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Boundedness of rough fractional multilinear integral operators on generalized Morrey spaces

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Abstract

We consider the boundedness of fractional multilinear integral operators with rough kernels $T_{\Omega,\alpha}^{A,m}$ on the generalized Morrey spaces $M_{p,\varphi}$. We find the sufficient conditions on the pair (φ_1,φ_2), which ensures the boundedness of the operators $T_{\Omega,\alpha}^{A,m}$ from M_{p,φ_1} to M_{p,φ_2} for $1<p<\infty$. In all cases the conditions for the boundedness of the operator $T_{\Omega,\alpha}^{A,m}$ is given in terms of Zygmund-type integral inequalities on (φ_1,φ_2), which do not make any assumption on the monotonicity of $\varphi_1(x,r)$, $\varphi_2(x,r)$ in r.

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Keywords: fractional multilinear integral; rough kernel; BMO; generalized Morrey space

1 Introduction and results

The classical Morrey spaces were originally introduced by Morrey in [1] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [1–9]. Mizuhara [10] introduced generalized Morrey spaces. Later, Guliyev [5] defined the generalized Morrey spaces $M_{p,\varphi}$ with normalized norm.

Suppose that $\Omega \in L_s(\mathbb{S}^{n-1})$ $(s>1)$ is homogeneous of degree zero on $\mathbb{R}^n$ with zero means value on $\mathbb{S}^{n-1}$, A is a function defined on $\mathbb{R}^n$. Following [11], the rough fractional multilinear integral operator $T_{\Omega,\alpha}^{A,m}$ is defined by

$$T_{\Omega,\alpha}^{A,m}(f)(x) = \int_{\mathbb{R}^n} \frac{R_m(A;x,y)}{|x-y|^{n-\alpha+m-1}} \Omega(x-y) f(y)\,dy, \tag{1.1}$$

where $0<\alpha<n$, and $R_m(A;x,y)$ is the mth remainder of Taylor series of A at x about y. More precisely,

$$R_m(A;x,y) = A(x) - \sum_{|\gamma|<m} \frac{1}{\gamma!} D^\gamma A(y)(x-y)^\gamma. \tag{1.2}$$

When $m=1$, then $T_{\Omega,\alpha}^{A} \equiv T_{\Omega,\alpha}^{A,1}$ is just the commutator of the fractional integral $T_{\Omega,\alpha} f(x)$ with function A,

$$T_{\Omega,\alpha}^{A}(f)(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} \big(A(x)-A(y)\big) f(y)\,dy.$$

The weighted (L_p, L_q)-boundedness of such a commutator is given by Ding and Lu in [12]. When $m \geq 2$, $T^A_{\Omega,\alpha}$ is a non-trivial generalization of the above commutator. In [13], Wu and Yang proved the following result.

Theorem A *Suppose that $\Omega \in L_s(\mathbb{S}^{n-1})$ and assume that A has derivatives of order $m-1$ in $BMO(\mathbb{R}^n)$. Let $m \geq 2$, $0 < \alpha < n$, $1 \leq s' < p < n/\alpha$, and $1/q = 1/p - \alpha/n$. Then there exists a constant C, independent of A and f, such that*

$$\|T^{A,m}_{\Omega,\alpha} f\|_{L_q(\mathbb{R}^n)} \leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \|f\|_{L_p(\mathbb{R}^n)}.$$

Here and in the sequel, we always denote by p' the conjugate index of any $p > 1$, that is, $1/p + 1/p' = 1$, and by C a constant which is independent of the main parameters and may vary from line to line.

The commutators are useful in many nondivergence elliptic equations with discontinuous coefficients [3, 14–16]. In the recent development of commutators, Pérez and Trujillo-González [17] generalized these multilinear commutators and proved the weighted Lebesgue estimates.

In [18], Guliyev proved the following result.

Theorem B *Let $0 < \alpha < n$, $1 < p < n/\alpha$, and $1/q = 1/p - \alpha/n$, $\Omega \in L_s(\mathbb{S}^{n-1})$, $1 < s \leq \infty$, $A \in BMO$, and (φ_1, φ_2) satisfies the condition*

$$\int_r^\infty \ln\left(e + \frac{t}{r}\right) \frac{\operatorname{ess\,sup}_{t<s<\infty} \varphi_1(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{q}}} \frac{dt}{t} \leq C\varphi_2(x,r),$$

where C does not depend on x and r. Then the operator $T^A_{\Omega,\alpha}$ is bounded from M_{p,φ_1} to M_{q,φ_2}.

It has been proved by many authors that most of the operators which are bounded on a Lebesgue space are also bounded in an appropriate Morrey space; see [19]. As far as we know, there is no research regarding the boundedness of the fractional multilinear integral operator on Morrey space.

