphi,u} : \mathcal{B}^\alpha \to H^\infty_\mu$ follows in this case.

Now assume that $\|\varphi\|_\infty = 1$. Let $\mathbb{D}_j = \{z \in \mathbb{D} : r_j \le |\varphi(z)| < r_{j+1}\}$ where $r_j = (j-n)/(j+\alpha-1)$ for $j \ge n$. Then from Lemma 1 in [16], which also holds for $m = 0$, *i.e.*, $n = 1$ in our case, we have that there is a $\delta > 0$ such that

$$\min_{z\in\mathbb{D}_j} j^{\alpha-1} j(j-1)\cdots(j-n+1)\big|\varphi(z)\big|^{j-n}\big(1-\big|\varphi(z)\big|\big)^{\alpha+n-1} \ge \delta,$$

for every $j \ge k+1$, where k is the smallest natural number such that $\mathbb{D}_k \ne \emptyset$.

Fix $N \ge k+1$. Then, clearly $N \ge n+1$ and we have

$$\|D^n_{\varphi,u} f\|_{H^\infty_\mu} \le \sup_{|\varphi(z)|<\frac{N-n}{N+\alpha-1}} \mu(z)\big|u(z)\big|\big|f^{(n)}\big(\varphi(z)\big)\big| + \sup_{|\varphi(z)|\ge\frac{N-n}{N+\alpha-1}} \mu(z)\big|u(z)\big|\big|f^{(n)}\big(\varphi(z)\big)\big|. \tag{5}$$

The finiteness of M_1 implies $u \in H^\infty_\mu$. Hence, as in the first case, we have

$$\sup_{|\varphi(z)|<\frac{N-n}{N+\alpha-1}} \mu(z)\big|u(z)\big|\big|f^{(n)}\big(\varphi(z)\big)\big| \preceq \|u\|_{H^\infty_\mu}\|f\|_{\mathcal{B}^\alpha}. \tag{6}$$

On the other hand, since $\mathbb{D} \setminus \{|\varphi(z)| < \frac{N-n}{N+\alpha-1}\} = \bigcup_{j\ge N} \mathbb{D}_j$, we get

$$\begin{aligned}
&\sup_{|\varphi(z)|\ge\frac{N-n}{N+\alpha-1}} \mu(z)\big|u(z)\big|\big|f^{(n)}\big(\varphi(z)\big)\big| \\
&\quad= \sup_{j\ge N}\sup_{z\in\mathbb{D}_j} \mu(z)\big|u(z)\big|\big|f^{(n)}\big(\varphi(z)\big)\big| \\
&\quad= \sup_{j\ge N}\sup_{z\in\mathbb{D}_j} \mu(z)\big|u(z)\big|\frac{j^{\alpha-1}j(j-1)\cdots(j-n+1)|\varphi(z)|^{j-n}|f^{(n)}(\varphi(z))|(1-|\varphi(z)|)^{\alpha+n-1}}{j^{\alpha-1}j(j-1)\cdots(j-n+1)(1-|\varphi(z)|)^{\alpha+n-1}|\varphi(z)|^{j-n}} \\
&\quad\preceq \frac{\|f\|_{\mathcal{B}^\alpha}}{\delta}\sup_{j\ge N} j^{\alpha-1}\big\|D^n_{\varphi,u}(p_j)\big\|_{H^\infty_\mu} \le \frac{M_1}{\delta}\|f\|_{\mathcal{B}^\alpha} < \infty.
\end{aligned} \tag{7}$$

From (5), (6) and (7), the boundedness of $D^n_{\varphi,u} : \mathcal{B}^\alpha \to H^\infty_\mu$ follows.

(c) ⇒ (d) First note that (c) implies that $u \in H^\infty_\mu$. Further, since

$$\sup_{z\in\mathbb{D}}\big(1-|z|^2\big)^\alpha\big|f'_w(z)\big| = \sup_{z\in\mathbb{D}}\big(1-|z|^2\big)^\alpha \frac{|\alpha\overline{w}|(1-|w|^2)}{|1-\overline{w}z|^{\alpha+1}} \le |\alpha|2^{\alpha+1}, \quad w \in \mathbb{D},$$

the family of functions $(f_w)_{w\in\mathbb{D}}$ is uniformly bounded in $\mathcal{B}^\alpha$. Furthermore

$$f_w(z) = \big(1-|w|^2\big)\sum_{j=0}^\infty \frac{\Gamma(j+\alpha)}{j!\Gamma(\alpha)}\overline{w}^j z^j, \quad z \in \mathbb{D}.$$

