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Nonconforming double set parameter finite element methods for a fourth order variational inequality with two-sided displacement obstacle

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Abstract

Two nonconforming finite elements constructed by double set parameter method are used to approximate a fourth order variational inequality with two-sided displacement obstacle. Because the exact solution does not belong to $H^4_{\mathrm{loc}}(\Omega)$ and each element space involves two sets of parameters, a series of novel approaches different from the exiting literature are developed in the procedure for presenting convergence analysis and deriving the optimal error estimates in broken energy norm.

MSC: 65N30; 65N15

Keywords: fourth order variational inequality; two-sided displacement obstacle; nonconforming elements; double set parameter method; error estimates

1 Introduction

Let $\Omega \subset R^2$ be a bounded convex polygon domain, $f \in L^2(\Omega)$, $\psi_1, \psi_2 \in C^2(\Omega) \cap C(\bar{\Omega})$, $\psi_1 < \psi_2$ on $\bar{\Omega}$ and $\psi_1 < 0 < \psi_2$ on $\partial\Omega$. Consider the following two-sided displacement obstacle problem:

$$\begin{cases} \text{Find } u \in K \text{ such that} \\ u = \arg\min_{v \in K} J(v), \end{cases} \tag{1}$$

where $J(v) = \frac{1}{2}a(v,v) - (f,v)$, $a(w,v) = \int_\Omega D^2 w : D^2 v \, dx\, dy = \int_\Omega (w_{xx}v_{xx} + w_{yy}v_{yy} + 2w_{xy}v_{xy})\, dx\, dy$, $(f,v) = \int_\Omega fv \, dx\, dy$, $K = \{v \in H_0^2(\Omega); \psi_1 \le v \le \psi_2 \text{ on } \Omega\}$.

It follows from the theory in [1, 2] that the solution of obstacle problem (1) is uniquely determined by the following fourth order variational inequality:

$$\begin{cases} \text{Find } u \in K \text{ such that} \\ a(u, v-u) \ge (f, v-u), \quad \forall v \in K. \end{cases} \tag{2}$$

Different from the second order displacement obstacle problems (solutions of that have the H^2 regularity, which allows the corresponding strong form of the variational inequal-

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ity to be used in the convergence analysis of finite element methods (FEMs) [3–9]), the fourth order problem (1) has a unique solution u belonging to $H^3(\Omega) \cap C^2(\Omega)$ (in general $u \notin H^4_{\mathrm{loc}}(\Omega)$ even for smooth data) [10–12]. This lack of the H^4 regularity means that the corresponding strong form of the variational inequality (2) is not available for the convergence analysis of FEMs, which leads to a cardinal difficulty in optimal order error estimates. Although FEMs for the fourth order variational inequality with one side displacement obstacle were investigated and optimal order error estimates were obtained in [13–16], the techniques, which depend greatly on the one side displacement obstacle condition, can not be applied to the two-sided case directly. The main reason is that the two-sided condition will lead to an important inequality in the error estimate invalid, which makes the convergence analysis complicated and difficult to be handled. Recently, a new unified convergence analysis for C^1-conforming FEs, classical nonconforming FEs, and discontinuous FEMs for problem (1) was developed in [17, 18]. In which the optimal error estimate in the energy norm with order $O(h)$ was established by using an auxiliary obstacle problem and an enriching operator (here and later h denotes the mesh parameter). Subsequently the idea was extended to a generalized FEM in [19] and a quadratic C^0 interior penalty method in [20].

It is well known that the degree of piecewise polynomials of a C^1-conforming element space must be very high to meet C^1 smoothness requirement. For example, Argyris element [21] with 5-degree polynomials and Bogner-Fox-Schmit element [22] with bicubic degree polynomials are conforming triangular and rectangular elements respectively. This phenomenon causes some computational difficulties. To reduce the order of polynomials on each element, nonconforming elements are an attractive option. Unfortunately, the degrees of freedom of classical nonconforming FEs are usually very difficult to satisfy the following two requirements simultaneously: (a) to pass through the generalized patch test or F-E-M-test (*cf.* [23, 24]); (b) to be simple and convenient so that the size of discrete system is small. For example, the degrees of freedom of Veubeke element [25] are complicated, while Zienkiewicz element [26, 27] is convergent only on three parallel direction meshes [28]. In order to circumvent or ameliorate the above deficiency, the double set parameter method is proposed [29], which has two sets of parameters chosen independently. In principle, the first set is selected to meet convergence requirement, while the second set is chosen to be simple to make the total number of unknowns in the resulting discrete system as small as possible. Then the first set of parameters is discretized into linear combinations with respect to the second set (real degrees of freedom) in a certain way so as to maintain the convergence order. Up to now, several nonconforming plate elements have been successfully constructed by the double set parameter method and applied to deal with some fourth order PDEs (*cf.* [30–36]). However, as far as we know, there is no consideration about problem (1) with this FEM.

The main aim of this paper is to apply two nonconforming elements constructed by the double set parameter method to approximate problem (1). In this situation, enriching operators meeting the requirements in [17] become very difficult to be developed for each element space involves two sets of parameters. Consequently, a new approach is adopted to get the optimal error estimate, in which we skillfully use two kinds of auxiliary obstacle problems and two enriching operators E_{1h} and E_{2h}. The first auxiliary problem proposed in [17] plays an important role in connecting the continuous and discrete obstacle problems. The second one introduced by ourselves can be regarded as a bridge between the discrete and the first auxiliary obstacle problems. E_{1h} (presented by [37]) establishes the relationship between the triangular Morley [38] and Argyris element spaces, and E_{2h} is newly constructed as a connection between the rectangular Morley [39] and Q_4 Bogner-Fox-Schmit element spaces. Finally, we derive an optimal error estimate of order $O(h)$ in the broken energy norm successfully by a different approach from the existing literature.

The remainder of this paper is organized as follows. In the next section, the two nonconforming finite elements constructed by the double set parameter method are described. In Section 3, we introduce two kinds of auxiliary obstacle problems and two enriching operators and obtain error estimates of order $O(h)$ in the energy norm.

The following standard notations for the Sobolev spaces will be used: for an integer $m > 0$, $H^m(\Omega)$ with norm $\|\cdot\|_{m,\Omega}$ and semi-norm $|\cdot|_{m,\Omega}$, $H^m(T)$ with norm $\|\cdot\|_{m,T}$ and semi-norm $|\cdot|_{m,T}$, $L^2(\Omega)$ with norm $\|\cdot\|_{0,\Omega}$ and $L^2(T)$ with norm $\|\cdot\|_{0,T}$, respectively. Besides, let $P_k(T)$ be the space consisting of piecewise polynomials of degree k, and $Q_k(T)$ be the space of polynomials whose degrees for x, y are equal to k on element T. Throughout the paper, C denotes a positive constant independent of the mesh parameter h and may be different at each appearance.