In this paper, we are going to prove that these results are valid for the rough fractional multilinear integral operator $T^{A,m}_{\Omega,\alpha}$ on generalized Morrey spaces. Our main results can be formulated as follows.

Theorem 1.1 *Let $0 < \alpha < n$, $1 \leq s' < p < n/\alpha$, and $1/q = 1/p - \alpha/n$. Suppose that $\Omega \in L_s(\mathbb{S}^{n-1})$ and (φ_1, φ_2) satisfy the condition*

$$\int_r^\infty \left(1 + \ln\frac{t}{r}\right) \frac{\operatorname{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau) \tau^{\frac{n}{p}}}{t^{\frac{n}{q}}} \frac{dt}{t} \leq C_0 \varphi_2(x,r), \tag{1.3}$$

where C_0 does not depend on x and r. If A has derivatives of order $m-1$ in $BMO(\mathbb{R}^n)$, then the operator $T^{A,m}_{\Omega,\alpha}$ is bounded from $M_{p,\varphi_1}(\mathbb{R}^n)$ to $M_{q,\varphi_2}(\mathbb{R}^n)$. Moreover, then there is a constant $C > 0$ independent of f such that

$$\|T^{A,m}_{\Omega,\alpha} f\|_{M_{q,\varphi_2}} \leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \|f\|_{M_{p,\varphi_1}}.$$

Remark 1.1 Note that in the case $m = 1$ from Theorem 1.1 we get Theorem B.

2 Generalized Morrey spaces $M_{p,\varphi}$

The classical Morrey spaces $M_{p,\lambda}$ were originally introduced by Morrey in [1] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [20, 21].

We denote by $M_{p,\lambda} \equiv M_{p,\lambda}(\mathbb{R}^n)$ the Morrey space, the space of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorm,

$$\|f\|_{M_{p,\lambda}} = \sup_{x\in\mathbb{R}^n, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))},$$

where $1 \le p < \infty$ and $0 \le \lambda \le n$.

Note that $M_{p,0} = L_p(\mathbb{R}^n)$ and $M_{p,n} = L_\infty(\mathbb{R}^n)$. If $\lambda < 0$ or $\lambda > n$, then $M_{p,\lambda} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

In [10], Mizuhara introduced the generalized Morrey spaces $\mathcal{M}_{p,\varphi}(\mathbb{R}^n)$ in the following form and discussed the boundedness of the Calderón-Zygmund singular integral operators.

Definition 2.1 Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. We denote by $\mathcal{M}_{p,\varphi} \equiv \mathcal{M}_{p,\varphi}(\mathbb{R}^n)$ the generalized Morrey space, the space of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{\mathcal{M}_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \|f\|_{L_p(B(x,r))}.$$

Note that the generalized Morrey spaces $M_{p,\varphi} \equiv M_{p,\varphi}(\mathbb{R}^n)$ with normalized norm,

$$\|f\|_{M_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \bigl|B(x,r)\bigr|^{-\frac{1}{p}} \|f\|_{L_p(B(x,r))},$$

were first defined by Guliyev in [5].

Also, in [5], by $WM_{p,\varphi} \equiv WM_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL_p^{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \bigl|B(x,r)\bigr|^{-\frac{1}{p}} \|f\|_{WL_p(B(x,r))} < \infty.$$

By the definition, we recover the Morrey space $M_{p,\lambda}$ and weak Morrey space $WM_{p,\lambda}$ under the choice $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$:

$$M_{p,\lambda} = M_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}, \qquad WM_{p,\lambda} = WM_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}.$$

There are many papers discussing the conditions on $\varphi(x,r)$ to obtain the boundedness of operators on the generalized Morrey spaces. For example, in [10], the function φ is supposed to be a positive growth function and satisfy the double condition: for all $r > 0$, $\varphi(2r) \le D\varphi(r)$, where $D \ge 1$ is a constant independent of r. This type of conditions on φ is studied by many authors; see, for example, [22, 23]. In [24], the following statement was proved by Nakai for the Riesz potential I_α:

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)\,dy}{|x-y|^{n-\alpha}}, \quad 0 < \alpha < n.$$

Theorem C *Let* $1 \le p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, *and let* $\varphi(x,r)$ *satisfy the conditions*

$$c^{-1}\varphi(x,r) \le \varphi(x,t) \le c\varphi(x,r), \tag{2.1}$$

whenever $r \le t \le 2r$, *where* c $(c \ge 1)$ *does not depend on* t, r, x, *and*

$$\int_r^\infty t^\alpha \varphi^q(x,t)\frac{dt}{t} \le C\varphi^p(x,r), \tag{2.2}$$

where C *does not depend on* x *and* r. *Then the operator* I_α *is bounded from* $M_{p,\varphi}$ *to* $M_{q,\varphi}$ *for* $p > 1$ *and from* $M_{1,\varphi}$ *to* $WM_{1,\varphi}$ *for* $p = 1$.