By Stirling's formula, we have $\frac{\Gamma(j+\alpha)}{j!\Gamma(\alpha)} \approx j^{\alpha-1}$ as $j \to \infty$. Using this fact, the linearity and continuity of the operator, we get

$$\left\|D^n_{\varphi,u} f_w\right\|_{H^\infty_\mu} \le C\left(1-|w|^2\right)\sum_{j=n}^{\infty}|w|^j j^{\alpha-1}\left\|D^n_{\varphi,u}(p_j)\right\|_{H^\infty_\mu} \preceq M_1 < \infty, \quad w \in \mathbb{D}.$$

Consequently, $\sup_{w\in\mathbb{D}} \|D^n_{\varphi,u} f_{\varphi(w)}\|_{H^\infty_\mu} \preceq M_1$, and along with the inequality $n^{\alpha-1}n!\|u\|_{H^\infty_\mu} \preceq M_1$, obtained by considering $\|D^n_{\varphi,u}(n^{\alpha-1}p_n)\|_{H^\infty_\mu}$, we also have

$$\max\left\{M_2, \|u\|_{H^\infty_\mu}\right\} \preceq M_1. \tag{8}$$

(d) $\Rightarrow$ (e) For $\lambda \in \mathbb{D}$, it follows from (d) and (1) that

$$M_2 \ge \left\|D^n_{\varphi,u} f_{\varphi(\lambda)}\right\|_{H^\infty_\mu} \ge \frac{\mu(\lambda)|u(\lambda)||\varphi(\lambda)|^n \prod_{j=0}^{n-1}(\alpha+j)}{(1-|\varphi(\lambda)|^2)^{n+\alpha-1}}. \tag{9}$$

For any fixed $r \in (0,1)$, from (9), we have

$$\sup_{|\varphi(\lambda)|>r} \frac{\mu(\lambda)|u(\lambda)|}{(1-|\varphi(\lambda)|^2)^{n+\alpha-1}} \le \sup_{|\varphi(\lambda)|>r} \frac{|\varphi(\lambda)|^n}{r^n}\frac{\mu(\lambda)|u(\lambda)|}{(1-|\varphi(\lambda)|^2)^{n+\alpha-1}} \preceq \frac{M_2}{r^n}. \tag{10}$$

On the other hand, from $u \in H^\infty_\mu$, we have

$$\begin{aligned}\sup_{|\varphi(\lambda)|\le r} \frac{\mu(\lambda)|u(\lambda)|}{(1-|\varphi(\lambda)|^2)^{n+\alpha-1}} &\le \frac{\sup_{|\varphi(\lambda)|\le r}\mu(\lambda)|u(\lambda)|}{(1-r^2)^{n+\alpha-1}} \\ &\le \frac{\|u\|_{H^\infty_\mu}}{(1-r^2)^{n+\alpha-1}} < \infty.\end{aligned} \tag{11}$$

Therefore, (10) and (11) yield the inequality of (e), as well as the asymptotic relation

$$M_3 \preceq \max\left\{M_2, \|u\|_{H^\infty_\mu}\right\}. \tag{12}$$

(e) $\Rightarrow$ (a) By Proposition 8 in [1], if $f \in \mathcal{B}^\alpha$ and $k \in \mathbb{N}$, we see that

$$\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^{k+\alpha-1}\left|f^{(k)}(z)\right| \le C\|f\|_{\mathcal{B}^\alpha},$$

for some constant C independent of f. Therefore, for $z \in \mathbb{D}$, we have

$$\begin{aligned}\mu(z)\left|\left(D^n_{\varphi,u}f\right)(z)\right| &= \mu(z)\left|u(z)\right|\left|f^{(n)}\left(\varphi(z)\right)\right| \\ &\le C\frac{\mu(z)|u(z)|}{(1-|\varphi(z)|^2)^{n+\alpha-1}}\|f\|_{\mathcal{B}^\alpha},\end{aligned} \tag{13}$$

where C is independent of f. Taking the supremum in (13) over $\mathbb{D}$ and then using the first condition in (e) we see that $D^n_{\varphi,u} : \mathcal{B}^\alpha \to H^\infty_\mu$ is bounded, and

$$\left\|D^n_{\varphi,u}\right\|_{\mathcal{B}^\alpha\to H^\infty_\mu} \preceq M_3. \tag{14}$$

If the operator $D^n_{\varphi,u} : \mathcal{B}^\alpha \to H^\infty_\mu$ is bounded, then from (3), (4), (8), (12), and (14), we obtain (2), completing the proof. □

3 Compactness and essential norm of $D^n_{\varphi,u} : \mathcal{B}^\alpha$ (or $\mathcal{B}^\alpha_0$) $\to H^\infty_\mu$

In this section we will give an estimate for the essential norm of the operator $D^n_{\varphi,u} : \mathcal{B}^\alpha \to H^\infty_\mu$, as well as of $D^n_{\varphi,u} : \mathcal{B}^\alpha_0 \to H^\infty_\mu$. For this purpose, we state several lemmas, which will be used in the proof of the main result.

