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A class of mixed orthogonal arrays obtained from projection matrix inequalities

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Abstract

By using the relationship between orthogonal arrays and decompositions of projection matrices and projection matrix inequalities, we present a method for constructing a class of new orthogonal arrays which have higher percent saturations. As an application of the method, many new mixed-level orthogonal arrays of run sizes 108 and 144 are constructed.

MSC: 62K15; 05B15

Keywords: matrix inequality; orthogonal array; orthogonal projection matrix; matrix image

1 Introduction

An $n \times m$ matrix A, having k_i columns with p_i levels, $i = 1, 2, \ldots, r$, $m = \sum_{i=1}^{r} k_i$, $p_i \neq p_j$ for $i \neq j$, is called an orthogonal array (OA) of strength d and size n if each $n \times d$ submatrix of A contains all possible $1 \times d$ row vectors with the same frequency. Unless stated otherwise, we use the notation $L_n(p_1^{k_1} \cdots p_r^{k_r})$ for an OA of strength 2. An orthogonal array is said to have mixed levels if $r \geq 2$. Orthogonal arrays have been used extensively in statistical design of experiments, computer science and cryptography. Constructions of OAs have been studied extensively in the literature; see Hedayet *et al.* [1, 2], Zhang *et al.* [3, 7], Pang *et al.* [5], Pang [6], Zhang *et al.* [7], Chen *et al.* [8], Du *et al.* [9], *etc.* Zhang *et al.* [3] present a method of construction of orthogonal arrays of strength two by using a relationship between orthogonal arrays and decompositions of projection matrices. In the construction of new mixed orthogonal arrays, two goals should be kept in mind, first, we want the orthogonal array to be as close to a saturated main-effect plan as possible so that there will be a large number of factors and second, we want the p_i, the number of levels, to be as large as possible so that the design has a high degree of flexibility (see Mandeli [10]). In this paper by using projection matrix inequalities and further exploring the relationship, matrix images of a class of orthogonal arrays can be found. Therefore we construct a class of new orthogonal arrays. If, as in Mandeli [10], we still define the percent saturation of an OA $L_n(p_1^{k_1} \cdots p_r^{k_r})$ to be $\sum_{i=1}^{r} k_i(p_i - 1)/(n - 1) \times 100\%$, then the orthogonal arrays constructed in this paper have higher percent saturations.

2 Basic concepts and main theorems

The following definitions, notations, and results are needed in the sequel.

Definition 1 A matrix A is said to be an orthogonal projection matrix if it is idempotent ($A^2 = A$) and symmetric ($A^T = A$).

Definition 2 Suppose that an experiment is carried out according to an array $A = (a_{ij})_{n\times m} = (a_1,\dots,a_m)$, and $Y = (Y_1,\dots,Y_n)^T$ is the experimental data vector. In the analysis of variance S_j^2, the sum of squares of the jth factor, is defined as

$$S_j^2 = \sum_{i=0}^{p_j-1} \frac{1}{|I_{ij}|}\left(\sum_{s\in I_{ij}} Y_s\right)^2 - \frac{1}{n}\left(\sum_{s=1}^{n} Y_s\right)^2, \tag{1}$$

where $I_{ij} = \{s : a_{sj} = i\}$ and $|I_{ij}|$ is the number of elements in I_{ij}. From (1), S_j^2 is a quadratic form in Y and there exists a unique symmetric matrix A_j such that $S_j^2 = Y^T A_j Y$. The matrix A_j is called the *matrix image* (MI) of the jth column a_j of A, denoted by $m(a_j) = A_j$. The MI of a subarray of A is defined as the sum of the MIs of all its columns. In particular, we denote the MI of A by $m(A)$. Let 1_r be the $r \times 1$ vector of 1's. Then $m(1_r) = P_r$ where $P_r = \frac{1}{r}1_r1_r^T$.