2 Two double set parameter elements

To begin with, for the sake of completeness, we introduce two nonconforming finite elements constructed by the double set parameter method in [30–32].

Assume that T_h is a family of regular triangular or rectangular subdivisions of Ω [26].

(I) Triangular element: let $T \in T_h$ be a triangle with vertices $a_i(x_i, y_i)$ $(i = 1,2,3)$. We denote by l_i, F_i, n_i, λ_i, Δ, respectively, the side opposite to a_i, the length of l_i, the unit outward normal vector on l_i, the area coordinates for T and the area of T. Let v_i, v_{ix}, v_{iy} be the function value of v and its first derivatives at a_i, and a_{12}, a_{23}, a_{31} be the midpoints of l_3, l_1, l_2, respectively. Furthermore, for $i = 1,2,3$ (mod(3)), we define

$$\begin{aligned} &b_i = y_{i+1} - y_{i-1}, \qquad c_i = x_{i-1} - x_{i+1}, \\ &r_i = \frac{(b_{i+1}b_{i-1} + c_{i+1}c_{i-1})}{\Delta}, \qquad t_i = \frac{F_i^2}{\Delta}. \end{aligned}$$

On element T, let the shape function space be

$$P(T) = P_2(T) = \operatorname{Span}\{\lambda_1, \lambda_2, \lambda_3, \lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_3\lambda_1\}, \tag{3}$$

and the degrees of freedom be

$$D_1(v) = \big(d_1(v), d_2(v), \ldots, d_6(v)\big)', \tag{4}$$

where $d_i(v) = v(a_i) \doteq v_i$, $d_{i+3}(v) = \frac{1}{F_i}\int_{l_i} \frac{\partial v}{\partial n_i}\,ds$, $i = 1,2,3$.

For $v \in P(T)$, suppose that

$$v = \beta_1\lambda_1 + \beta_2\lambda_2 + \beta_3\lambda_3 + \beta_4\lambda_1\lambda_2 + \beta_5\lambda_2\lambda_3 + \beta_6\lambda_3\lambda_1. \tag{5}$$

Substituting (5) into (4), we have $D_1(v) = C_1 b$ with

$$C_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ k_1t_1 & k_1r_3 & k_1r_2 & \frac{k_1}{2}t_1 & -\frac{k_1}{2}t_1 & \frac{k_1}{2}t_1 \\ k_2r_3 & k_2t_2 & k_2r_1 & \frac{k_2}{2}t_2 & \frac{k_2}{2}t_2 & -\frac{k_2}{2}t_2 \\ k_3r_2 & k_3r_1 & k_3t_3 & -\frac{k_3}{2}t_3 & \frac{k_3}{2}t_3 & \frac{k_3}{2}t_3 \end{pmatrix}, \tag{6}$$

where $k_i = -\frac{1}{2F_i}$ $(i = 1, 2, 3)$, $b = (\beta_1, \beta_2, \ldots, \beta_6)' \in R^6$.

It is easy to see that $\det C_1 = \frac{k_1k_2k_3t_1t_2t_3}{2} \neq 0$. Thus $b = C_1^{-1}D_1(v)$, *i.e.*, there holds

$$\forall v \in P(T), \quad v = \bigl(P_1, C_1^{-1}D_1(v)\bigr), \tag{7}$$

where $P_1 = (\lambda_1, \lambda_2, \lambda_3, \lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_3\lambda_1)'$.

Then the associated interpolation operator on T is defined by

$$I_{1T} : v \in H^2(T) \longmapsto I_{1T}v \in P(T), \quad I_{1T}v = \bigl(P_1, C_1^{-1}D_1(v)\bigr). \tag{8}$$

Nodal parameters are chosen as

$$Q_1(v) = (v_1, v_{1x}, v_{1y}, v_2, v_{2x}, v_{2y}, v_3, v_{3x}, v_{3y})'. \tag{9}$$

We discrete the degrees of freedom $D_1(v)$ in terms of the nodal parameters $Q_1(v)$ as follows.

$d_i(v)$ $(i = 1, 2, 3)$ by the exact values of v on the three vertices of element, *i.e.*,

$$d_i(v) = v_i,$$

$d_i(v)$ $(i = 4, 5, 6)$ by trapezoidal formula:

$$\begin{cases} d_4(v) = k_1(b_1(v_{2x} + v_{3x}) + c_1(v_{2y} + v_{3y})) + O(h^2|v|_{3,T}), \\ d_5(v) = k_2(b_2(v_{3x} + v_{1x}) + c_2(v_{3y} + v_{1y})) + O(h^2|v|_{3,T}), \\ d_6(v) = k_3(b_3(v_{1x} + v_{2x}) + c_3(v_{1y} + v_{2y})) + O(h^2|v|_{3,T}). \end{cases}$$

The above discretizations can be written in matrix form as

$$\forall v \in H^3(T), \quad D_1(v) = G_1Q_1(v) + E_1(v), \tag{10}$$

where $E_1(v) = (0, 0, 0, \varepsilon(v), \varepsilon(v), \varepsilon(v))'$, $\varepsilon(v) = O(h^2|v|_{3,T})$, and

$$G_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_1b_1 & k_1c_1 & 0 & k_1b_1 & k_1c_1 \\ 0 & k_2b_2 & k_2c_2 & 0 & 0 & 0 & 0 & k_2b_2 & k_2c_2 \\ 0 & k_3b_3 & k_3c_3 & 0 & k_3b_3 & k_3c_3 & 0 & 0 & 0 \end{pmatrix}. \tag{11}$$

Note that for all $v \in P(T)$, $\frac{\partial v}{\partial n}$ is the first order polynomial on l_i $(i = 1, 2, 3)$, and the trapezoidal rule of numerical integration is exact for linear polynomials, thus $E_1(v)$ vanishes, *i.e.*,

$$\forall v \in P(T), \quad D_1(v) = G_1 Q_1(v). \tag{12}$$

We take the real shape function still as (5), but with

$$b = C_1^{-1} G_1 Q_1(v), \tag{13}$$

then the corresponding FE space is defined by

$$\begin{aligned} V_{1h} = \bigl\{ & v; v|_T = \bigl(P_1, C_1^{-1} G_1 Q_1(v)\bigr), \forall T \in T_h, \\ & v(a) = v_x(a) = v_y(a) = 0, \forall \text{ node } a \in \partial\Omega \bigr\}. \end{aligned} \tag{14}$$