Lemma 3.1 [16] *Let $\alpha > 0$, $m \geq n+1$, where $n \in \mathbb{N}$. Define the function $H_{m,\alpha} : [0,1] \to [0,\infty)$ by*

$$H_{m,\alpha}(x) = \frac{m!}{(m-n-1)!} x^{m-n-1}(1-x)^{n+\alpha}.$$

Then the following statements hold:

(i)

$$\max_{0\leq x\leq 1} H_{m,\alpha}(x) = H_{m,\alpha}(r_m) = \begin{cases} (n+1)!, & m = n+1, \\ \frac{m!}{(m-n-1)!}(\frac{m-n-1}{m+\alpha-1})^{m-n-1}(\frac{n+\alpha}{m+\alpha-1})^{\alpha+n}, & m > n+1, \end{cases}$$

where

$$r_m = \begin{cases} 0, & m = n+1, \\ \frac{m-n-1}{m+\alpha-1}, & m > n+1. \end{cases}$$

(ii) *For $m > n+1$, $H_{m,\alpha}$ is decreasing on $[r_m, r_{m+1}]$, and so*

$$\min_{r_m\leq x\leq r_{m+1}} H_{m,\alpha}(x) = H_{m,\alpha}(r_{m+1}) = \frac{m!}{(m-n-1)!}\left(\frac{m-n}{m+\alpha}\right)^{m-n-1}\left(\frac{n+\alpha}{m+\alpha}\right)^{\alpha+n}.$$

Consequently,

$$\lim_{m\to\infty} m^{\alpha-1} \min_{r_m\leq x\leq r_{m+1}} H_{m,\alpha}(x) = \frac{(n+\alpha)^{n+\alpha}}{e^{n+\alpha}}.$$

Denote by $K_r f(z) = f(rz)$ for $r \in (0,1)$ and $z \in \mathbb{D}$. Then K_r is a compact operator on $\mathcal{B}^\alpha$ for every $\alpha > 0$, and $\|K_r\| \leq 1$ (see, *e.g.*, Proposition 1.3 in [24] and [27]). Let I denote the identity operator. The following three lemmas can be found in [25] (see also [16]).

Lemma 3.2 *Let $0 < \alpha < 1$. Then there is a sequence $(r_k)_{k\in\mathbb{N}}$, with $0 < r_k < 1$ tending to 1, such that the sequence of compact operators $L_j = \frac{1}{j}\sum_{k=1}^j K_{r_k}$, $j \in \mathbb{N}$, on $\mathcal{B}^\alpha_0$ satisfies the following.*

(i) *For any $t \in (0,1)$, $\lim_{j\to\infty} \sup_{\|f\|_{\mathcal{B}^\alpha}\leq 1} \sup_{|z|\leq t} |((I-L_j)f)'(z)| = 0$.*

(ii) $\lim_{j\to\infty} \sup_{\|f\|_{\mathcal{B}^\alpha}\leq 1} \sup_{z\in\mathbb{D}} |(I-L_j)f(z)| = 0$.

(iii) $\limsup_{j\to\infty} \|I-L_j\| \leq 1$.

Furthermore, these statements hold as well for the sequence of biadjoints L_j^{**} on $\mathcal{B}^\alpha$.

Lemma 3.3 *Let $\alpha = 1$. Then there is a sequence $(r_k)_{k\in\mathbb{N}}$, with $0 < r_k < 1$ tending to 1, such that the sequence of compact operators $L_j = \frac{1}{j}\sum_{k=1}^j K_{r_k}$, $j \in \mathbb{N}$, on $\mathcal{B}_0$ satisfies the following.*

(i) *For any $t \in [0,1)$, $\lim_{j\to\infty} \sup_{\|f\|_{\mathcal{B}}\leq 1} \sup_{|z|\leq t} |((I-L_j)f)'(z)| = 0$.*

(iia) $\lim_{j\to\infty} \sup_{\|f\|_{\mathcal{B}}\leq 1} \sup_{|z|>s} |(I-L_j)f(z)|(\log\frac{1}{1-|z|^2})^{-1} \leq 1$, *for s sufficiently close to* 1.

(iib) $\lim_{j\to\infty} \sup_{\|f\|_{\mathcal{B}}\leq 1} \sup_{|z|\leq s} |(I-L_j)f(z)| = 0$ *for the above s.*

(iii) $\limsup_{j\to\infty} \|I-L_j\| \leq 1$.