Let $(r) = (0,\dots,r-1)^T$, $e_i(r) = (0\cdots 0\overset{i}{1}0\cdots 0)^T_{1\times r}$ and I_r be the identity matrix of order r. Then $m((r)) = \tau_r$ where $\tau_r = I_r - P_r$. The following permutation matrices are very useful:

$$N_r = e_1(r)e_2^T(r) + \cdots + e_{r-1}(r)e_r^T(r) + e_r(r)e_1^T(r)$$

and

$$K(p,q) = \sum_{i=1}^{p}\sum_{j=1}^{q} e_i(p)e_j^T(q) \otimes e_j(q)e_i^T(p),$$

where $\otimes$ is the usual Kronecker product in the theory of matrices. Sometimes, it is necessary and easy to use the following properties of these two permutation matrices to obtain the orthogonal arrays needed:

$$N_r(r) = 1_r + (r), \quad \bmod r,$$
$$K(p,q)\big(1_q \otimes (p)\big) = (p) \otimes 1_q, \qquad K(p,q)\big((q)\otimes(p)\big) = (p)\otimes(q)$$

and

$$K(p,q)(P_q \otimes \tau_p)K^T(p,q) = \tau_p \otimes P_q, \qquad K(p,q)(\tau_q \otimes \tau_p)K^T(p,q) = \tau_p \otimes \tau_q.$$

Lemma 1 *For any permutation matrix S and any array L, $m(S(L\otimes 1_r)) = S(m(L)\otimes P_r)S^T$, and $m(S(1_r\otimes L)) = S(P_r \otimes m(L))S^T$.*

Lemma 2 *Let A be an OA of strength* 1, *i.e.*,

$$A = (a_1,\dots,a_m) = \big(S_1\big(1_{r_1}\otimes(p_1)\big),\dots,S_m\big(1_{r_m}\otimes(p_m)\big)\big),$$

where $r_jp_j = n$ and S_i is a permutation matrix, for $i = 1,\dots,m$. The following statements are equivalent.

(1) *A is an OA of strength* 2.
(2) $m(A)$ *is a projection matrix.*
(3) $m(a_i)m(a_j) = 0$ $(i \neq j)$.
(4) *The projection matrix* τ_n *can be decomposed as* $\tau_n = m(a_1) + \cdots + m(a_m) + \triangle$, *where* $\mathrm{rk}(\triangle) = n - 1 - \sum_{j=1}^{m}(p_j - 1)$ *is the rank of the matrix* $\triangle$.

Lemma 3 *Suppose L and H are OAs satisfying* $m(L)m(H) = 0$. *Then* $K = (L, H)$ *is also an OA.*

Lemma 4 *Let* (L, H) *and K be OAs of run size n. Then* (K, H) *is also an OA if* $m(K) \leq m(L)$, *where* $A \leq B$ *means that the difference* $B - A$ *is nonnegative.*

Lemmas 1, 2, 3, and 4 can be found in Zhang *et al.* [3].
Now we state the following theorems.

Theorem 1 *If* $L = [(r) \otimes 1_p, H]$ *is an OA, then* $m(H) \leq I_r \otimes \tau_p$. *If* $L = [1_p \otimes (r), K]$ *is an OA, then* $m(K) \leq \tau_p \otimes I_r$.

Proof From Lemma 2, we have $m(L) = \tau_r \otimes P_p + m(H)$. Since $m(L) \leq \tau_{rp}$, $\tau_{rp} = \tau_r \otimes P_p + I_r \otimes \tau_p$ and $m((r) \otimes 1_p) = \tau_r \otimes P_p$, it follows that $m(H) \leq I_r \otimes \tau_p$.
Similarly, we can prove that $m(K) \leq \tau_p \otimes I_r$. □

Corollary *If p is a prime and* $D(r, m; p)$ *is a p-level difference matrix, then both* $D(r, m; p) \oplus (p)$ *and* $(p) \oplus D(r, m; p)$ *are OAs, and* $m(D(r, m; p) \oplus (p)) \leq I_r \otimes \tau_p$ *and* $m((p) \oplus D(r, m; p)) \leq \tau_p \otimes I_r$.

Proof From Bose and Bush [11], we see that

$$L = \left[(r) \otimes 1_p, D(r, m; p) \oplus (p)\right]$$

is an OA. From Theorem 1, it follows that $m(D(r, m; p) \oplus (p)) \leq I_r \otimes \tau_p$. Similarly, we can prove that $(p) \oplus D(r, m; p)$ is an OA and $m((p) \oplus D(r, m; p)) \leq \tau_p \otimes I_r$. □

Theorem 2 *If p is a prime and* $D(r, m; p)$ *is a p-level difference matrix, then* $D(r, m; p) \oplus 0_q \oplus (p)$ *is an OA, and* $m(D(r, m; p) \oplus 0_q \oplus (p)) \leq I_r \otimes P_q \otimes \tau_p$.