The associated interpolation operator Π_{1h} on V_{1h} is defined by $\Pi_{1h}|_T = \Pi_{1T}$, where Π_{1T} : $v \in H^3(T) \longmapsto \Pi_{1T} v \in P(T)$ satisfies

$$\Pi_{1T} v = \bigl(P_1, C_1^{-1} G_1 Q_1(v)\bigr). \tag{15}$$

Obviously,

$$I_{1T} v - \Pi_{1T} v = \bigl(P_1, C_1^{-1} E_1(v)\bigr). \tag{16}$$

(II) Rectangular element: suppose $T \in T_h$ is a rectangular with sides parallelled to axes of coordinates. Let (x_T, y_T), $a_i(x_i, y_i)$ and $l_i = \overline{a_i a_{i+1}}$ $(i = 1, 2, 3, 4)$ be its center, vertices and edges, the edges length be $2h_x$ and $2h_y$, respectively. Let $\hat{T} = [-1, 1] \times [-1, 1]$ be the reference element on ξ-η plane with center point $(0, 0)$, and its four vertices be $\hat{a}_1 = (-1, -1)$, $\hat{a}_2 = (1, -1)$, $\hat{a}_3 = (1, 1)$ and $\hat{a}_4 = (-1, 1)$, four edges be $\hat{l}_1 = \overline{\hat{a}_1 \hat{a}_2}$, $\hat{l}_2 = \overline{\hat{a}_2 \hat{a}_3}$, $\hat{l}_3 = \overline{\hat{a}_3 \hat{a}_4}$ and $\hat{l}_4 = \overline{\hat{a}_4 \hat{a}_1}$, v_i, v_{ix}, v_{iy} be the function value of v and its first derivatives at a_i, respectively. Then there exists an affine mapping $F : \hat{T} \longrightarrow T$,

$$\begin{cases} x = x_T + h_x \xi, \\ y = y_T + h_y \eta \end{cases} \tag{17}$$

satisfying $F(\hat{T}) = T$, $v(x, y) = \hat{v}(\xi, \eta)$.

On element T, the shape function space is taken as

$$P(T) = P_2(T) \cup \bigl\{x^3, y^3\bigr\} = \operatorname{Span}\{p_1, p_2, \ldots, p_8\}, \tag{18}$$

where

$$\begin{aligned} & p_1 = \frac{1}{4}(1 - \xi)(1 - \eta), \qquad p_2 = \frac{1}{4}(1 + \xi)(1 - \eta), \\ & p_3 = \frac{1}{4}(1 + \xi)(1 + \eta), \qquad p_4 = \frac{1}{4}(1 - \xi)(1 + \eta), \\ & p_5 = 1 - \xi^2, \qquad p_6 = 1 - \eta^2, \qquad p_7 = \xi\bigl(1 - \xi^2\bigr), \qquad p_8 = \eta\bigl(1 - \eta^2\bigr). \end{aligned}$$

The degrees of freedom are chosen as

$$D_2(v) = \big(d_1(v), d_2(v), \dots, d_8(v)\big)', \tag{19}$$

where

$$\begin{aligned}
&d_i(v) = v(a_i) \doteq v_i, \quad i = 1,2,3,4,\\
&d_5(v) = -\frac{h_y}{h_x}\int_{l_1}\frac{\partial v}{\partial n}\,ds = \int_{-1}^{1}\frac{\partial \hat{v}}{\partial \eta}(\xi,-1)\,d\xi,\\
&d_6(v) = \frac{h_x}{h_y}\int_{l_2}\frac{\partial v}{\partial n}\,ds = \int_{-1}^{1}\frac{\partial \hat{v}}{\partial \xi}(1,\eta)\,d\eta,\\
&d_7(v) = -\frac{h_y}{h_x}\int_{l_3}\frac{\partial v}{\partial n}\,ds = \int_{-1}^{1}\frac{\partial \hat{v}}{\partial \eta}(\xi,1)\,d\xi,\\
&d_8(v) = \frac{h_x}{h_y}\int_{l_4}\frac{\partial v}{\partial n}\,ds = \int_{-1}^{1}\frac{\partial \hat{v}}{\partial \xi}(-1,\eta)\,d\eta.
\end{aligned}$$

For $v \in P(T)$, it can be expressed as

$$v = \sum_{i=1}^{8}\beta_i p_i = (P_2, b), \tag{20}$$

where $b = (\beta_1, \beta_2, \dots, \beta_8)' \in R^8$, $P_2 = (p_1, p_2, \dots, p_8)'$.

Substituting (20) into (19), we have $D_2(v) = C_2 b$ with

$$C_2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
-\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 2 & 0 & -2\\
-\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -2 & 0 & -2 & 0\\
-\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & -2 & 0 & -2\\
-\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & 2 & 0 & -2 & 0
\end{pmatrix}. \tag{21}$$

It is easy to check that $\det C_2 = -64 \neq 0$, thus $b = C_2^{-1}D_2(v)$, *i.e.*, there holds

$$\forall v \in P(T), \quad v = \big(P_2, C_2^{-1}D_2(v)\big). \tag{22}$$

The associated interpolation operator on T is defined by

$$I_{2T} : v \in H^2(T) \longmapsto I_{2T}v \in P(T), \quad I_{2T}v = \big(P_2, C_2^{-1}D_2(v)\big). \tag{23}$$

Then we take the nodal parameters as

$$Q_2(v) = (v_1, v_{1x}, v_{1y}, v_2, v_{2x}, v_{2y}, v_3, v_{3x}, v_{3y}, v_4, v_{4x}, v_{4y})'. \tag{24}$$

Discretize $D_2(v)$ into a linear combination of the nodal parameters $Q_2(v)$ as follows.

$d_i(v) = v_i$ $(i = 1, 2, 3, 4)$; $d_i(v)$ $(i = 5, 6, 7, 8)$ with the trapezoidal rule of numerical integration:

$$\begin{cases} d_5(v) = h_y(v_{1y} + v_{2y}) + O(h^2|v|_{3,T}), \\ d_6(v) = h_x(v_{2x} + v_{3x}) + O(h^2|v|_{3,T}), \\ d_7(v) = h_y(v_{3y} + v_{4y}) + O(h^2|v|_{3,T}), \\ d_8(v) = h_x(v_{4x} + v_{1x}) + O(h^2|v|_{3,T}). \end{cases}$$

Then, for $v \in H^3(T)$, the above discretizations can be written in matrix form as

$$D_2(v) = G_2Q_2(v) + E_2(v), \tag{25}$$

where $E_2(v) = (0, 0, 0, 0, \varepsilon(v), \varepsilon(v), \varepsilon(v), \varepsilon(v))'$ and

$$G_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & h_y & 0 & 0 & h_y & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_x & 0 & 0 & h_x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_y & 0 & 0 & h_y \\ 0 & h_x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_x & 0 \end{pmatrix}. \tag{26}$$

