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Complete moment convergence of moving average process generated by a class of random variables

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Abstract
In this paper, we establish the complete moment convergence of a moving average process generated by the class of random variables satisfying a Rosenthal-type maximal inequality and a weak mean dominating condition with a mean dominating variable.

MSC: 60F15

Keywords: complete moment convergence; moving average process; Rosenthal-type maximal inequality; weak mean domination; slowly varying

1 Introduction

Let $\{Y_i, -\infty < i < \infty\}$ be a doubly infinite sequence of random variables with zero means and finite variances and $\{a_i, -\infty < i < \infty\}$ an absolutely summable sequence of real numbers. Define a moving average process $\{X_n, n \geq 1\}$ by

$$X_n = \sum_{i=-\infty}^{\infty} a_i Y_{i+n}, \quad n \geq 1. \tag{1.1}$$

The concept of complete moment convergence is as follows: Let $\{Y_n, n \geq 1\}$ be a sequence of random variables and $a_n > 0$, $b_n > 0$. If $\sum_{n=1}^{\infty} a_n E\{b_n^{-1}|Y_n| - \epsilon\}^+ < \infty$ for all $\epsilon > 0$, then we call that $\{Y_n, n \geq 1\}$ satisfies the complete moment convergence. It is well known that the complete moment convergence can imply the complete convergence.

Chow [1] first showed the following complete moment convergence for a sequence of i.i.d. random variables by generalizing the result of Baum and Katz [2].

Theorem 1.1 *Suppose that $\{Y_n, n \geq 1\}$ is a sequence of i.i.d. random variables with $EY_1 = 0$. For $1 \leq p < 2$ and $r > p$, if $E\{|Y_1|^r + |Y_1| \log(1 + |Y_1|)\} < \infty$, then $\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} E(|\sum_{i=1}^{n} Y_i| - \epsilon n^{\frac{1}{p}})^+ < \infty$ for any $\epsilon > 0$.*

Recently, under dependence assumptions many authors studied extensively the complete moment convergence of a moving average process; see for example, Li and Zhang [3] for NA random variables, Zhou [4] for φ-mixing random variables, and Zhou and Lin [5] for ρ-mixing random variables.

We recall that a sequence $\{Y_n, n \geq 1\}$ of random variables satisfies a weak mean dominating condition with a mean dominating random variable Y if there is some positive constant C such that

$$\frac{1}{n}\sum_{k=1}^{n} P(|Y_k| > x) \leq CP(|Y| > x) \tag{1.2}$$

for all $x > 0$ and all $n \geq 1$ (see Kuczmaszewska [6]).

One of the most interesting inequalities in probability theory and mathematical statistics is the Rosenthal-type maximal inequality. For a sequence $\{Y_i, 1 \leq i \leq n\}$ of i.i.d. random variables with $E|Y_1|^q < \infty$ for $q \geq 2$ there exists a positive constant C_q depending only on q such that

$$E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}(Y_i - EY_i)\right|\right)^q \leq C_q\left\{\sum_{i=1}^{n} E|Y_i|^q + \left(\sum_{i=1}^{n} EY_i^2\right)^{q/2}\right\}. \tag{1.3}$$

The above inequality has been obtained for dependent random variables by many authors. See, for example, Peligrad [7] for a strong stationary ρ-mixing sequence, Peligrad and Gut [8] for a ρ^*-mixing sequence, Stoica [9] for a martingale difference sequence, and so forth.

In this paper we will establish the complete moment convergence for a moving average process generated by the class of random variables satisfying a Rosenthal-type maximal inequality and a weak mean dominating condition.

2 Some lemmas

The following lemmas will be useful to prove the main results.

Recall that a real valued function h, positive and measurable on $[0, \infty)$, is said to be slowly varying at infinity if for each $\lambda > 0$

$$\lim_{x\to\infty}\frac{h(\lambda x)}{h(x)} = 1.$$

Lemma 2.1 (Zhou [4]) *If h is a slowly varying function at infinity and m a positive integer, then*

(1) $\sum_{n=1}^{m} n^t h(n) \leq Cm^{t+1}h(m)$ *for* $t > -1$,
(2) $\sum_{n=m}^{\infty} n^t h(n) \leq Cm^{t+1}h(m)$ *for* $t < -1$.