The following statements, containing the Mizuhara and Nakai results obtained in [10, 24], were proved by Guliyev in [5, 25] (see also [26]).

Theorem D *Let* $1 \le p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, *and* (φ_1, φ_2) *satisfy the condition*

$$\int_r^\infty t^\alpha \varphi_1(x,t)\frac{dt}{t} \le C\varphi_2(x,r), \tag{2.3}$$

where C *does not depend on* x *and* r. *Then the operator* I_α *is bounded from* M_{p,φ_1} *to* M_{q,φ_2} *for* $p > 1$ *and from* M_{1,φ_1} *to* WM_{q,φ_2} *for* $p = 1$.

Recently, in [27] and [28], Guliyev *et al.* introduced a weaker condition for the boundedness of Riesz potential from M_{p,φ_1} to M_{q,φ_2}.

Theorem E *Let* $1 \le p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, *and* (φ_1, φ_2) *satisfy the condition*

$$\int_t^\infty \frac{\operatorname{ess\,sup}_{r<s<\infty} \varphi_1(x,s)s^{\frac{n}{p}}}{r^{\frac{n}{q}+1}}\,dr \le C\varphi_2(x,t), \tag{2.4}$$

where C *does not depend on* x *and* t. *Then the operator* I_α *is bounded from* M_{p,φ_1} *to* M_{q,φ_2} *for* $1 < p < q < \infty$ *and from* M_{1,φ_1} *to* WM_{q,φ_2} *for* $1 < q < \infty$.

By an easy computation, we can check that if the pair (φ_1, φ_2) satisfies the double condition, then it will satisfy condition (2.3). Moreover, if (φ_1, φ_2) satisfies condition (2.3), it will also satisfy condition (2.4). But the opposite is not true. We refer to [24] and Remark 4.7 in [28] for details.

3 Some preliminaries

Let $B = B(x_0, r_B)$ denote the ball with the center x_0 and radius r_B. For a given weight function ω and a measurable set E, we also denote the Lebesgue measure of E by $|E|$. For any given $\Omega \subseteq \mathbb{R}^n$ and $0 < p < \infty$, denote by $L_p(\Omega)$ the space of all function f satisfying

$$\|f\|_{L_p(\Omega)} = \left(\int_\Omega |f(x)|^p \, dx \right)^{\frac{1}{p}} < \infty.$$

First we recall the definition of the space $BMO(\mathbb{R}^n)$.

Definition 3.1 Suppose that $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$, let

$$\|f\|_* = \sup_{x \in \mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_{B(x,r)}| \, dy < \infty,$$

where

$$f_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y) \, dy.$$

Define

$$BMO(\mathbb{R}^n) = \{ f \in L_1^{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_* < \infty \}.$$

If one regards two functions whose difference is a constant as one, then space $BMO(\mathbb{R}^n)$ is a Banach space with respect to norm $\| \cdot \|_*$.

Remark 3.1 [29]

(1) The John-Nirenberg inequality: there are constants $C_1, C_2 > 0$, such that, for all $f \in BMO(\mathbb{R}^n)$ and $\beta > 0$,

$$\left| \{ x \in B : |f(x) - f_B| > \beta \} \right| \le C_1 |B| e^{-C_2 \beta / \|f\|_*}, \quad \forall B \subset \mathbb{R}^n.$$

(2) The John-Nirenberg inequality implies that

$$\|f\|_* \approx \sup_{x \in \mathbb{R}^n, r>0} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_{B(x,r)}|^p \, dy \right)^{\frac{1}{p}} \tag{3.1}$$

for $1 < p < \infty$.

(3) Let $f \in BMO(\mathbb{R}^n)$. Then there is a constant $C > 0$ such that

$$|f_{B(x,r)} - f_{B(x,t)}| \le C \|f\|_* \ln \frac{t}{r} \quad \text{for } 0 < 2r < t, \tag{3.2}$$

where C is independent of f, x, r, and t.

Lemma 3.1 [30] *Let b be a function in $BMO(\mathbb{R}^n)$, $1 \le p < \infty$, and $r_1, r_2 > 0$. Then*

$$\left(\frac{1}{|B(x,r_1)|} \int_{B(x,r_1)} |b(y) - b_{B(x,r_2)}|^p \, dy \right)^{\frac{1}{p}} \le C \left(1 + \left| \ln \frac{r_1}{r_2} \right| \right) \|b\|_*,$$

where $C > 0$ is independent of b, x, r_1, and r_2.