Note that for all $v \in P(T)$, $\frac{\partial v}{\partial n}$ is the first order polynomial of x on l_1 and l_3, and of y on l_2 and l_4, thus

$$D_2(v) = G_2Q_2(v). \tag{27}$$

We take the real shape function still as (20), but with

$$b = C_2^{-1}G_2Q_2(v), \tag{28}$$

then the corresponding FE space is defined by

$$\begin{aligned} V_{2h} = \bigl\{&v; v|_T = \bigl(P_2, C_2^{-1}G_2Q_2(v)\bigr), \forall T \in T_h, \\ &v(a) = v_x(a) = v_y(a) = 0, \forall \text{ node } a \in \partial\Omega\bigr\}. \end{aligned} \tag{29}$$

The associated interpolation operator Π_{2h} on V_{2h} is defined by $\Pi_{2h}|_T = \Pi_{2T}$, where Π_{2T} : $v \in H^3(T) \longmapsto \Pi_{2T}v \in P(T)$ satisfies

$$\Pi_{2T}v = \bigl(P_2, C_2^{-1}G_2Q_2(v)\bigr). \tag{30}$$

Obviously,

$$I_{2T}v - \Pi_{2T}v = \bigl(P_2, C_2^{-1}E_2(v)\bigr). \tag{31}$$

In addition, define $\|\cdot\|_h = (\sum_{T\in T_h} |\cdot|_{2,T}^2)^{\frac{1}{2}}$, it can be checked that $\|\cdot\|_h$ is a norm over V_{kh}.

3 Error estimates

Consider the double set parameter FE approximation of (1):

$$\begin{cases} \text{Find } u_{kh} \in K_{kh} \text{ such that} \\ u_{kh} = \arg\min_{v \in K_{kh}} J_h(v), \end{cases} \tag{32}$$

where $K_{kh} = \{v \in V_{kh}; \psi_1(a) \le v(a) \le \psi_2(a), \forall a \in D_h\}$, $k = 1,2$, D_h denotes the set of the vertices of $T \in T_h$, for any $v, w \in K_h$, $J_h(v) = \frac{1}{2}a_h(v,v) - (f,v)$, $a_h(w,v) = \sum_{T\in T_h} \int_T D^2 w : D^2 v \, dx\, dy$.

Since $a_h(\cdot,\cdot)$ is symmetric, bounded and coercive on V_{kh}, and $K_{kh} \subset V_{kh}$, the obstacle problem (32) has a unique solution u_{kh} determined by the following discrete variational inequality:

$$\begin{cases} \text{Find } u_{kh} \in K_{kh} \text{ such that} \\ a_h(u_{kh}, v - u_{kh}) \ge (f, v - u_{kh}), \quad \forall v \in K_{kh}. \end{cases} \tag{33}$$

In order to obtain optimal error estimates, we firstly introduce two kinds of auxiliary obstacle problems as follows.

(i) An auxiliary obstacle problem which is built as a bridge between the continuous problem (1) and the discrete obstacle problem (32):

$$\begin{cases} \text{Find } \tilde{u}_h \in \tilde{K}_h \text{ such that} \\ \tilde{u}_h = \arg\min_{v \in \tilde{K}_h} J(v), \end{cases} \tag{34}$$

where $\tilde{K}_h = \{v \in H_0^2(\Omega); \psi_1(a) \le v(a) \le \psi_2(a), \forall a \in D_h\}$.

Equation (34) has a unique solution $\tilde{u}_h$ determined by

$$\begin{cases} \text{Find } \tilde{u}_h \in \tilde{K}_h \text{ such that} \\ a(\tilde{u}_h, v - \tilde{u}_h) \ge (f, v - \tilde{u}_h), \quad \forall v \in \tilde{K}_h. \end{cases} \tag{35}$$

It was shown in [17] that the solution u of (1) and $\tilde{u}_h$ has the following estimate:

$$|u - \tilde{u}_h|_{2,\Omega} \le Ch, \tag{36}$$

and there exists $h_0 > 0$ such that for $h \le h_0$,

$$\hat{u}_h = \tilde{u}_h + \delta_{h,1}\phi_1 - \delta_{h,2}\phi_2 \in K, \tag{37}$$

where $\phi_i \in C_0^\infty(\bar{\Omega})$ and $\delta_{h,i}$ satisfy

$$\delta_{h,i} \le Ch^2, \quad i = 1,2.$$

(ii) Let $\bar{V}_{kh}$ ($k = 1,2$) be the triangular and rectangular Morley element spaces respectively, *i.e.*, $\bar{V}_{kh} = \{v, v|_T = (P_k, C_k^{-1} D_k(v)), \int_l [\frac{\partial v}{\partial n}]\, ds = 0, l \subset \partial T, \forall T \in T_h, v(a) = 0, \forall \text{ node } a \in \partial\Omega\}$, where $[v]$ is the jump of v across the edge l, and $[v] = v$ if $l \subset \partial\Omega$. Let I_{kh} be $I_{kh}|_T = I_{kT}$.

Then the second kind of auxiliary obstacle problem is introduced to establish a relationship between the discrete obstacle problem (32) and the first auxiliary obstacle problem (34):

$$\begin{cases} \text{Find } \bar{u}_{kh} \in \bar{K}_{kh} \text{ such that} \\ \bar{u}_{kh} = \arg\min_{v \in \bar{K}_{kh}} J_h(v), \end{cases} \tag{38}$$

where $\bar{K}_{kh} = \{v \in \bar{V}_{kh}; \psi_1(a) \le v(a) \le \psi_2(a), \forall a \in D_h\}$.

Equation (38) has a unique solution $\bar{u}_{kh}$ determined by the following discrete variational inequality:

$$\begin{cases} \text{Find } \bar{u}_{kh} \in \bar{K}_{kh} \text{ such that} \\ a_h(\bar{u}_{kh}, v - \bar{u}_{kh}) \ge (f, v - \bar{u}_{kh}), \quad \forall v \in \bar{K}_{kh}. \end{cases} \tag{39}$$