Proof From Lemma 1 and Theorem 1, we have $m(D(r, m; p) \oplus (p) \oplus 0_q) \leq I_r \otimes \tau_p \otimes P_q$,

$$\begin{aligned} & m\big(D(r, m; p) \oplus 0_q \oplus (p)\big) \\ & \quad = m\big(\big(I_r \otimes K(q, p)\big)\big(D(r, m; p) \oplus (p) \oplus 0_q\big)\big) \\ & \quad \leq \big(I_r \otimes K(q, p)\big)(I_r \otimes \tau_p \otimes P_q)\big(I_r \otimes K(q, p)\big)^T \\ & \quad = I_r \otimes P_q \otimes \tau_p. \end{aligned}$$

□

Theorem 3 *If* $\tau_p \otimes \tau_q = \sum_{i=1}^{k} S_i(\tau_p \otimes P_q)S_i^T$ *is an orthogonal decomposition of* $\tau_p \otimes \tau_q$, *then* $I_r \otimes \tau_p \otimes \tau_q = \sum_{i=1}^{k}(I_r \otimes S_i)(I_r \otimes \tau_p \otimes P_q)(I_r \otimes S_i)^T$ *is an orthogonal decomposition of* $I_r \otimes \tau_p \otimes \tau_q$. *If there exists an OA H such that* $m(H) \leq I_r \otimes \tau_p$, *then* $L = [(I_r \otimes S_1)(H \otimes 1_q), \ldots, (I_r \otimes S_k)(H \otimes 1_q)]$ *is also an OA.*

Proof Since $\tau_p \otimes \tau_q = \sum_{i=1}^{k} S_i(\tau_p \otimes P_q)S_i^T$, we have

$$\begin{aligned} I_r \otimes \tau_p \otimes \tau_q &= I_r \otimes \left(\sum_{i=1}^{k} S_i(\tau_p \otimes P_q)S_i^T\right) \\ &= \sum_{i=1}^{k}(I_rI_rI_r) \otimes \bigl(S_i(\tau_p \otimes P_q)S_i^T\bigr). \end{aligned}$$

Using the property $(ABC) \otimes (DEF) = (A \otimes D)(B \otimes E)(C \otimes F)$, we obtain

$$I_r \otimes \tau_p \otimes \tau_q = \sum_{i=1}^{k}(I_r \otimes S_i)(I_r \otimes \tau_p \otimes P_q)(I_r \otimes S_i)^T.$$

Because of the orthogonality in each step, the above decomposition of $I_r \otimes \tau_p \otimes \tau_q$ is orthogonal. By Lemma 1, we have

$$m\bigl((I_r \otimes S_i)(H \otimes 1_q)\bigr) = (I_r \otimes S_i)(H \otimes 1_q)(I_r \otimes S_i)^T \le (I_r \otimes S_i)(I_r \otimes \tau_p \otimes P_q)(I_r \otimes S_i)^T.$$

By Lemmas 3 and 4, we see that $(I_r \otimes S_i)(H \otimes 1_q)$ and $(I_r \otimes S_j)(H \otimes 1_q)$ is orthogonal $(i \neq j)$. It follows from Lemma 2 that L is an OA. □

3 Some examples

These matrix inequalities in Theorems 1, 2, and 3 are very useful for construction of orthogonal arrays. We illustrate their applications with some examples. We begin with OAs $L_9(3^4)$ and $L_{16}(4^5)$ and their properties:

$$L_9\bigl(3^4\bigr) = \bigl((3) \otimes 1_3, 1_3 \otimes (3), a, b\bigr) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 & 2 & 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 2 & 0 & 1 & 1 & 2 & 0 \end{bmatrix}^T.$$