Lemma 2.2 (Gut [10]) *Let $\{X_n, n \geq 1\}$ be a sequence of random variables satisfying a weak dominating condition with a mean dominating random variable X, i.e., there exists some positive constant C*

$$\frac{1}{n}\sum_{i=1}^{n} P(|X_i| > x) \leq CP(|X| > x) \quad \text{for all } x > 0 \text{ and all } n \geq 1.$$

Let $r > 0$ and for some $A > 0$

$$X_i' = X_i I(|X_i| \leq A), \qquad X_i'' = X_i I(|X_i| > A),$$

$$X_i^* = X_i I(|X_i| \leq A) - AI(X_i < -A) + AI(X_i > A),$$

and

$$X' = XI(|X| \le A), \qquad X'' = XI(|X| > A),$$
$$X^* = XI(|X| \le A) - AI(X < -A) + AI(X > A).$$

Then for some $C > 0$

(1) *if* $E|X|^r < \infty$, *then* $(n^{-1})\sum_{i=1}^n E|X_i|^r \le CE|X|^r$,
(2) $(n^{-1})\sum_{i=1}^n E|X_i'|^r \le C(E|X'|^r + A^r P(|X| > A))$ *for any* $A > 0$,
(3) $(n^{-1})\sum_{i=1}^n E|X_i''|^r \le CE|X''|^r$ *for any* $A > 0$,
(4) $(n^{-1})\sum_{i=1}^n E|X_i^*|^r \le CE|X^*|^r$ *for any* $A > 0$.

3 Main result

Theorem 3.1 *Let h be a function slowly varying at infinity, $p \ge 1$, $\alpha > \frac{1}{2}$ and $\alpha p > 1$. Assume that $\{a_i, -\infty < i < \infty\}$ is an absolutely summable sequence of real numbers and that $\{Y_i, -\infty < i < \infty\}$ is a sequence of mean zero random variables satisfying a weak mean dominating condition with a mean dominating random variable Y, i.e. there exists some positive constant C*

$$\frac{1}{n}\sum_{i=j+1}^{j+n} P(|Y_i| > x) \le CP(|Y| > x) \quad \textit{for all } x > 0, -\infty < j < \infty$$

and all $n \ge 1$ and $E|Y|^p h(|Y|^{\frac{1}{\alpha}}) < \infty$.

Suppose that $\{X_n, n \ge 1\}$ is a moving average process, where $X_n = \sum_{i=-\infty}^{\infty} a_i Y_{i+n}$, $n \ge 1$ is defined as (1.1).

Assume that for any $q \ge 2$, there exists a positive C_q depending only on q such that

$$E\left(\max_{1\le i\le n}\left|\sum_{j=1}^{i}(Y_{xj} - EY_{xj})\right|^q\right) \le C_q\left\{\sum_{j=1}^{n} E|Y_{xj}|^q + \left(\sum_{j=1}^{n} EY_{xj}^2\right)^{q/2}\right\}, \tag{3.1}$$

where $Y_{xj} = -xI(Y_j < -x) + Y_j I(|Y_j| \le x) + xI(Y_j > x)$ for all $x > 0$.

Then for all $\epsilon > 0$

$$\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} h(n) E\left\{\max_{1\le i\le n}\left|\sum_{j=1}^{i} X_j\right| - \epsilon n^{\alpha}\right\}^+ < \infty \tag{3.2}$$

and

$$\sum_{n=1}^{\infty} n^{\alpha p-2} h(n) E\left\{\sup_{i\ge n}\left|i^{-\alpha}\sum_{j=1}^{i} X_j\right| - \epsilon\right\}^+ < \infty. \tag{3.3}$$

Proof of (3.2) Let $\tilde{Y}_{xj} = Y_j - Y_{xj}$ and $l(n) = n^{\alpha p-2-\alpha} h(n)$.

Recall that $\sum_{k=1}^n X_k = \sum_{k=1}^n \sum_{i=-\infty}^{\infty} a_i Y_{i+k} = \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_j$ by (1.1).