Furthermore, these statements hold as well for the sequence of biadjoints L_j^{**} on $\mathcal{B}$.

Lemma 3.4 *Let $\alpha > 1$. Then there is a sequence $(r_k)_{k\in\mathbb{N}}$, with $0 < r_k < 1$ tending to 1, such that the sequence of compact operators $L_j = \frac{1}{j}\sum_{k=1}^{j} K_{r_k}$, $j \in \mathbb{N}$, on $\mathcal{B}_0^\alpha$ satisfies the following.*

(i) *For any $t \in [0,1)$, $\lim_{j\to\infty}\sup_{\|f\|_{\mathcal{B}^\alpha}\le 1}\sup_{|z|\le t}|((I-L_j)f)'(z)| = 0$.*

(ii) *For any $t \in [0,1)$, $\lim_{j\to\infty}\sup_{\|f\|_{\mathcal{B}^\alpha}\le 1}\sup_{|z|\le t}|(I-L_j)f(z)| = 0$.*

(iii) $\limsup_{j\to\infty}\|I-L_j\| \le 1$.

Furthermore, these statements hold as well for the sequence of biadjoints L_j^{**} on $\mathcal{B}^\alpha$.

To study the compactness, we also need the following lemma, which can be proved in a standard way (see, for example, Proposition 3.11 in [23]).

Lemma 3.5 *Let n be a nonnegative integer, $\alpha > 0$, μ a weight, $u \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then $D_{\varphi,u}^n : \mathcal{B}^\alpha$ (or $\mathcal{B}_0^\alpha$) $\to H_\mu^\infty$ is compact if and only if $D_{\varphi,u}^n : \mathcal{B}^\alpha$ (or $\mathcal{B}_0^\alpha$) $\to H_\mu^\infty$ is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathcal{B}^\alpha$, which converges to zero uniformly on compact subsets of $\mathbb{D}$,*

$$\lim_{k\to\infty}\left\|D_{\varphi,u}^n f_k\right\|_{H_\mu^\infty} = 0.$$

Now we are ready to state and prove the main results in this section.

Theorem 3.6 *Let n be a positive integer, $\alpha > 0$, μ a weight, $u \in H(\mathbb{D})$, and φ be an analytic self-map of $\mathbb{D}$. Suppose that $D_{\varphi,u}^n : \mathcal{B}^\alpha \to H_\mu^\infty$ is bounded. Then*

$$\left\|D_{\varphi,u}^n\right\|_{e,\mathcal{B}^\alpha\to H_\mu^\infty} \approx \left\|D_{\varphi,u}^n\right\|_{e,\mathcal{B}_0^\alpha\to H_\mu^\infty} \approx \limsup_{j\to\infty} j^{\alpha-1}\left\|D_{\varphi,u}^n(p_j)\right\|_{H_\mu^\infty}. \tag{15}$$

Proof First note that the inequality

$$\left\|D_{\varphi,u}^n\right\|_{e,\mathcal{B}_0^\alpha\to H_\mu^\infty} \le \left\|D_{\varphi,u}^n\right\|_{e,\mathcal{B}^\alpha\to H_\mu^\infty} \tag{16}$$

obviously holds.

Now we give a lower estimate for the essential norm $\|D_{\varphi,u}^n\|_{e,\mathcal{B}_0^\alpha\to H_\mu^\infty}$. Without loss of generality, we assume that $j \ge n$. Choose the sequence of functions $q_j = j^{\alpha-1}p_j \in \mathcal{B}_0^\alpha$, $j \in \mathbb{N}$. Then $\|q_j\|_{\mathcal{B}^\alpha} \approx 1$, and $(q_j)_{j\in\mathbb{N}}$ converges to zero weakly on $\mathcal{B}_0^\alpha$ as $j \to \infty$ (see, for example, Theorem 7.5 in [30]). Since by a well-known theorem, for any compact operator $\widehat{K} : X \to Y$, where X and Y are Banach spaces, the weak convergence $x_n \xrightarrow{w} x_0$ implies the norm convergence $\widehat{K}x_n \to \widehat{K}x_0$ [31], we have

$$\lim_{j\to\infty}\|Kq_j\|_{H_\mu^\infty} = 0, \tag{17}$$

for any given compact operator K from $\mathcal{B}_0^\alpha$ to H_μ^∞.