For this OA $L_9(3^4)$, there exists a 9×9 permutation T_2 (as follows) such that $((3) \otimes 1_3, 1_3 \otimes (3)) = T_2(a,b)$. From Lemma 2, we have $m(L_9(3^4)) = \tau_9$, $m(a,b) = \tau_9 - \tau_3 \otimes P_3 - P_3 \otimes \tau_3 = \tau_3 \otimes \tau_3$ and $m((3) \otimes 1_3, 1_3 \otimes (3)) = T_2(\tau_3 \otimes \tau_3)T_2^T$. Hence

$$\tau_9 = \sum_{i=1}^{2} T_i(\tau_3 \otimes \tau_3)T_i^T,$$

where $T_1 = I_9$ and

$$T_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

On the other hand, we have $\tau_9 - \tau_3 \otimes P_3 - P_3 \otimes \tau_3 = \tau_3 \otimes \tau_3$. Also, $\tau_3 \otimes \tau_3 = \sum_{i=1}^{2} Q_i(\tau_3 \otimes P_3)Q_i^T$, where $Q_1 = \operatorname{diag}(I_3, N_3, N_3^2)K(3,3)$ and $Q_2 = \operatorname{diag}(I_3, N_3^2, N_3)K(3,3)$.

Consider the orthogonal array $L_{16}(4^5)$:

$$L_{16}(4^5) = \bigl((4)\otimes 1_4, 1_4 \otimes (4), c, d, f\bigr) = \begin{bmatrix} 0&0&0&0&1&1&1&1&2&2&2&2&3&3&3&3\\ 0&1&2&3&0&1&2&3&0&1&2&3&0&1&2&3\\ 0&1&2&3&1&0&3&2&2&3&0&1&3&2&1&0\\ 0&1&2&3&2&3&0&1&3&2&1&0&1&0&3&2\\ 0&1&2&3&3&2&1&0&1&0&3&2&2&3&0&1 \end{bmatrix}^T.$$

From Lemma 2, we have $\tau_{16} = \sum_{i=1}^{2} S_i(\tau_4 \otimes P_4)S_i^T + \tau_4 \otimes \tau_4$, where $S_1 = I_{16}$ and $S_2 = K(4,4)$.

Moreover, we have $\tau_4 \otimes \tau_4 = \sum_{i=3}^{5} S_i(\tau_4 \otimes P_4)S_i^T$, where $S_3 = \operatorname{diag}(Q'_1, Q'_2, Q'_3, Q'_4)K(4,4)$, $S_4 = \operatorname{diag}(Q'_1, Q'_3, Q'_4, Q'_2)K(4,4)$, $S_5 = \operatorname{diag}(Q'_1, Q'_4, Q'_2, Q'_3)K(4,4)$, $Q'_1 = I_4$, $Q'_2 = I_2 \otimes N_2$, $Q'_3 = N_2 \otimes I_2$ and $Q'_4 = N_2 \otimes N_2$.

Example 1 (Construction of orthogonal arrays of run size 108) Orthogonally decompose the projection matrix τ_{108} as follows:

$$\begin{aligned} \tau_{108} &= I_{12} \otimes \tau_9 + \tau_{12} \otimes P_9 \\ &= I_{12} \otimes \tau_3 \otimes \tau_3 + I_{12} \otimes \tau_3 \otimes P_3 + I_{12} \otimes P_3 \otimes \tau_3 \\ &\quad + (P_6 \otimes \tau_2 + P_3 \otimes \tau_2 \otimes \tau_2 + \tau_3 \otimes I_4 + P_3 \otimes \tau_2 \otimes P_2) \otimes P_9 \\ &= I_{12} \otimes \tau_3 \otimes \tau_3 + I_{12} \otimes \tau_3 \otimes P_3 + (P_6 \otimes \tau_2 + P_3 \otimes \tau_2 \otimes \tau_2) \otimes P_9 \\ &\quad + I_{12} \otimes P_3 \otimes \tau_3 + (\tau_3 \otimes I_4 + P_3 \otimes \tau_2 \otimes P_2) \otimes P_9 \\ &= I_{12} \otimes \tau_3 \otimes \tau_3 + \bigl(I_{12} \otimes \tau_3 + (P_6 \otimes \tau_2 + P_3 \otimes \tau_2 \otimes \tau_2) \otimes P_3\bigr) \otimes P_3 \\ &\quad + Q\bigl(\bigl(I_{12} \otimes \tau_3 + (\tau_3 \otimes I_4 + P_3 \otimes \tau_2 \otimes P_2) \otimes P_3\bigr) \otimes P_3\bigr)Q^T, \end{aligned}$$

where $Q = I_{12} \otimes K(3,3)$.