Below we present some conclusions as regards $R_m(A;x,y)$.

Lemma 3.2 [21] *Suppose b be a function on $\mathbb{R}^n$ with the mth derivatives in $L_q(\mathbb{R}^n)$, $q > n$. Then*

$$\left|R_m(b;x,y)\right| \le C|x-y|^m \sum_{|\gamma|=m} \left(\frac{1}{B(x,5\sqrt{n}|x-y|)} \int_{B(x,5\sqrt{n}|x-y|)} \left|D^\gamma b(z)\right| dz\right)^{1/q}.$$

Lemma 3.3 *Let $x \in B(x_0,r)$, $y \in B(x_0,2^{j+1}r) \setminus B(x_0,2^j r)$. Then*

$$\left|R_m(A;x,y)\right| \le C|x-y|^{m-1}\left(j \sum_{|\gamma|=m-1} \left\|D^\gamma A\right\|_* + \sum_{|\gamma|=m-1} \left|D^\gamma A(y) - \left(D^\gamma A\right)_{B(x_0,r)}\right|\right). \tag{3.3}$$

Proof For fixed $x \in \mathbb{R}^n$, let

$$\bar{A}(x) = A(x) - \sum_{|\gamma|=m-1} \frac{1}{\gamma!}\left(D^\gamma A\right)_{B(x,5\sqrt{n}|x-y|)} x^\gamma.$$

So

$$\begin{aligned} \left|R_m(A;x,y)\right| &= \left|R_m(\bar{A};x,y)\right| \\ &\le \left|R_{m-1}(\bar{A};x,y)\right| + \sum_{|\gamma|=m-1} \frac{1}{\gamma!}\left|\left(D^\gamma \bar{A}(y)\right)\right||x-y|^{m-1}. \end{aligned} \tag{3.4}$$

By Lemma 3.2, we get

$$\left|R_{m-1}(\bar{A};x,y)\right| \le C|x-y|^{m-1} \sum_{|\gamma|=m-1} \left\|D^\gamma A\right\|_*. \tag{3.5}$$

Since $x \in B(x_0,r)$, $y \in B(x_0,2^{j+1}r) \setminus B(x_0,2^j r)$, it is easy to see that $2^{j-1}r \le |x-y| \le 2^{j+2}r$. In this way, we have

$$B\left(x_0,2^{j-1}r\right) \subset B\left(x,5\sqrt{n}|x-y|\right) \subset 100\sqrt{n}B\left(x_0,2^j r\right).$$

Then

$$\frac{|100\sqrt{n}B(x_0,2^j r)|}{|B(x,5\sqrt{n}|x-y|)|} \le \frac{|100\sqrt{n}B(x_0,2^j r)|}{|B(x_0,2^{j-1}r)|} \le C.$$

Therefore

$$\begin{aligned} &\left|\left(D^\gamma A\right)_{B(x,5\sqrt{n}|x-y|)} - \left(D^\gamma A\right)_{B(x_0,2^j r)}\right| \\ &\quad\le \frac{1}{|B(x,5\sqrt{n}|x-y|)|} \int_{B(x,5\sqrt{n}|x-y|)} \left|D^\gamma A(y) - \left(D^\gamma A\right)_{B(x_0,2^j r)}\right| dy \\ &\quad\le \frac{1}{|100\sqrt{n}B(x_0,2^j r)|} \int_{100\sqrt{n}B(x_0,2^j r)} \left|D^\gamma A(y) - \left(D^\gamma A\right)_{B(x_0,2^j r)}\right| dy \\ &\quad\le C\left\|D^\gamma A\right\|_*. \end{aligned}$$

Note that

$$\begin{aligned}&\left|\left(D^\gamma A\right)_{B(x_0,2^jr)}-\left(D^\gamma A\right)_{B(x_0,r)}\right|\\&\quad\le\sum_{k=1}^{j}\left|\left(D^\gamma A\right)_{B(x_0,2^kr)}-\left(D^\gamma A\right)_{B(x_0,2^{k-1}r)}\right|\\&\quad\le 2^nj\left\|D^\gamma A\right\|_*.\end{aligned}$$

Then

$$\begin{aligned}&\left|\left(D^\gamma A\right)_{B(x,5\sqrt{n}|x-y|)}-\left(D^\gamma A\right)_{B(x_0,r)}\right|\\&\quad\le\left|\left(D^\gamma A\right)_{B(x,5\sqrt{n}|x-y|)}-\left(D^\gamma A\right)_{B(x_0,2^jr)}\right|+\left|\left(D^\gamma A\right)_{B(x_0,2^jr)}-\left(D^\gamma A\right)_{B(x_0,r)}\right|\\&\quad\le C_j\left\|D^\gamma A\right\|_*.\end{aligned}$$