Hence

$$\left\|D_{\varphi,u}^n - K\right\|_{\mathcal{B}_0^\alpha\to H_\mu^\infty} \succeq \left\|\left(D_{\varphi,u}^n - K\right)q_j\right\|_{H_\mu^\infty} \ge \left\|D_{\varphi,u}^n q_j\right\|_{H_\mu^\infty} - \|Kq_j\|_{H_\mu^\infty}.$$

Letting $j \to \infty$ in the last relation and using (17), we obtain

$$\left\|D_{\varphi,u}^n - K\right\|_{\mathcal{B}_0^\alpha\to H_\mu^\infty} \succeq \limsup_{j\to\infty}\left\|D_{\varphi,u}^n q_j\right\|_{H_\mu^\infty} = \limsup_{j\to\infty} j^{\alpha-1}\left\|D_{\varphi,u}^n(p_j)\right\|_{H_\mu^\infty},$$

and consequently

$$\left\|D^n_{\varphi,u}\right\|_{e,\mathcal{B}^\alpha_0\to H^\infty_\mu}=\inf_K\left\|D^n_{\varphi,u}-K\right\|_{\mathcal{B}^\alpha_0\to H^\infty_\mu}\succeq\limsup_{j\to\infty}j^{\alpha-1}\left\|D^n_{\varphi,u}(p_j)\right\|_{H^\infty_\mu}. \tag{18}$$

Now, we give the upper estimates for the essential norm $\|D^n_{\varphi,u}\|_{e,\mathcal{B}^\alpha\to H^\infty_\mu}$. For the case of $\sup_{z\in\mathbb{D}}|\varphi(z)|<1$, there is a number $\delta\in(0,1)$ such that $\sup_{z\in\mathbb{D}}|\varphi(z)|<\delta$. In this case, the operator $D^n_{\varphi,u}:\mathcal{B}^\alpha\to H^\infty_\mu$ is compact. Indeed, choose a bounded sequence $(f_j)_{j\in\mathbb{N}}$ in $\mathcal{B}^\alpha$ which converges to zero uniformly on compact subsets of $\mathbb{D}$. From Cauchy's integral formula, $(f_j^{(n)})_{j\in\mathbb{N}}$ also converges to zero on compact subsets of $\mathbb{D}$ as $j\to\infty$. Hence

$$\begin{aligned}\lim_{j\to\infty}\left\|D^n_{\varphi,u}f_j\right\|_{H^\infty_\mu}&=\lim_{j\to\infty}\sup_{z\in\mathbb{D}}\mu(z)\left|u(z)f_j^{(n)}\big(\varphi(z)\big)\right|\\&\le\|u\|_{H^\infty_\mu}\lim_{j\to\infty}\sup_{z\in\mathbb{D}}\left|f_j^{(n)}\big(\varphi(z)\big)\right|\\&=\|u\|_{H^\infty_\mu}\lim_{j\to\infty}\sup_{|w|\le\delta}\left|f_j^{(n)}(w)\right|=0.\end{aligned}$$

From this and by Lemma 3.5 we see that the operator $D^n_{\varphi,u}:\mathcal{B}^\alpha\to H^\infty_\mu$ is compact. This also shows that

$$\left\|D^n_{\varphi,u}\right\|_{e,\mathcal{B}^\alpha\to H^\infty_\mu}=0. \tag{19}$$

From (16), (18), and (19), we get the desired result in the case $\sup_{z\in\mathbb{D}}|\varphi(z)|<1$.

Next, we assume that $\sup_{z\in\mathbb{D}}|\varphi(z)|=1$. Let $(L_j)_{j\in\mathbb{N}}$ be the sequence of operators given in Lemmas 3.2-3.4. Since L_j^{**} is compact on $\mathcal{B}^\alpha$, for every $j\in\mathbb{N}$, and $D^n_{\varphi,u}$ is bounded from $\mathcal{B}^\alpha$ to H^∞_μ, then $D^n_{\varphi,u}L_j^{**}$ is also compact from $\mathcal{B}^\alpha$ to H^∞_μ. Hence

$$\begin{aligned}\left\|D^n_{\varphi,u}\right\|_{e,\mathcal{B}^\alpha\to H^\infty_\mu}&\le\limsup_{j\to\infty}\left\|D^n_{\varphi,u}-D^n_{\varphi,u}L_j^{**}\right\|_{\mathcal{B}^\alpha\to H^\infty_\mu}\\&=\limsup_{j\to\infty}\left\|D^n_{\varphi,u}\big(I-L_j^{**}\big)\right\|_{\mathcal{B}^\alpha\to H^\infty_\mu}\\&=\limsup_{j\to\infty}\sup_{\|f\|_{\mathcal{B}^\alpha}\le1}\left\|D^n_{\varphi,u}\big(I-L_j^{**}\big)f\right\|_{H^\infty_\mu}\\&=\limsup_{j\to\infty}\sup_{\|f\|_{\mathcal{B}^\alpha}\le1}\sup_{z\in\mathbb{D}}\mu(z)\left|u(z)\big(\big(I-L_j^{**}\big)f\big)^{(n)}\big(\varphi(z)\big)\right|.\end{aligned}$$