Now we want to find OAs H_1, H_2 and H_3 such that $m(H_1) \le I_{12} \otimes \tau_3 \otimes \tau_3$, and $m(H_2) \le I_{12} \otimes \tau_3 + (P_6 \otimes \tau_2 + P_3 \otimes \tau_2 \otimes \tau_2) \otimes P_3$, and $m(H_3) \le I_{12} \otimes \tau_3 + (\tau_3 \otimes I_4 + P_3 \otimes \tau_2 \otimes P_2) \otimes P_3$.

Let $\oplus$ be the Kronecker sum (mod 3) in ordinary matrix theory and $D(12,12,3)$ be a difference matrix as follows:

$$D(12,12,3) = \begin{pmatrix} 0&0&0&1&1&0&0&1&0&2&2&0\\ 0&0&0&0&2&0&2&0&2&0&0&1\\ 0&0&1&0&0&2&1&2&0&0&1&0\\ 0&0&2&2&0&1&0&0&1&1&0&0\\ 0&1&2&2&0&0&1&1&2&0&2&2\\ 0&1&2&1&2&1&2&2&2&2&1&0\\ 0&1&0&0&2&2&0&2&1&1&2&2\\ 0&1&1&2&1&2&2&0&0&2&0&2\\ 0&2&1&2&1&0&0&2&2&1&1&1\\ 0&2&1&0&0&1&2&1&1&2&2&1\\ 0&2&2&1&2&2&1&1&0&1&0&1\\ 0&2&0&1&1&1&1&0&1&0&1&2 \end{pmatrix}.$$

It follows from the corollary of Theorem 1 that $H = L_{36}(3^{12}) = D(12,12,3) \oplus (3)$ is an OA and $m(H) \le I_{12} \otimes \tau_3$. From the property $\tau_3 \otimes \tau_3 = \sum_{i=1}^{2} Q_i(\tau_3 \otimes P_3)Q_i^T$ and Theorem 3, we have an orthogonal array H_1:

$$H_1 = L_{108}(3^{24}) = [(I_r \otimes Q_1)(H \otimes 1_3), (I_r \otimes Q_2)(H \otimes 1_3)]$$

and $m(H_1) \le I_{12} \otimes \tau_3 \otimes \tau_3$.

From the orthogonal array $L_{36}(6^2 3^4 2^9)$ in Zhang *et al.* [4], we can get an orthogonal array $L_{36}^1(6^2 3^4 2^9) = K(12,3)L_{36}(6^2 3^4 2^9) = [L_{12}(2^9) \otimes 1_3, L_{36}(6^2 3^4)]$ satisfying $m(L_{36}^1(6^2 3^4 \times 2^9)) \le \tau_{36} = I_{12} \otimes \tau_3 + \tau_{12} \otimes P_3$. Deleting the orthogonal array $L_{12}(2^9) \otimes 1_3$ from $L_{36}^1(6^2 3^4 2^9)$, we obtain an OA $H_2 = L_{36}(6^2 3^4)$ whose MI satisfies $m(H_2) \le I_{12} \otimes \tau_3 + (P_6 \otimes \tau_2 + P_3 \otimes \tau_2 \otimes \tau_2) \otimes P_3$.

By the definition of an OA, there exists a permutation matrix T such that $TL_{36}(6^2 3^4 2^9)$ contains the two columns $1_6 \otimes (2) \otimes 1_3$ and $1_3 \otimes ((2) \oplus (2)) \otimes 1_3$. Deleting these two columns from $TL_{36}(6^2 3^4 2^9)$, we obtain an OA $H_3 = L_{36}(6^2 3^4 2^7)$ satisfying $m(H_3) \le I_{12} \otimes \tau_3 + (\tau_3 \otimes I_4 + P_3 \otimes \tau_2 \otimes P_2) \otimes P_3$.