Thus

$$\begin{aligned}\left|D^\gamma\tilde{A}(y)\right|&=\left|D^\gamma A(y)-\left(D^\gamma A\right)_{B(x,5\sqrt{n}|x-y|)}\right|\\&\le\left|D^\gamma A(y)-\left(D^\gamma A\right)_{B(x_0,r)}\right|+\left|\left(D^\gamma A\right)_{B(x,5\sqrt{n}|x-y|)}-\left(D^\gamma A\right)_{B(x_0,r)}\right|\\&\le\left|D^\gamma A(y)-\left(D^\gamma A\right)_{B(x_0,r)}\right|+C_j\left\|D^\gamma A\right\|_*.\end{aligned}\tag{3.6}$$

Combining with (3.4), (3.5), and (3.6), then (3.3) is proved. □

Finally, we present a relationship between essential supremum and essential infimum.

Lemma 3.4 [31] *Let f be a real-valued nonnegative function and measurable on E. Then*

$$\left(\operatorname*{ess\,inf}_{x\in E}f(x)\right)^{-1}=\operatorname*{ess\,sup}_{x\in E}\frac{1}{f(x)}.$$

4 A local Guliyev-type estimates

In the following theorem we get Guliyev-type local estimate (see, for example, [5, 25]) for the operator $T^{A,m}_{\Omega,\alpha}$.

Theorem 4.1 *Let $\Omega\in L_s(\mathbb{S}^{n-1})$, $1\le s'<p<n/\alpha$, and let $1/q=1/p-\alpha/n$. Let A be a function defined on $\mathbb{R}^n$. Suppose that it has derivatives of order $m-1$ in $BMO(\mathbb{R}^n)$, then the inequality*

$$\left\|T^{A,m}_{\Omega,\alpha}(f)\right\|_{L_q(B(x_0,r))}\le C\sum_{|\gamma|=m-1}\left\|D^\gamma A\right\|_*r^{\frac{n}{q}}\int_{2r}^{\infty}\|f\|_{L_p(B(x_0,t))}t^{-\frac{n}{q}-1}\,dt\tag{4.1}$$

holds for any ball $B(x_0,r)$, and for all $f\in L^{\mathrm{loc}}_p(\mathbb{R}^n)$, where the constant C is independent of f, r, and x_0.

Proof We write f as $f=f_1+f_2$, where $f_1(y)=f(y)_{\chi_{B(x_0,2r)}(y)}$, $\chi_{B(x_0,2r)}$ denotes the characteristic function of $B(x_0,2r)$. Then

$$\left\|T^{A,m}_{\Omega,\alpha}(f)\right\|_{L_q(B(x_0,r))}\le\left\|T^{A,m}_{\Omega,\alpha}(f_1)\right\|_{L_q(B(x_0,r))}+\left\|T^{A,m}_{\Omega,\alpha}(f_2)\right\|_{L_q(B(x_0,r))}.$$

Since $f_1 \in L_p(\mathbb{R}^n)$, by the boundedness of $T^A_{\Omega,\alpha}$ from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ (Theorem A) we get

$$\begin{aligned}\|T^{A,m}_{\Omega,\alpha}(f_1)\|_{L_q(B(x_0,r))} &\le \|T^{A,m}_{\Omega,\alpha}(f_1)\|_{L_q(\mathbb{R}^n)} \\ &\le C\sum_{|\gamma|=m-1}\|D^\gamma A\|_*\|f_1\|_{L_p(\mathbb{R}^n)} \\ &= C\sum_{|\gamma|=m-1}\|D^\gamma A\|_*\|f\|_{L_p(B(x_0,2r))}.\end{aligned}$$

Moreover,

$$\begin{aligned}\|f\|_{L_p(B(x_0,2r))} &\le Cr^{\frac{n}{s'}-\alpha}\|f\|_{L_p(B(x_0,2r))}\int_{2r}^\infty \frac{dt}{t^{n-\alpha+1}} \\ &\le Cr^{\frac{n}{q}}\|1\|_{L_{\frac{s'p}{p-s'}}(B(x_0,r))}\int_{2r}^\infty \|f\|_{L_p(B(x_0,t))}\frac{dt}{t^{n-\alpha+1}} \\ &\le Cr^{\frac{n}{q}}\int_{2r}^\infty \|f\|_{L_p(B(x_0,t))}\|1\|_{L_{\frac{s'p}{p-s'}}(B(x_0,t))}\frac{dt}{t^{n-\alpha+1}}.\end{aligned}$$