If $\alpha > 1$, by the assumption that $\sum_{i=-\infty}^{\infty} |a_i| < \infty$ and Lemma 2.2 we have, for $x > n^{\alpha}$,

$$x^{-1}\left|E\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+n}Y_{xj}\right| \le Cx^{-1}n\bigl\{E|Y|I\bigl[|Y|\le x\bigr]+xP\bigl(|Y|>x\bigr)\bigr\}$$
$$\le Cn^{1-\alpha}\to 0 \quad \text{as } n\to\infty. \tag{3.4i}$$

If $\frac{1}{2}<\alpha\le 1$, $\alpha p>1$ implies $p>1$. By the assumption $EY_i=0$ for all $-\infty<i<\infty$ and Lemma 2.2 we obtain

$$\begin{aligned}
x^{-1}\left|E\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+n}Y_{xj}\right| &= x^{-1}\left|E\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+n}\tilde{Y_{xj}}\right| \\
&\le Cx^{-1}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n}E|Y_j|I\bigl[|Y_j|>x\bigr] \\
&\le Cx^{-1}nE|Y|I\bigl[|Y|>x\bigr]\le Cx^{\frac{1}{\alpha}-1}E|Y|I\bigl[|Y|>x\bigr] \\
&\le CE|Y|^pI\bigl[|Y|>x\bigr]\to 0 \quad \text{as } x\to\infty.
\end{aligned} \tag{3.4ii}$$

It follows from (3.4i) and (3.4ii) that for $x>n^{\alpha}$ large enough,

$$x^{-1}\left|E\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+n}Y_{xj}\right| < \frac{\epsilon}{4}, \tag{3.5}$$

which yields

$$\begin{aligned}
&\sum_{n=1}^{\infty}l(n)E\left\{\max_{1\le k\le n}\left|\sum_{j=1}^{k}X_j\right|-\epsilon n^{\alpha}\right\}^{+} \\
&\quad\le\sum_{n=1}^{\infty}l(n)\int_{\epsilon n^{\alpha}}^{\infty}P\left(\max_{1\le k\le n}\left|\sum_{j=1}^{k}X_j\right|\ge x\right)dx \quad \bigl(\text{letting } x=\epsilon x'\bigr) \\
&\quad\le\epsilon\sum_{n=1}^{\infty}l(n)\int_{n^{\alpha}}^{\infty}P\left(\max_{1\le k\le n}\left|\sum_{j=1}^{k}X_j\right|\ge\epsilon x'\right)dx' \\
&\quad\le C\sum_{n=1}^{\infty}l(n)\int_{n^{\alpha}}^{\infty}P\left(\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+k}\tilde{Y_{xj}}\right|\ge\frac{\epsilon x}{2}\right)dx \\
&\qquad+C\sum_{n=1}^{\infty}l(n)\int_{n^{\alpha}}^{\infty}P\left(\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+k}(Y_{xj}-EY_{xj})\right|\ge\frac{\epsilon x}{4}\right)dx \\
&\quad=I_1+I_2.
\end{aligned} \tag{3.6}$$

Now we will by an estimate show that $I_1<\infty$. It is clear that $|\tilde{Y_{xj}}|\le|Y_j|I[|Y_j|>x]$. Hence for I_1, by Markov's inequality and Lemma 2.2, we have

$$\begin{aligned}
I_1 &\le C\sum_{n=1}^{\infty}l(n)\int_{n^{\alpha}}^{\infty}x^{-1}E\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+k}\tilde{Y_{xj}}\right|dx \\
&\le C\sum_{n=1}^{\infty}l(n)\int_{n^{\alpha}}^{\infty}x^{-1}\sum_{-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n}E|\tilde{Y_{xj}}|\,dx
\end{aligned}$$

$$\begin{aligned}
&\le C\sum_{n=1}^{\infty} nl(n)\int_{n^\alpha}^{\infty} x^{-1}E|Y|I\bigl[|Y|>x\bigr]\,dx \\
&= C\sum_{n=1}^{\infty} nl(n)\sum_{m=n}^{\infty}\int_{m^\alpha}^{(m+1)^\alpha} x^{-1}E|Y|I\bigl[|Y|>x\bigr]\,dx \\
&\le C\sum_{n=1}^{\infty} nl(n)\sum_{m=n}^{\infty} m^{-1}E|Y|I\bigl[|Y|>m^\alpha\bigr] \\
&= C\sum_{m=1}^{\infty} m^{-1}E|Y|I\bigl[|Y|>m^\alpha\bigr]\sum_{n=1}^{m} n^{\alpha p-1-\alpha}h(n).
\end{aligned} \tag{3.7}$$