Hence we can construct a new OA $L_{108}(6^4 3^{32} 2^7)$ as follows:

$$L_{108}(6^4 3^{32} 2^7) = [H_1, H_2 \otimes 1_3, Q(H_3 \otimes 1_3)]. \tag{2}$$

The percent saturation for this OA is 84.2%.

Also, similarly constructing of H_2 and by use of the orthogonal arrays $L_{36}(3^2 2^{20})$, $L_{36}(3^3 2^{13})$, $L_{36}(6^1 3^2 2^{11})$ and $L_{36}(3^1 2^{27})$ in Zhang *et al.* [3], $L_{36}(6^1 3^1 2^{18})$ in Hedayet *et al.* [2], $L_{36}(3^{12} 2^{11})$ and $L_{36}(6^1 3^8 2^{10})$ in Zhang *et al.* [4], we can obtain seven OAs $L_{36}(3^2 2^{11})$, $L_{36}(3^1 2^{18})$, $L_{36}(6^1 3^2 2^2)$, $L_{36}(6^1 3^1 2^9)$, $L_{36}(6^2 3^4)$, $L_{36}(3^{12} 2^2)$ and $L_{36}(6^1 3^8 2^1)$ whose MIs are less than or equal to $I_{12} \otimes \tau_3 + (P_6 \otimes \tau_2 + P_3 \otimes \tau_2 \otimes \tau_2) \otimes P_3$.

On the other hand, by the definition of an OA, any OA of run size 36 with two factors having two levels can contain the two columns $1_6 \otimes (2) \otimes 1_3$ and $1_3 \otimes ((2) \oplus (2)) \otimes 1_3$ through row permutations. For example, there exists OA $L_{36}(9^1 3^{16})$ in Example 2, $L_{36}(18^1 2^2)$, $L_{36}(3^{12} 2^{11})$, $L_{36}(2^{35})$, $L_{36}(3^1 2^{27})$, $L_{36}(6^1 3^{12} 2^2)$, $L_{36}(3^2 2^{20})$, $L_{36}(3^3 2^{13})$, $L_{36}(6^1 3^2 2^{11})$ in Zhang *et al.* [3], $L_{36}(6^1 3^1 2^{18})$ in Hedayet *et al.* [2], $L_{36}(6^1 3^8 2^{10})$, $L_{36}(6^2 3^9 2^3)$, $L_{36}(6^2 3^5 2^2)$, $L_{36}(6^3 2^8)$ in Zhang *et al.* [4], and $L_{36}(6^2 3^1 2^{10})$, $L_{36}(6^3 3^1 2^4)$, $L_{36}(6^3 3^2 2^3)$ in Xu [12]. Deleting these two columns $1_6 \otimes (2) \otimes 1_3$ and $1_3 \otimes ((2) \oplus (2)) \otimes 1_3$ from these arrays, we can obtain 17 OAs $L_{36}(9^1 3^{14})$, $L_{36}(18^1)$, $L_{36}(3^{12} 2^9)$, $L_{36}(2^{33})$, $L_{36}(3^1 2^{25})$, $L_{36}(6^1 3^{12})$, $L_{36}(3^2 2^{18})$, $L_{36}(3^3 2^{11})$, $L_{36}(6^1 3^2 2^9)$, $L_{36}(6^1 3^1 2^{16})$, $L_{36}(6^1 3^8 2^8)$, $L_{36}(6^2 3^9 2^1)$, $L_{36}(6^2 3^5)$, $L_{36}(6^3 2^6)$, $L_{36}(6^2 3^1 2^8)$, L_{36} $(6^3 3^1 2^2)$, and $L_{36}(6^3 3^2 2^1)$ whose MIs are less than or equal to $I_{12} \otimes \tau_3 + (\tau_3 \otimes I_4 + P_3 \otimes \tau_2 \otimes P_2) \otimes P_3$.

Replacing H_2 in (2) by those seven OAs and H_3 in (2) by these 17 OAs, respectively, we can construct $(7 + 1) \times (17 + 1) - 1 = 143$ OAs such as $L_{108}(6^5 3^{28} 2^6)$, *etc.* The orthogonal arrays constructed not only are new but also have higher percent saturations.