For each positive integer $i\ge n$, we define $\mathbb{D}_i=\{z\in\mathbb{D}:r_i\le|\varphi(z)|<r_{i+1}\}$, where r_i is given in Lemma 3.1. Let k be the smallest positive integer such that $\mathbb{D}_k\ne\emptyset$. Since $\sup_{z\in\mathbb{D}}|\varphi(z)|=1$, $\mathbb{D}_i$ is not empty for every integer $i\ge k$ and $\mathbb{D}=\bigcup_{i=k}^\infty\mathbb{D}_i$, we have

$$\sup_{\|f\|_{\mathcal{B}^\alpha}\le1}\sup_{z\in\mathbb{D}}\mu(z)\left|u(z)\big(\big(I-L_j^{**}\big)f\big)^{(n)}\big(\varphi(z)\big)\right|=I_1+I_2,$$

where

$$I_1=\sup_{\|f\|_{\mathcal{B}^\alpha}\le1}\sup_{k\le i\le N-1}\sup_{z\in\mathbb{D}_i}\mu(z)\left|u(z)\big(\big(I-L_j^{**}\big)f\big)^{(n)}\big(\varphi(z)\big)\right|$$

and

$$I_2 = \sup_{\|f\|_{\mathcal{B}^\alpha}\le 1} \sup_{N\le i} \sup_{z\in\mathbb{D}_i} \mu(z)\left|u(z)\left(\left(I-L_j^{**}\right)f\right)^{(n)}\left(\varphi(z)\right)\right|.$$

Here N is a positive integer determined as follows.

By Lemma 3.1, $\lim_{i\to\infty} \frac{i^{1-\alpha}}{H_{i,\alpha}(r_{i+1})} = \frac{e^{n+\alpha}}{(n+\alpha)^{n+\alpha}}$. Hence, for any given $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$\frac{i^{1-\alpha}}{H_{i,\alpha}(r_{i+1})} \le \frac{e^{n+\alpha}}{(n+\alpha)^{n+\alpha}} + \varepsilon$$

when $i \ge N$. For such N it follows that

$$\begin{aligned} I_2 &= \sup_{\|f\|_{\mathcal{B}^\alpha}\le 1} \sup_{N\le i} \sup_{z\in\mathbb{D}_i} \mu(z)\left|u(z)\left(\left(I-L_j^{**}\right)f\right)^{(n)}\left(\varphi(z)\right)\right| \\ &= \sup_{\|f\|_{\mathcal{B}^\alpha}\le 1} \sup_{N\le i} \sup_{z\in\mathbb{D}_i} \mu(z)\left|u(z)\left(\left(I-L_j^{**}\right)f\right)^{(n)}\left(\varphi(z)\right)\right| \frac{H_{i,\alpha}(|\varphi(z)|)}{i^{1-\alpha}} \frac{i^{1-\alpha}}{H_{i,\alpha}(|\varphi(z)|)} \\ &\preceq \left(\frac{e^{n+\alpha}}{(n+\alpha)^{n+\alpha}} + \varepsilon\right) \sup_{\|f\|_{\mathcal{B}^\alpha}\le 1} \left\|\left(I-L_j^{**}\right)f\right\|_{\mathcal{B}^\alpha} \sup_{N\le i} \sup_{z\in\mathbb{D}_i} \mu(z)\left|u(z)\right| \frac{i!}{(i-n)!} \frac{|\varphi(z)|^{i-n}}{i^{1-\alpha}} \\ &\preceq \left\|I-L_j^{**}\right\| \sup_{N\le i} i^{\alpha-1} \left\|D_{\varphi,u}^n(p_i)\right\|_{H_\mu^\infty}. \end{aligned}$$

Thus

$$\limsup_{j\to\infty} I_2 \preceq \sup_{i\ge N} i^{\alpha-1} \left\|D_{\varphi,u}^n(p_i)\right\|_{H_\mu^\infty}. \tag{20}$$