Thus

$$\begin{aligned}&\|T^{A,m}_{\Omega,\alpha}(f_1)\|_{L_q(B(x_0,r))} \\ &\quad\le C\sum_{|\gamma|=m-1}\|D^\gamma A\|_* r^{\frac{n}{q}}\int_{2r}^\infty \|f\|_{L_p(B(x_0,t))}t^{-\frac{n}{q}}\frac{dt}{t}.\end{aligned} \tag{4.2}$$

Let $\triangle_i = (B(x_0,2^{j+1}r))\setminus(B(x_0,2^jr))$, and let $x \in B(x_0,r)$. By Lemma 3.3 we get

$$\begin{aligned}|T^{A,m}_{\Omega,\alpha}(f_2)(x)| &\le \left|\int_{(B(x_0,2r))^c}\frac{R_m(A;x,y)}{|x-y|^{n-\alpha+m-1}}\Omega(x-y)f(y)\,dy\right| \\ &\le C\sum_{j=1}^\infty\int_{\triangle_i}\frac{|\Omega(x-y)f(y)|}{|x-y|^{n-\alpha}} \\ &\quad\times\left(j\sum_{|\gamma|=m-1}\|D^\gamma A\|_* + \sum_{|\gamma|=m-1}|D^\gamma A(y)-(D^\gamma A)_{B(x_0,r)}|\right)dy \\ &\le C\sum_{|\gamma|=m-1}\|D^\gamma A\|_*\sum_{j=1}^\infty j\int_{\triangle_i}\frac{|\Omega(x-y)f(y)|}{|x-y|^{n-\alpha}}\,dy \\ &\quad+ C\sum_{|\gamma|=m-1}\sum_{j=1}^\infty\int_{\triangle_i}\frac{|\Omega(x-y)f(y)|}{|x-y|^{n-\alpha}}|D^\gamma A(y)-(D^\gamma A)_{B(x_0,r)}|\,dy \\ &= I_1 + I_2.\end{aligned} \tag{4.3}$$

By Hölder's inequality we have

$$\int_{\triangle_i}\frac{|\Omega(x-y)f(y)|}{|x-y|^{n-\alpha}}\,dy \le \left(\int_{\triangle_i}|\Omega(x-y)|^s\,dy\right)^{\frac{1}{s}}\left(\int_{\triangle_i}\frac{|f(y)|^{s'}}{|x-y|^{(n-\alpha)s'}}\,dy\right)^{\frac{1}{s'}}.$$

When $x \in B(x_0, s)$ and $y \in \triangle_i$, then by a direct calculation, we can see that $2^{j-1}r \le |y-x| < 2^{j+1}r$. Hence

$$\left(\int_{\triangle_i} \left|\Omega(x-y)\right|^s dy\right)^{\frac{1}{s}} \le C\|\Omega\|_{L^s(\mathbb{S}^{n-1})}\left|B\left(x_0, 2^{j+1}r\right)\right|^{\frac{1}{s}}. \tag{4.4}$$

We also note that if $x \in B(x_0, r)$, $y \in B(x_0, 2r)^c$, then $\frac{1}{2}|x_0-y| \le |x-y| \le \frac{3}{2}|x_0-y|$. Consequently

$$\left(\int_{\triangle_i} \frac{|f(y)|^{s'}}{|x-y|^{(n-\alpha)s'}}\, dy\right)^{\frac{1}{s'}} \le \frac{1}{|B(x_0, 2^{j+1}r)|^{1-\alpha/n}}\left(\int_{B(x_0,2^{j+1}r)} \left|f(y)\right|^{s'} dy\right)^{\frac{1}{s'}}. \tag{4.5}$$

Then

$$I_1 \le C\sum_{|\gamma|=m-1}\left\|D^\gamma A\right\|_* \sum_{j=1}^{\infty} j\left(2^{j+1}r\right)^{\alpha-n}\left(\int_{B(x_0,2^{j+1}r)} \left|f(y)\right|^{s'} dy\right)^{\frac{1}{s'}}. \tag{4.6}$$

Since $s' < p$, it follows from Hölder's inequality that

$$\left(\int_{B(x_0,2^{j+1}r)} \left|f(y)\right|^{s'} dy\right)^{\frac{1}{s'}} \le C\|f\|_{L_p(B(x_0,2^{j+1}r))}\|1\|_{L_{\frac{s'p}{p-s'}}(B(x_0,2^{j+1}r))}. \tag{4.7}$$