Example 2 (Construction of orthogonal arrays of run size 144) From the above properties of OAs $L_9(3^4)$ and $L_{16}(4^5)$, we have

$$\tau_9 = \sum_{i=1}^{2} T_i(\tau_3 \otimes \tau_3)T_i^T,$$

where $T_1 = I_9$ and T_2 is the above permutation matrix and

$$\tau_{16} = \sum_{i=1}^{2} S_i(\tau_4 \otimes P_4)S_i^T + \tau_4 \otimes \tau_4.$$

By using the properties $I_9 = I_9 I_9 I_9$, $T_i I_9 T_i^T = I_9$, $S_i P_{16} S_i^T = P_{16}$ $(i = 1, 2)$, $(ABC) \otimes (DEF) = (A \otimes D)(B \otimes E)(C \otimes F)$, we can orthogonally decompose τ_{144} as follows:

$$\begin{aligned}\tau_{144} &= I_9 \otimes \tau_{16} + \tau_9 \otimes P_{16} \\ &= \sum_{i=1}^{2} (T_i \otimes S_i)\big((I_9 \otimes \tau_4 + \tau_3 \otimes \tau_3 \otimes P_4) \otimes P_4\big)(T_i \otimes S_i)^T + I_9 \otimes \tau_4 \otimes \tau_4.\end{aligned}$$

From the $L_{16}(4^5)$, we have

$$\tau_4 \otimes \tau_4 = \sum_{i=3}^{5} S_i(\tau_4 \otimes P_4)S_i^T.$$

Hence, τ_{144} can be further decomposed as

$$\begin{aligned}\tau_{144} = &\sum_{i=1}^{2} (T_i \otimes S_i)\big((I_9 \otimes \tau_4 + \tau_3 \otimes \tau_3 \otimes P_4) \otimes P_4\big)(T_i \otimes S_i)^T \\ &+ \sum_{i=3}^{5} (I_9 \otimes S_i)(I_9 \otimes \tau_4 \otimes P_4)(I_9 \otimes S_i)^T.\end{aligned}$$

Now we want to find OAs H_1 and H_2 such that $m(H_1) \le I_9 \otimes \tau_4 + \tau_3 \otimes \tau_3 \otimes P_4$ and $m(H_2) \le I_9 \otimes \tau_4$.

By the definition of an OA and through row permutation, any OA with two 3-level columns can contain the two columns $(3) \otimes 1_{12}$, $1_3 \otimes (3) \otimes 1_4$. For example, from the result in Bose and Bush [11], we see that $K_1 = [1_3 \otimes L_{12}(12), (3) \oplus D(12,12,3)]$ is an OA, where $D(12,12,3)$ is a difference matrix in Example 1.

It is clear that K_1 contains two columns $(3) \otimes 1_{12}$ and $(3) \oplus (3) \otimes 1_4$, and $(\operatorname{diag}(I_3, N_3^2, N_3) \otimes I_4)[(3) \otimes 1_{12}, (3) \oplus (3) \otimes 1_4] = [(3) \otimes 1_{12}, 1_3 \otimes (3) \otimes 1_4]$.

Set $K_2 = (\operatorname{diag}(I_3, N_3^2, N_3) \otimes I_4)K_1$, we get an OA $K_2 = L_{36}(12^1 3^{12})$, which contains two columns $(3) \otimes 1_{12}$ and $1_3 \otimes (3) \otimes 1_4$. Deleting these two columns from K_2, we obtain an OA $H_1 = L_{36}(12^1 3^{10})$, whose MI satisfies $m(H_1) = \tau_{36} - \tau_3 \otimes P_{12} - P_3 \otimes \tau_3 \otimes P_4 = I_9 \otimes \tau_4 + \tau_3 \otimes \tau_3 \otimes P_4$.