By Lemmas 3.2, 3.3, 3.4, and Cauchy's integral formula, we have

$$\begin{aligned} \limsup_{j\to\infty} I_1 &= \limsup_{j\to\infty} \sup_{\|f\|_{\mathcal{B}^\alpha}\le 1} \sup_{k\le i\le N-1} \sup_{z\in\mathbb{D}_i} \mu(z)\left|u(z)\left(\left(I-L_j^{**}\right)f\right)^{(n)}\left(\varphi(z)\right)\right| \\ &\le \|u\|_{H_\mu^\infty} \limsup_{j\to\infty} \sup_{\|f\|_{\mathcal{B}^\alpha}\le 1} \sup_{|\varphi(z)|<r_N} \left|\left(\left(I-L_j^{**}\right)f\right)^{(n)}\left(\varphi(z)\right)\right| = 0, \end{aligned}$$

which together with (20) implies that

$$\begin{aligned} &\limsup_{j\to\infty} \sup_{\|f\|_{\mathcal{B}^\alpha}\le 1} \sup_{z\in\mathbb{D}} \mu(z)\left|u(z)\left(\left(I-L_j^{**}\right)f\right)^{(n)}\left(\varphi(z)\right)\right| \\ &\quad\preceq \sup_{i\ge N} i^{\alpha-1} \left\|D_{\varphi,u}^n(p_i)\right\|_{H_\mu^\infty}. \end{aligned}$$

Therefore

$$\left\|D_{\varphi,u}^n\right\|_{e,\mathcal{B}^\alpha\to H_\mu^\infty} \preceq \sup_{i\ge N} i^{\alpha-1} \left\|D_{\varphi,u}^n(p_i)\right\|_{H_\mu^\infty}.$$

From the last relation we get

$$\left\|D_{\varphi,u}^n\right\|_{e,\mathcal{B}^\alpha\to H_\mu^\infty} \preceq \limsup_{i\to\infty} i^{\alpha-1} \left\|D_{\varphi,u}^n(p_i)\right\|_{H_\mu^\infty}. \tag{21}$$

From (16), (18), and (21), the asymptotic relations in (15) follow, completing the proof of the theorem. □

From Theorem 3.6, letting $\alpha = 1$ we deduce the following result.

Corollary 3.7 *Let n be a positive integer, μ a weight, $u \in H(\mathbb{D})$, and φ be an analytic self-map of $\mathbb{D}$. Suppose that $D^n_{\varphi,u} : \mathcal{B} \to H^\infty_\mu$ is bounded. Then*

$$\|D^n_{\varphi,u}\|_{e,\mathcal{B}\to H^\infty_\mu} \approx \|D^n_{\varphi,u}\|_{e,\mathcal{B}_0\to H^\infty_\mu} \approx \limsup_{j\to\infty}\|D^n_{\varphi,u}(p_j)\|_{H^\infty_\mu}.$$

Theorem 3.8 *Let n be a positive integer, $\alpha > 0$, μ a weight, $u \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. If $D^n_{\varphi,u} : \mathcal{B}^\alpha \to H^\infty_\mu$ is bounded, then the following statements are equivalent.*

(a) *The operator $D^n_{\varphi,u} : \mathcal{B}^\alpha \to H^\infty_\mu$ is compact.*
(b) *The operator $D^n_{\varphi,u} : \mathcal{B}^\alpha_0 \to H^\infty_\mu$ is compact.*
(c) $\lim_{j\to\infty} j^{\alpha-1}\|D^n_{\varphi,u}(p_j)\|_{H^\infty_\mu} = 0$.
(d) $\lim_{|\varphi(w)|\to 1} \|D^n_{\varphi,u}f_{\varphi(w)}\|_{H^\infty_\mu} = 0$.
(e) $\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|u(z)|}{(1-|\varphi(z)|^2)^{n+\alpha-1}} = 0$.

Proof The equivalence of statements (a)-(c) follows from Theorem 3.6.

(c) ⇒ (d) From (c), we see that, for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that

$$j^{\alpha-1}\|D^n_{\varphi,u}(p_j)\|_{H^\infty_\mu} < \varepsilon/2,$$

for all $j \geq N$.

Let $(z_k)_{k\in\mathbb{N}} \subset \mathbb{D}$ be an arbitrary sequence such that $|\varphi(z_k)| \to 1$ as $k \to \infty$ (if such a sequence does not exist then the equality in (d) vacuously holds). Similarly to the proof of Theorem 2.1, we have

$$\begin{aligned}\|D^n_{\varphi,u}f_{\varphi(z_k)}\|_{H^\infty_\mu} &\leq C(1-|\varphi(z_k)|^2)\sum_{j=n}^{\infty}|\varphi(z_k)|^j j^{\alpha-1}\|D^n_{\varphi,u}(p_j)\|_{H^\infty_\mu}\\ &= C(1-|\varphi(z_k)|^2)\sum_{j=n}^{N-1}|\varphi(z_k)|^j j^{\alpha-1}\|D^n_{\varphi,u}(p_j)\|_{H^\infty_\mu}\\ &\quad + C(1-|\varphi(z_k)|^2)\sum_{j=N}^{\infty}|\varphi(z_k)|^j j^{\alpha-1}\|D^n_{\varphi,u}(p_j)\|_{H^\infty_\mu}\\ &\leq 2C(1-|\varphi(z_k)|^N)M_0 + C\varepsilon, \end{aligned} \tag{22}$$

for $k \in \mathbb{N}$, where $M_0 = \max_{n\leq j\leq N-1} j^{\alpha-1}\|D^n_{\varphi,u}(p_j)\|_{H^\infty_\mu}$.