Next, in order to establish a connection between the auxiliary obstacle problems (34) and (38), we present two enriching operators $E_{kh} : v \in \bar{V}_{kh} \longmapsto E_{kh}v \in \tilde{V}_{kh}$ $(k = 1, 2)$ as follows:

$$\begin{cases} E_{1h}v(a) = v(a), \\ \frac{\partial(E_{1h}v)}{\partial n}(m) = \frac{\partial v}{\partial n}(m), \\ \partial^\alpha(E_{1h}v)(a) = \frac{1}{|\Upsilon_a|}\sum_{T\in\Upsilon_a} \partial^\alpha v|_T(a), \quad |\alpha| = 1, \\ \partial^\alpha(E_{1h}v)(a) = 0, \quad |\alpha| = 2 \end{cases} \tag{40}$$

and

$$\begin{cases} E_{2h}v(a) = v(a), \\ \frac{\partial(E_{2h}v)}{\partial n}(m) = \frac{\partial v}{\partial n}(m), \\ E_{2h}v(c) = v(c), \\ E_{2h}v(m) = \frac{1}{|\Upsilon_m|}\sum_{T\in\Upsilon_m} v|_T(m), \\ \partial^\alpha(E_{2h}v)(a) = \frac{1}{|\Upsilon_a|}\sum_{T\in\Upsilon_a} \partial^\alpha v|_T(a), \quad |\alpha| = 1, \\ \partial^\alpha(E_{2h}v)(a) = 0, \quad \alpha = (1,1), \end{cases} \tag{41}$$

where $\tilde{V}_{1h}$ and $\tilde{V}_{2h}$ are the Argyris and Q_4 Bogner-Fox-Schmit element spaces associated with T_h [26], respectively, $m \in M_h$, $c \in C_h$, $a \in D_h$, M_h and C_h are the sets of midpoints of edges and centers of elements in T_h, Υ_m and Υ_a are the sets of elements in T_h sharing m as a common midpoint of edge and a as a common vertex, respectively, $|\Upsilon_m|$ and $|\Upsilon_a|$ are the numbers of elements in Υ_m and Υ_a.

Because $\tilde{V}_{1h}$ and $\tilde{V}_{2h}$ are C^1-conforming spaces, $\tilde{V}_{kh} \subset H_0^2(\Omega)$, thus for all $v \in \bar{K}_{kh}$, there holds $E_{kh}v \in \tilde{K}_h$. Furthermore, we have the following.

Lemma 1 *For all* $u \in H^3(\Omega)$, $v \in \bar{V}_{kh}$ $(k = 1, 2)$, *the following estimates hold*:

$$\|v - E_{kh}v\|_{0,\Omega} \le Ch^2\|v\|_h, \tag{42}$$

$$\left(\sum_{T\in T_h} |v - E_{kh}v|_{1,T}^2\right)^{\frac{1}{2}} \le Ch\|v\|_h, \tag{43}$$

$$|E_{kh}v|_{2,\Omega} \le C\|v\|_h, \tag{44}$$

$$\|u - E_{kh}I_{kh}u\|_{0,\Omega} + h|u - E_{kh}I_{kh}u|_{1,\Omega} + h^2|u - E_{kh}I_{kh}u|_{2,\Omega} \le Ch^3|u|_{3,\Omega}. \tag{45}$$

Proof Because the above properties of operator E_{1h} were proven in [37], we only need to present the proof for E_{2h}.

In fact, from (41), we have

$$\begin{aligned} v|_T - (E_{2h}v)|_T &= \sum_{i=1}^{4} \bigl(v|_T - (E_{2h}v)|_T\bigr)(m_i)q_i \\ &\quad + \sum_{i=1}^{4} \sum_{|\alpha|=1} \partial^\alpha \bigl(v|_T - (E_{2h}v)|_T\bigr)(a_i)q_{\alpha,i} \\ &\quad + \sum_{i=1}^{4} \frac{\partial^2 (v|_T - (E_{2h}v)|_T)}{\partial x\,\partial y}(a_i)q_{2,i}, \end{aligned} \tag{46}$$

where q_i, $q_{\alpha,i}$ and $q_{2,i}$ are the nodal basis functions corresponding to the nodal parameters $v(m_i)$, $\partial^\alpha v(a_i)$ and $\frac{\partial^2 v}{\partial x\,\partial y}(a_i)$ of the Q_4 Bogner-Fox-Schmit element respectively.

Then there holds

$$\begin{aligned} \bigl\|v - (E_{2h}v)\bigr\|_{0,T} &\le \sum_{i=1}^{4} \bigl|\bigl(v|_T - (E_{2h}v)|_T\bigr)(m_i)\bigr| \|q_i\|_{0,T} \\ &\quad + \sum_{i=1}^{4} \sum_{|\alpha|=1} \bigl|\partial^\alpha \bigl(v|_T - (E_{2h}v)|_T\bigr)(a_i)\bigr| \|q_{\alpha,i}\|_{0,T} \\ &\quad + \sum_{i=1}^{4} \biggl|\frac{\partial^2 (v|_T - (E_{2h}v)|_T)}{\partial x\,\partial y}(a_i)\biggr| \|q_{2,i}\|_{0,T} \\ &\doteq I_1 + I_2 + I_3. \end{aligned} \tag{47}$$

Now we start to estimate I_j one by one for $j = 1, 2, 3$.

For I_1, let $l_i \subset \Omega$, it follows from (41) that

$$\bigl|\bigl(v|_T - (E_{2h}v)|_T\bigr)(m_i)\bigr| = \biggl|\frac{1}{2} \sum_{T' \in \Upsilon_{m_i}} (v|_T - v|_{T'})(m_i)\biggr| = \frac{1}{2}\bigl|(v|_T - v|_{T'})(m_i)\bigr|.$$

Denote $(v|_T - v|_{T'}) = \{v\}$, note that $\{v\}$ is the third order polynomial of x on l_1 and l_3, and of y on l_2 and l_4. Taking l_1 for example and using Taylor expansion, we have

$$\begin{aligned} \{v\}(x) &= \{v\}\bigl(x^*\bigr) + \bigl(x - x^*\bigr)\{v\}_x\bigl(x^*\bigr) \\ &\quad + \frac{1}{2}\bigl(x - x^*\bigr)^2\{v\}_{xx}\bigl(x^*\bigr) + \frac{1}{6}\bigl(x - x^*\bigr)^3\{v\}_{xxx}\bigl(x^*\bigr), \end{aligned} \tag{48}$$

where $x^* = \frac{x_1+x_2}{2}$, x_1 and x_2 are x-axis coordinate components of the midpoints and two endpoints of l_1, respectively.

Since $v|_T$ and $v|_{T'}$ agree at the two endpoints of l_1, *i.e.*, $\{v\}(x_1) = \{v\}(x_2) = 0$, there holds

$$\{v\}\bigl(x^*\bigr) = -\frac{1}{8}(x_1 - x_2)^2\{v\}_{xx}\bigl(x^*\bigr),$$

which together with the standard inverse estimate yields

$$\bigl|\{v\}(m_1)\bigr| \le \frac{h_x^2}{2}\bigl(|v|_{2,\infty,T} + |v|_{2,\infty,T'}\bigr) \le Ch|v|_{2,T} + Ch|v|_{2,T'}. \tag{49}$$