Then

$$\begin{aligned}
I_1 &\le C\sum_{|\gamma|=m-1}\left\|D^\gamma A\right\|_* \sum_{j=1}^{\infty} j\left(2^{j+1}r\right)^{\alpha-n}\left(\int_{B(x_0,2^{j+1}s)} \left|f(y)\right|^{q'} dy\right)^{\frac{1}{q'}} \\
&\le C\sum_{|\gamma|=m-1}\left\|D^\gamma A\right\|_* \sum_{j=1}^{\infty}\left(1+\ln\frac{2^{j+1}r}{r}\right)\left(2^{j+1}r\right)^{\alpha-n}\|f\|_{L_p(B(x_0,2^{j+1}r))}\|1\|_{L_{\frac{s'p}{p-s'}}(B(x_0,2^{j+1}r))} \\
&\le C\sum_{|\gamma|=m-1}\left\|D^\gamma A\right\|_* \sum_{j=1}^{\infty}\int_{2^{j+1}r}^{2^{j+2}l}\left(1+\ln\frac{t}{r}\right)\|f\|_{L_p(B(x_0,t))}\|1\|_{L_{\frac{s'p}{p-s'}}(B(x_0,t))}\frac{dt}{r^{1-\alpha+n/s'}} \\
&\le C\sum_{|\gamma|=m-1}\left\|D^\gamma A\right\|_* \int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\|f\|_{L_p(B(x_0,t))}\|1\|_{L_{\frac{s'p}{p-s'}}(B(x_0,t))}\frac{dt}{t^{1-\alpha+n/s'}}.
\end{aligned}$$

Then

$$I_1 \le C\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\|f\|_{L_p(B(x_0,t))}t^{-\frac{n}{q}-1}\, dt. \tag{4.8}$$

On the other hand, by Hölder's inequality and (4.4), (4.5), we have

$$\begin{aligned}
&\int_{\triangle_i}\frac{|\Omega(x-y)f(y)|}{|x-y|^{n-\alpha}}\left|D^\gamma A(y)-\left(D^\gamma A\right)_{B(x_0,r)}\right| dy \\
&\quad\le \left(\int_{\triangle_i}\left|\Omega(x-y)\right|^s dy\right)^{\frac{1}{s}}\left(\int_{\triangle_i}\frac{|D^\gamma A(y)-(D^\gamma A)_{B(x_0,r)}f(y)|^{s'}}{|x-y|^{(n-\alpha)s'}}\, dy\right)^{\frac{1}{s'}} \\
&\quad\le C\sum_{|\gamma|=m-1}\sum_{j=1}^{\infty}\left(2^{j+1}r\right)^{\alpha-n}\left(\int_{B(x_0,2^{j+1}r)}\left|D^\gamma A(y)-\left(D^\gamma A\right)_{B(x_0,r)}\right|^{s'}\left|f(y)\right|^{s'} dy\right)^{\frac{1}{s'}}.
\end{aligned}$$

Applying Hölder's inequality we get

$$\left(\int_{B(x_0,2^{j+1}r)} \left|D^\gamma A(y) - \left(D^\gamma A\right)_{B(x_0,r)}\right|^{s'} \left|f(y)\right|^{s'} dy\right)^{\frac{1}{s'}}$$
$$\leq C\|f\|_{L_p(B(x_0,2^{j+1}r))} \left\|D^\gamma A(\cdot) - \left(D^\gamma A\right)_{B(x_0,r)}\right\|_{L^{\frac{ps'}{p-s'}}(B(x_0,2^{j+1}r))}.$$

Consequently,

$$I_2 \leq C \sum_{|\gamma|=m-1} \sum_{j=1}^{\infty} \int_{2^{j+1}r}^{2^{j+2}r} \left(2^{j+1}r\right)^{\alpha-n} \|f\|_{L_p(B(x_0,t))}$$
$$\times \left\|D^\gamma A(\cdot) - \left(D^\gamma A\right)_{B(x_0,r)}\right\|_{L^{\frac{ps'}{p-s'}}(B(x_0,t))} dt$$
$$\leq C \sum_{|\gamma|=m-1} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))}$$
$$\times \left\|D^\gamma A(\cdot) - \left(D^\gamma A\right)_{B(x_0,r)}\right\|_{L^{\frac{ps'}{p-s'}}(B(x_0,t))} \frac{dt}{t^{1-\alpha+n/s'}}.$$

Then it follows from Lemma 3.1 that

$$\left\|D^\gamma A(\cdot) - \left(D^\gamma A\right)_{B(x_0,r)}\right\|_{L^{\frac{ps'}{p-s'}}(B(x_0,t))}$$
$$\leq \left(\int_{B(x_0,t)} \left|D^\gamma A(y) - \left(D^\gamma A\right)_{B(x_0,r)}\right|^{\frac{ps'}{p-s'}} dy\right)^{\frac{p-s'}{ps'}}$$
$$\leq C\left\|D^\gamma A\right\|_* \left(1 + \ln\frac{t}{r}\right) r^{\frac{n(p-s')}{ps'}}$$
$$\leq C\left\|D^\gamma A\right\|_* \left(1 + \ln\frac{t}{r}\right) r^{\frac{n}{s'}-\alpha} r^{-\frac{n}{q}}.$$