Since $|\varphi(z_k)| \to 1$ as $k \to \infty$, from (22), we deduce that

$$\limsup_{k\to\infty}\|D^n_{\varphi,u}f_{\varphi(z_k)}\|_{H^\infty_\mu} \leq C\varepsilon.$$

Since ε is an arbitrary positive number, the implication follows.

(d) ⇒ (e) Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $\lim_{k\to\infty}|\varphi(z_k)| = 1$ (if such a sequence does not exist then the implication vacuously holds). Since the sequence $(f_{\varphi(z_k)})_{k\in\mathbb{N}}$

is bounded in $\mathcal{B}^\alpha$ and converges to 0 uniformly on compact subsets of $\mathbb{D}$, by (9) and Lemma 3.5, we have

$$\frac{\mu(z_k)|u(z_k)||\varphi(z_k)|^n \prod_{j=0}^{n-1}(\alpha+j)}{(1-|\varphi(z_k)|^2)^{n+\alpha-1}} \le \left\|D^n_{\varphi,u} f_{\varphi(z_k)}\right\|_{H^\infty_\mu} \to 0 \quad \text{as } k \to \infty.$$

Therefore

$$\lim_{k\to\infty} \frac{\mu(z_k)|u(z_k)|}{(1-|\varphi(z_k)|^2)^{n+\alpha-1}} = \lim_{k\to\infty} \frac{\mu(z_k)|u(z_k)||\varphi(z_k)|^n}{(1-|\varphi(z_k)|^2)^{n+\alpha-1}} = 0, \tag{23}$$

which implies (e).

(e) $\Rightarrow$ (a) Assume $(f_k)_{k\in\mathbb{N}}$ is a bounded sequence in $\mathcal{B}^\alpha$ converging to 0 uniformly on compact subsets of $\mathbb{D}$. By the assumption, for any $\varepsilon > 0$, there exists a $\delta \in (0,1)$ such that

$$\frac{\mu(z)|u(z)|}{(1-|\varphi(z)|^2)^{n+\alpha-1}} < \varepsilon \tag{24}$$

when $\delta < |\varphi(z)| < 1$.

Therefore, since $u \in H^\infty_\mu$ we have

$$\begin{aligned} \left\|D^n_{\varphi,u} f_k\right\|_{H^\infty_\mu} &= \sup_{z\in\mathbb{D}} \mu(z)\left|\left(D^n_{\varphi,u} f_k\right)(z)\right| \\ &\le \sup_{z\in\Omega_\delta} \mu(z)\left|u(z)\right|\left|f_k^{(n)}\left(\varphi(z)\right)\right| \\ &\quad + C \sup_{z\in\mathbb{D}\setminus\Omega_\delta} \frac{\mu(z)|u(z)|}{(1-|\varphi(z)|^2)^{n+\alpha-1}} \|f_k\|_{\mathcal{B}^\alpha} \\ &\le \|u\|_{H^\infty_\mu} \sup_{z\in\Omega_\delta} \left|f_k^{(n)}\left(\varphi(z)\right)\right| + C\varepsilon \|f_k\|_{\mathcal{B}^\alpha}, \end{aligned} \tag{25}$$

where $\Omega_\delta = \{z \in \mathbb{D} : |\varphi(z)| \le \delta\}$.

Since $(f_k)_{k\in\mathbb{N}}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$, by Cauchy's estimate so do the sequences $(f_k^{(n)})_{k\in\mathbb{N}}$ for every $n \in \mathbb{N}$. Letting $k \to \infty$ in (25) and using the fact that ε is an arbitrary positive number, we obtain $\lim_{k\to\infty} \|D^n_{\varphi,u} f_k\|_{H^\infty_\mu} = 0$. By Lemma 3.5, we deduce that $D^n_{\varphi,u} : \mathcal{B}^\alpha \to H^\infty_\mu$ is compact. □

Competing interests
The authors declare that they do not have competing interests.

Authors' contributions
Both authors contributed equally to the writing of this paper. They read and approved the final version of the manuscript.

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