On the other hand, if $l_i \subset \partial\Omega$, (41) implies

$$\big(v|_T - (E_{2h}v)|_T\big)(m_i) = 0, \tag{50}$$

then in view of (49) and (50), employing the fact $\|q_i\|_{0,T} \le Ch$, we have

$$I_1 \le Ch^2\big(|v|_{2,T} + |v|_{2,T'}\big).$$

As to I_2, it follows from (41) and a standard inverse estimate that

$$\begin{aligned}
&\sum_{|\alpha|=1}\big|\partial^\alpha\big(v|_T - (E_{2h}v)|_T\big)(a_i)\big| \\
&\quad = \frac{1}{|\Upsilon_{a_i}|}\sum_{T'\in\Upsilon_{a_i}}\sum_{|\alpha|=1}\big|\partial^\alpha(v|_T - v|_{T'})(a_i)\big| \\
&\quad \le C\bigg(\sum_{T'\in\Upsilon_{a_i}}\sum_{|\alpha|=1}\big|\partial^\alpha(v|_T - v|_{T'})(a_i)\big|^2\bigg)^{\frac{1}{2}} \\
&\quad \le C\bigg(\sum_{l\in\Phi_{a_i}}\frac{1}{|l|}\sum_{|\alpha|=1}\big\|\partial^\alpha(v|_{T'} - v|_{T''})\big\|_{0,l}^2\bigg)^{\frac{1}{2}} \\
&\quad \le C\bigg(\sum_{l\in\Phi_{a_i}}\frac{1}{|l|}\bigg(\bigg\|\frac{\partial(v|_{T'} - v|_{T''})}{\partial s}\bigg\|_{0,l}^2 + \bigg\|\frac{\partial(v|_{T'} - v|_{T''})}{\partial n}\bigg\|_{0,l}^2\bigg)\bigg)^{\frac{1}{2}},
\end{aligned} \tag{51}$$

where Φ_{a_i} is the set of edges in T_h sharing a_i as a common vertex, $T', T'' \in \Upsilon_{a_i}$ satisfy $T' \cap T'' = l$, $\frac{\partial}{\partial n}$ and $\frac{\partial}{\partial s}$ denotes the outward normal derivative and tangential derivative along l, respectively. Here and later, $|l|$ denotes the length of l.

Since $v|_{T'}$ and $v|_{T''}$ agree at the two endpoints of l, we get $\int_l \frac{\partial(v|_{T'}-v|_{T''})}{\partial s}\,ds = 0$, thus

$$\begin{aligned}
&\bigg\|\frac{\partial(v|_{T'} - v|_{T''})}{\partial s}\bigg\|_{0,l} \\
&\quad = \bigg\|\frac{\partial(v|_{T'} - v|_{T''})}{\partial s} - \frac{1}{|l|}\bigg(\int_l \frac{\partial(v|_{T'} - v|_{T''})}{\partial s}\,ds\bigg)\bigg\|_{0,l} \\
&\quad \le Ch^{\frac{1}{2}}\bigg|\frac{\partial(v|_{T'} - v|_{T''})}{\partial s}\bigg|_{1,T'\cup T''} \le Ch^{\frac{1}{2}}\big(|v|_{2,T'} + |v|_{2,T''}\big).
\end{aligned} \tag{52}$$

Similarly, $\frac{\partial(v|_{T'}-v|_{T''})}{\partial n}(m) = 0$ leads to $\int_l \frac{\partial(v|_{T'}-v|_{T''})}{\partial n}\,ds = 0$ and

$$\bigg\|\frac{\partial(v|_{T'} - v|_{T''})}{\partial n}\bigg\|_{0,l} \le Ch^{\frac{1}{2}}\big(|v|_{2,T'} + |v|_{2,T''}\big). \tag{53}$$

Then substituting (52) and (53) into (51) yields

$$\sum_{|\alpha|=1}\big|\partial^\alpha\big(v|_T - (E_{2h}v)|_T\big)(a_i)\big| \le C\bigg(\sum_{T'\in\Upsilon_{a_i}}|v|_{2,T'}^2\bigg)^{\frac{1}{2}}, \tag{54}$$

which together with $\|q_{\alpha,i}\|_{0,T} \le Ch^2$ ($|\alpha| = 1$) gives

$$I_2 \le Ch^2 \biggl(\sum_{T' \in \Upsilon_{a_i}} |v|_{2,T'}^2 \biggr)^{\frac{1}{2}}.$$

In order to estimate I_3, employing (41) and the standard inverse estimate, we have

$$\left| \frac{\partial^2 (v|_T - (E_{2h}v)|_T)}{\partial x \, \partial y}(a_i) \right| \le \left| \frac{\partial^2 (v|_T)}{\partial x \, \partial y}(a_i) \right| \le |v|_{2,\infty,T} \le Ch^{-1} |v|_{2,T}, \tag{55}$$

which in conjunction with $\|q_{2,i}\|_{0,T} \le Ch^3$ implies

$$I_3 \le Ch^2 |v|_{2,T}.$$

Therefore, (42) follows by (47) and estimates of I_1, I_2 and I_3 directly.

Furthermore, applying (42), the standard inverse estimate and triangle inequality, we obtain the desired results (43) and (44) immediately.

At last, it follows from (23) and (41) that

$$(E_{2h}I_{2h}u)(a_i) = (I_{2h}u)(a_i) = u(a_i), \qquad \frac{\partial (E_{2h}I_{2h}u)}{\partial n}(m_i) = \frac{\partial (I_{2h}u)}{\partial n}(m_i) = \frac{\partial u}{\partial n}(m_i),$$

i.e., for all $u|_T \in P(T)$, there holds $u|_T = E_{2h}I_{2h}u|_T$, then (45) can be obtained by the interpolation theorem. The proof is completed. □

Now, we are ready to present the convergence analysis of double set parameter nonconforming FEM for (1).

Theorem 1 *Assume that u and u_{kh} are the solutions of* (1) *and* (32), *respectively*, $u \in H^3(\Omega) \cap H_0^2(\Omega)$, *then we have*

$$\|u - u_{kh}\|_h \le Ch, \quad k = 1, 2. \tag{56}$$

Proof Using the discrete variational inequality (33), we get

$$\begin{aligned} \|\Pi_{kh}u - u_{kh}\|_h^2 &\le a_h(\Pi_{kh}u - u_{kh}, \Pi_{kh}u - u_{kh}) \\ &= a_h(\Pi_{kh}u - u, \Pi_{kh}u - u_{kh}) \\ &\quad + a_h(u, \Pi_{kh}u - u_{kh}) - a_h(u_{kh}, \Pi_{kh}u - u_{kh}) \\ &\le C\|\Pi_{kh}u - u\|_h \|\Pi_{kh}u - u_{kh}\|_h \\ &\quad + a_h(u, \Pi_{kh}u - u_{kh}) - (f, \Pi_{kh}u - u_{kh}) \\ &\le C\|\Pi_{kh}u - u\|_h^2 + \frac{1}{2}\|\Pi_{kh}u - u_{kh}\|_h^2 \\ &\quad + \bigl[a_h(u, \Pi_{kh}u - u_{kh}) - (f, \Pi_{kh}u - u_{kh})\bigr], \end{aligned}$$

which implies

$$\|\Pi_{kh}u - u_{kh}\|_h^2 \le C\|\Pi_{kh}u - u\|_h^2 + \bigl[a_h(u, \Pi_{kh}u - u_{kh}) - (f, \Pi_{kh}u - u_{kh})\bigr]. \tag{57}$$