Then

$$I_2 \leq C \sum_{|\gamma|=m-1} \left\|D^\gamma A\right\|_* \int_{2r}^{\infty} \left(1 + \ln\frac{t}{r}\right) \|f\|_{L_p(B(x_0,t))} t^{-\frac{n}{q}-1}\, dt. \tag{4.9}$$

Combining the estimates of I_1 and I_2, we have

$$\sup_{x\in B(x_0,r)} \left|T_{\Omega,\alpha}^{A,m}(f_2)(x)\right|$$
$$\leq C \sum_{|\gamma|=m-1} \left\|D^\gamma A\right\|_* \int_{2r}^{\infty} \left(1 + \ln\frac{t}{r}\right) \|f\|_{L_p(B(x_0,t))} t^{-\frac{n}{q}-1}\, dt.$$

Then we get

$$\left\|T_{\Omega,\alpha}^{A,m}(f_2)\right\|_{L_q(B(x_0,r))}$$
$$\leq C \sum_{|\gamma|=m-1} \left\|D^\gamma A\right\|_* r^{\frac{n}{q}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} t^{-\frac{n}{q}-1}\, dt. \tag{4.10}$$

This completes the proof of Theorem 4.1. □

5 Proof of Theorem 1.1

Since $f \in M_{p,\varphi_1}(\mathbb{R}^n)$, by Lemma 3.4 and the non-decreasing, with respect to t, of the norm $\|f\|_{L_p(B(x_0,t))}$, we get

$$\begin{aligned}
&\frac{\|f\|_{L_p(B(x_0,t))}}{\operatorname{ess\,inf}_{0<t<\tau<\infty}\varphi_1(x_0,\tau)\tau^{\frac{n}{p}}} \\
&\quad \leq \operatorname*{ess\,sup}_{0<t<\tau<\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{\varphi_1(x_0,\tau)\tau^{\frac{n}{p}}} \\
&\quad \leq \sup_{\tau>0} \frac{\|f\|_{L_p(B(x_0,\tau))}}{\varphi_1(x_0,\tau)\tau^{\frac{n}{p}}} \\
&\quad \leq \|f\|_{M_{p,\varphi_1}}.
\end{aligned}$$

Since (φ_1, φ_2) satisfies (1.3), we have

$$\begin{aligned}
&\int_r^\infty \left(1+\ln\frac{t}{r}\right)\|f\|_{L_p(B(x_0,t))} t^{-\frac{n}{q}-1}\,dt \\
&\quad = \int_r^\infty \frac{\|f\|_{L_p(B(x_0,t))}}{\operatorname{ess\,inf}_{t<\tau<\infty}\varphi_1(x_0,\tau)\tau^{\frac{n}{p}}}\left(1+\ln\frac{t}{r}\right)\frac{\operatorname{ess\,inf}_{t<\tau<\infty}\varphi_1(x_0,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{q}}}\frac{dt}{t} \\
&\quad \leq \|f\|_{M_{p,\varphi_1}} \int_r^\infty \left(1+\ln\frac{t}{r}\right)\frac{\operatorname{ess\,inf}_{t<\tau<\infty}\varphi_1(x_0,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{q}}}\frac{dt}{t} \\
&\quad \leq C\|f\|_{M_{p,\varphi_1}}\varphi_2(x_0,t).
\end{aligned}$$

Then by (4.1) we get

$$\begin{aligned}
&\left\| T_{\Omega,\alpha}^{A,m}(f)\right\|_{M_{q,\varphi_2}} \\
&\quad = \sup_{x_0\in\mathbb{R}^n,t>0}\frac{1}{\varphi_2(x_0,t)}\left(\frac{1}{|B(x_0,t)|}\int_{B(x_0,t)}\left|T_{\Omega,\alpha}^{A,m}(f)(y)\right|^q dy\right)^{1/q} \\
&\quad \leq C\sum_{|\gamma|=m-1}\left\|D^\gamma A\right\|_* \sup_{x_0\in\mathbb{R}^n,t>0}\frac{1}{\varphi_2(x_0,t)}\int_r^\infty\left(1+\ln\frac{t}{r}\right)\|f\|_{L_p(B(x_0,t))}t^{-\frac{n}{q}-1}\,dt \\
&\quad \leq C\sum_{|\gamma|=m-1}\left\|D^\gamma A\right\|_*\|f\|_{M_{p,\varphi_1}}.
\end{aligned}$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

This work was carried out in collaboration between all authors. AA and VHH raised these interesting problems in the research. AA, VHH, and ZVS proved the theorems, interpreted the results, and wrote the article. All authors defined the research theme, and read and approved the manuscript.

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