Now we need to find another OA H_2 such that $m(H_2) \le I_9 \otimes \tau_4$. By using row permutations of the orthogonal array $L_{36}(9^1 2^{16})$ (*cf.* [13]), we can obtain an OA $L_{36}(9^1 2^{16})$ as follows:

$$L_{36}(9^1 2^{16}) = \begin{bmatrix} 0 & x & x & x & -y & x & y & x & -y & -y & -z & -y & y & x & -x & -z & y \\ 1 & x & x & x & z & z & y & -x & x & z & -x & z & z & -x & -y & z & z \\ 2 & x & -y & -x & y & y & y & -x & -x & x & -y & -y & z & z & -x & -x & y \\ 3 & x & -z & y & x & -x & y & x & -y & z & y & y & x & -y & x & -x & -y \\ 4 & x & x & -x & -y & -y & y & z & -x & y & -y & y & -x & y & x & x & z \\ 5 & x & -x & z & -z & -x & y & x & y & y & y & -y & -y & -x & -x & x & y \\ 6 & x & -x & -x & -x & x & y & z & x & -x & -y & x & -z & z & y & y & -z \\ 7 & x & y & -y & z & -z & y & -z & y & -z & y & -x & y & z & -z & z & -y \\ 8 & x & -x & x & y & -z & y & -x & z & -y & x & y & -y & x & x & -y & y \end{bmatrix},$$

where $i = (i, i, i, i)^T$, for $i = 0, 1, \ldots, 8$, $x = (0, 0, 1, 1)^T$, $-x = (1, 1, 0, 0)^T$, $y = (0, 1, 0, 1)^T$, $-y = (1, 0, 1, 0)^T$, $z = (0, 1, 1, 0)^T$ and $-z = (1, 0, 0, 1)^T$.

Deleting the column $(9) \otimes 1_4$ from the $L_{36}(9^1 2^{16})$, we obtain an OA $H_2 = L_{36}(2^{16})$. By using Theorem 1, we see that $m(H_2) \leq I_9 \otimes \tau_4$.

From Theorem 3 the decomposition of τ_{144}, we construct a new OA $L_{144}(12^2 3^{20} 2^{48})$ as follows:

$$\begin{aligned} &L_{144}\bigl(12^2 3^{20} 2^{48}\bigr) \\ &\quad = \bigl[(T_1 \otimes S_1)(H_1 \otimes 1_4), (T_2 \otimes S_2)(H_1 \otimes 1_4), (I_9 \otimes S_3)(H_2 \otimes 1_4), \\ &\qquad (I_9 \otimes S_4)(H_2 \otimes 1_4), (I_9 \otimes S_5)(H_2 \otimes 1_4)\bigr]. \end{aligned} \tag{3}$$

The percent saturation for this OA is 76.4%. In fact, the percent saturations for the arrays constructed by Wang and Wu [14] are between 49.7% and 53.8% and the highest percent saturation for the arrays with 144 runs constructed by Mandeli [10] is 62.2%.

Also, by the definition of an OA, any OA of run size 36 with two factors having three levels can contain the two columns $(3) \otimes 1_{12}$ and $1_3 \otimes (3) \otimes 1_4$ through row permutations. For example, there exist OAs $L_{36}(6^2 3^8 2^1)$, $L_{36}(6^1 3^2 2^{11})$, $L_{36}(3^2 2^{20})$ in Zhang *et al.* [3], $L_{36}(12^1 3^{12})$, $L_{36}(6^1 3^{12} 2^2)$, $L_{36}(3^{13} 2^4)$, $L_{36}(3^{12} 2^{11})$, $L_{36}(6^2 3^5 2^2)$, $L_{36}(6^1 3^8 2^{10})$, $L_{36}(6^2 3^4 2^9)$, $L_{36}(6^1 3^9 2^3)$ in Zhang *et al.* [4], $L_{36}(6^3 3^2 2^3)$, $L_{36}(6^3 3^3 2^1)$ in Xu [12], $L_{36}(6^3 3^7)$ in Finney [15], *etc.* having $(3) \otimes 1_{12}$ and $1_3 \otimes (3) \otimes 1_4$. Deleting these two columns, we can obtain 14 OAs $L_{36}(6^3 3^5)$, $L_{36}(6^2 3^6 2^1)$, $L_{36}(6^1 2^{11})$, $L_{36}(6^1 3^{10} 2^2)$, and so on. Replacing H_1 in (3) by these OAs, respectively, we can construct 195 OAs such as $L_{144}(6^6 3^{10} 2^{48})$, *etc.* The orthogonal arrays constructed not only are new but also have higher percent saturations.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The research and writing of this manuscript was a collaborative effort of all the authors. All authors read and approved the final manuscript.

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