Then employing (57) and a triangle inequality yields

$$\begin{aligned}\|u-u_{kh}\|_h^2 &\leq C\|u-\Pi_{kh}u\|_h^2+\bigl[a_h(u,\Pi_{kh}u-u_{kh})-(f,\Pi_{kh}u-u_{kh})\bigr]\\ &\doteq \triangle_1+\triangle_2.\end{aligned} \tag{58}$$

Notice that

$$\triangle_1=\|u-\Pi_{kh}u\|_h^2\leq\|u-I_{kh}u\|_h^2+\|I_{kh}u-\Pi_{kh}u\|_h^2.$$

Then it follows from the interpolation theorem that

$$\|u-I_{kh}u\|_h^2\leq Ch^2|u|_{3,\Omega}^2.$$

On the other hand, applying (16) and (31) yields

$$|I_{kh}u-\Pi_{kh}u|_{2,T}=\bigl|\bigl(P_k,C_k^{-1}E_k(u)\bigr)\bigr|_{2,T}\leq C|P_k|_{2,T}\bigl|\varepsilon(u)\bigr|,$$

which together with $|P_k|_{2,T}\leq Ch^{-1}$ and $|\varepsilon(u)|\leq Ch^2|u|_{3,T}$ implies

$$\|I_{kh}u-\Pi_{kh}u\|_h^2\leq Ch^2|u|_{3,\Omega}^2. \tag{59}$$

Immediately we get

$$\triangle_1\leq Ch^2|u|_{3,\Omega}^2.$$

Next we focus on the estimate of $\triangle_2$, which is the key difficulty in convergence analysis. Note that

$$\begin{aligned}\triangle_2&=a_h(u,\Pi_{kh}u-u_{kh})-(f,\Pi_{kh}u-u_{kh})\\ &=\bigl[a_h(u,\Pi_{kh}u-I_{kh}u)-(f,\Pi_{kh}u-I_{kh}u)\bigr]\\ &\quad+\bigl[a_h(u,I_{kh}u-\bar{u}_{kh})-(f,I_{kh}u-\bar{u}_{kh})\bigr]\\ &\quad+\bigl[a_h(u,\bar{u}_{kh}-u_{kh})-(f,\bar{u}_{kh}-u_{kh})\bigr]\\ &\doteq N_1+N_2+N_3.\end{aligned} \tag{60}$$

Now we start to estimate N_j one by one for $j=1,2,3$.

For N_1, applying Green's formula gives

$$\begin{aligned}N_1&=\bigl[a_h(u,\Pi_{kh}u-I_{kh}u)-(f,\Pi_{kh}u-I_{kh}u)\bigr]\\ &=-\sum_{T\in T_h}\int_T\nabla\triangle u\nabla(\Pi_{kh}u-I_{kh}u)\,dx\,dy\\ &\quad-\sum_{T\in T_h}\int_T f(\Pi_{kh}u-I_{kh}u)\,dx\,dy\\ &\quad+\sum_{T\in T_h}\int_{\partial T}\biggl(\triangle u-\frac{\partial^2u}{\partial s^2}\biggr)\frac{\partial(\Pi_{kh}u-I_{kh}u)}{\partial n}\,ds\end{aligned}$$

$$+ \sum_{T\in T_h} \int_{\partial T} \frac{\partial^2 u}{\partial s\,\partial n} \frac{\partial(\Pi_{kh}u - I_{kh}u)}{\partial s}\,ds$$
$$\doteq M_1 + M_2 + M_3 + M_4. \tag{61}$$

Then it follows from $|P_k|_{1,T} \le C$, $\|P_k\|_{0,T} \le Ch$ and the Schwarz inequality that

$$\begin{aligned} M_1 &\le \left| -\sum_{T\in T_h} \int_T \nabla\triangle u \nabla(\Pi_{kh}u - I_{kh}u)\,dx\,dy \right| \\ &\le \sum_{T\in T_h} |u|_{3,T} |\Pi_{kh}u - I_{kh}u|_{1,T} \le \sum_{T\in T_h} C|u|_{3,T}|P_k|_{1,T}\left|\varepsilon(u)\right| \\ &\le \sum_{T\in T_h} Ch_T^2|u|_{3,T}^2 \le Ch^2|u|_{3,\Omega}^2 \end{aligned} \tag{62}$$

and

$$\begin{aligned} M_2 &\le \left| -\sum_{T\in T_h} \int_T f(\Pi_{kh}u - I_{kh}u)\,dx\,dy \right| \\ &\le \sum_{T\in T_h} \|f\|_{0,T} \|\Pi_{kh}u - I_{kh}u\|_{0,T} \le \sum_{T\in T_h} C\|f\|_{0,T}\|P_k\|_{0,T}\left|\varepsilon(u)\right| \\ &\le \sum_{T\in T_h} Ch_T^3|u|_{3,T}\|f\|_{0,T} \le Ch^3|u|_{3,\Omega}\|f\|_{0,\Omega}. \end{aligned} \tag{63}$$

At the same time, for all $l \subset \partial T \cap T'$, $T, T' \in T_h$, employing the definitions of Π_{kh} and I_{kh} yields

$$\int_l \left[\frac{\partial(\Pi_{kh}u - I_{kh}u)}{\partial n}\right] ds = \int_l \left[\frac{\partial \Pi_{kh}u}{\partial n}\right] ds - \int_l \left[\frac{\partial I_{kh}u}{\partial n}\right] ds = 0 \tag{64}$$

and

$$\int_l \left[\frac{\partial(\Pi_{kh}u - I_{kh}u)}{\partial s}\right] ds = \int_l \left[\frac{\partial \Pi_{kh}u}{\partial s}\right] ds - \int_l \left[\frac{\partial I_{kh}u}{\partial s}\right] ds = 0. \tag{65}$$

Then let $P_0 v = \frac{1}{|l|}\int_l v\,ds$, applying to (64) and (59), we have

$$\begin{aligned} M_3 &= \sum_{T\in T_h}\sum_{l\in\partial T} \int_l \left(\triangle u - \frac{\partial^2 u}{\partial s^2}\right) \frac{\partial(\Pi_{kh}u - I_{kh}u)}{\partial n}\,ds \\ &= \sum_{T\in T_h}\sum_{l\in\partial T} \int_l \left(\left(\triangle u - \frac{\partial^2 u}{\partial s^2}\right) - P_0\left(\triangle u - \frac{\partial^2 u}{\partial s^2}\right)\right) \\ &\quad \times \left(\frac{\partial(\Pi_{kh}u - I_{kh}u)}{\partial n} - P_0\frac{\partial(\Pi_{kh}u - I_{kh}u)}{\partial n}\right) ds \\ &\le Ch|u|_{3,\Omega}\|\Pi_{kh}u - I_{kh}u\|_h \le Ch^2|u|_{3,\Omega}^2. \end{aligned} \tag{66}$$

Similarly, (65) and (59) imply

$$M_4 \le Ch^2|u|_{3,\Omega}^2. \tag{67}$$

Therefore, substituting (62)-(63) and (66)-(67) into (61) yields

$$N_1 \le Ch^2|u|_{3,\Omega}\big(|u|_{3,\Omega} + \|f\|_{0,\Omega}\big). \tag{68}$$

As to N_2, it follows from (40) and (41) that

$$\int_l \left[\frac{\partial(v - E_{kh}v)}{\partial n}\right] ds = \int_l \left[\frac{\partial(v - E_{kh}v)}{\partial s}\right] ds = 0, \quad \forall v \in \bar{V}_{kh},$$

which together with (43) and (44) in Lemma 1 yields

$$\begin{aligned}
a_h(u, v - E_{kh}v) &= -\sum_{T\in T_h}\int_T \nabla\triangle u\nabla(v - E_{kh}v)\,dx\,dy \\
&\quad + \sum_{T\in T_h}\int_{\partial T}\left(\triangle u - \frac{\partial^2 u}{\partial s^2}\right)\frac{\partial(v - E_{kh}v)}{\partial n}\,ds \\
&\quad + \sum_{T\in T_h}\int_{\partial T}\frac{\partial^2 u}{\partial s\,\partial n}\frac{\partial(v - E_{kh}v)}{\partial s}\,ds \\
&= -\sum_{T\in T_h}\int_T \nabla\triangle u\nabla(v - E_{kh}v)\,dx\,dy \\
&\quad + \sum_{T\in T_h}\sum_{l\in\partial T}\int_l\left(\left(\triangle u - \frac{\partial^2 u}{\partial s^2}\right) - P_0\left(\triangle u - \frac{\partial^2 u}{\partial s^2}\right)\right) \\
&\quad \times \left(\frac{\partial(v - E_{kh}v)}{\partial n} - P_0\frac{\partial(v - E_{kh}v)}{\partial n}\right) ds \\
&\quad + \sum_{T\in T_h}\sum_{l\in\partial T}\int_l\left(\frac{\partial^2 u}{\partial s\,\partial n} - P_0\frac{\partial^2 u}{\partial s\,\partial n}\right) \\
&\quad \times \left(\frac{\partial(v - E_{kh}v)}{\partial s} - P_0\frac{\partial(v - E_{kh}v)}{\partial s}\right) ds \\
&\le |u|_{3,\Omega}\left(\sum_{T\in T_h}|v - E_{kh}v|_{1,T}^2\right)^{\frac{1}{2}} + Ch|u|_{3,\Omega}\|v - E_{kh}v\|_h \\
&\le Ch|u|_{3,\Omega}\|v\|_h.
\end{aligned} \tag{69}$$

Then using inequalities (2), (35) and (39), the definition of E_{kh}, Lemma 1, (36), (69), the fact that $\hat{u}_h \in K$, $\delta_{h,i} \le Ch^2$, and a similar argument as the one in [17], we can obtain

$$\big[a_h(u, I_{kh}u - \bar{u}_{kh}) - (f, I_{kh}u - \bar{u}_{kh})\big] \le Ch\|I_{kh}u - \bar{u}_{kh}\|_h + Ch^2 \tag{70}$$

and

$$\|u - \bar{u}_{kh}\|_h \le Ch. \tag{71}$$

Hence employing the interpolation theorem and a triangle inequality implies

$$\begin{aligned}
N_2 &= \big[a_h(u, I_{kh}u - \bar{u}_{kh}) - (f, I_{kh}u - \bar{u}_{kh})\big] \\
&\le Ch\big(\|I_{kh}u - u\|_h + \|u - \bar{u}_{kh}\|_h + h\big) \le Ch^2.
\end{aligned} \tag{72}$$

For N_3, from the construction of V_{kh}, (12) and (27), we know that

$$u_{kh}|_T = \bigl(P_k, C_k^{-1} G_k Q_k(u_{kh})\bigr) = \bigl(P_k, C_k^{-1} D_k(u_{kh})\bigr), \tag{73}$$

which along with (7) and (22) implies $u_{kh} \in \bar{V}_{kh}$.

Then employing (39) and (71) yields

$$\begin{aligned} N_3 &= a_h(u, \bar{u}_{kh} - u_{kh}) - (f, \bar{u}_{kh} - u_{kh}) \\ &\leq a_h(u, \bar{u}_{kh} - u_{kh}) - a_h(\bar{u}_{kh}, \bar{u}_{kh} - u_{kh}) \\ &= a_h(u - \bar{u}_{kh}, \bar{u}_{kh} - u) + a_h(u - \bar{u}_{kh}, u - u_{kh}) \\ &\leq -\|u - \bar{u}_{kh}\|_h^2 + \|u - \bar{u}_{kh}\|_h \|u - u_{kh}\|_h \\ &\leq \|u - \bar{u}_{kh}\|_h \|u - u_{kh}\|_h \leq \frac{1}{2}\|u - \bar{u}_{kh}\|_h^2 + \frac{1}{2}\|u - u_{kh}\|_h^2 \\ &\leq Ch^2 + \frac{1}{2}\|u - u_{kh}\|_h^2. \end{aligned} \tag{74}$$

At last, substituting (68), (72) and (74) into (60) yields

$$\triangle_2 \leq Ch^2 + \frac{1}{2}\|u - u_{kh}\|_h^2.$$

Therefore the desired result (56) follows from the estimates of $\triangle_1$ and $\triangle_2$ immediately. The proof is completed. □

Remark 1 With the help of two kinds of auxiliary obstacle problems (34) and (38), and two enriching operators E_{1h} and E_{2h}, we successfully deduce the optimal error estimates in broken energy norm of the two double set parameter nonconforming element approximations to problem (1). From the proofs of Theorem 1, one can check that the analysis approaches of this paper are indeed very different from [17, 18], and the results presented herein are also valid to the one-obstacle problem discussed in [13–16].

Remark 2 It should be pointed out that (12), (27), (16) and (31) play an important role in the convergence analysis. Unfortunately, not all nonconforming elements constructed by the double set parameter method satisfy these properties [29, 33–36]. This means that it is not an easy thing to derive the optimal error estimates of double set parameter nonconforming element approximation to problem (1).

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

PL carried out the theoretical studies and drafted the manuscript. SD participated in the design of the study and improved the final version. All authors read and approved the final draft.

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