If $p>1$, note that $\alpha p-1-\alpha>-1$. By Lemma 2.1 and (3.7) we obtain

$$\begin{aligned}
I_1 &\le C\sum_{m=1}^{\infty} m^{\alpha p-1-\alpha}h(m)E|Y|I\bigl[|Y|>m^\alpha\bigr] \\
&= C\sum_{m=1}^{\infty} m^{\alpha p-1-\alpha}h(m)\sum_{k=m}^{\infty} E|Y|I\bigl[k^\alpha<|Y|\le(k+1)^\alpha\bigr] \\
&= C\sum_{k=1}^{\infty} E|Y|I\bigl[k^\alpha<|Y|\le(k+1)^\alpha\bigr]\sum_{m=1}^{k} m^{\alpha p-1-\alpha}h(m) \\
&\le C\sum_{k=1}^{\infty} k^{\alpha p-\alpha}h(k)E|Y|I\bigl[k^\alpha<|Y|\le(k+1)^\alpha\bigr] \\
&\le CE|Y|^p h\bigl(|Y|^{\frac{1}{\alpha}}\bigr)<\infty.
\end{aligned} \tag{3.8}$$

If $p=1$, by (3.7), we also obtain

$$\begin{aligned}
I_1 &\le C\sum_{m=1}^{\infty} m^{-1}E|Y|I\bigl[|Y|>m^\alpha\bigr]\sum_{n=1}^{m} n^{-1}h(n) \\
&\le C\sum_{m=1}^{\infty} m^{-1}E|Y|I\bigl[|Y|>m^\alpha\bigr]\sum_{n=1}^{m} n^{-1+\alpha\delta}h(n) \quad \text{for any } \delta>0 \\
&\le C\sum_{m=1}^{\infty} m^{\alpha\delta-1}h(m)E|Y|I\bigl[|Y|>m^\alpha\bigr] \\
&\le CE|Y|^{1+\delta}h\bigl(|Y|^{\frac{1}{\alpha}}\bigr)<\infty.
\end{aligned} \tag{3.9}$$

So, by (3.8) and (3.9) we get

$$I_1<\infty \quad \text{for } p\ge 1. \tag{3.10}$$

For I_2, by Markov's inequality, Hölder's inequality, and (3.1) we get for any $q\ge 2$

$$\begin{aligned}
I_2 &\le C\sum_{n=1}^{\infty} l(n)\int_{n^\alpha}^{\infty} x^{-q}E\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty} a_i\sum_{j=i+1}^{i+k}(Y_{xj}-EY_{xj})\right|^q dx \\
&\le C\sum_{n=1}^{\infty} l(n)\int_{n^\alpha}^{\infty} x^{-q}
\end{aligned}$$

$$
\begin{aligned}
&\times E\Bigg[\sum_{i=-\infty}^{\infty}\big(|a_i|^{1-\frac{1}{q}}\big)\Bigg(|a_i|^{\frac{1}{q}}\max_{1\le k\le n}\Bigg|\sum_{j=i+1}^{i+k}(Y_{xj}-EY_{xj})\Bigg|\Bigg)\Bigg]^q\,dx \\
&\le C\sum_{n=1}^{\infty}l(n)\int_{n^\alpha}^{\infty}x^{-q} \\
&\quad\times\Bigg(\sum_{i=-\infty}^{\infty}|a_i|\Bigg)^{q-1}\Bigg(\sum_{i=-\infty}^{\infty}|a_i|E\max_{1\le k\le n}\Bigg|\sum_{j=i+1}^{i+k}(Y_{xj}-EY_{xj})\Bigg|^q\Bigg)\,dx \\
&\le C\sum_{n=1}^{\infty}l(n)\int_{n^\alpha}^{\infty}x^{-q}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n}E|Y_{xj}-EY_{xj}|^q\,dx \\
&\quad+C\sum_{n=1}^{\infty}l(n)\int_{n^\alpha}^{\infty}x^{-q}\sum_{i=-\infty}^{\infty}|a_i|\Bigg(\sum_{j=i+1}^{i+n}E|Y_{xj}-EY_{xj}|^2\Bigg)^{\frac{q}{2}}\,dx \\
&=: I_{21}+II_{22}.
\end{aligned}
\tag{3.11}
$$

For I_{21}, we consider the following two cases.

If $p > 1$, take $q > \max\{2, p\}$, then by the assumption that $\sum_{i=-\infty}^{\infty}|a_i| < \infty$, C_r inequality and Lemmas 2.1 and 2.2 we get

$$
\begin{aligned}
I_{21} &\le C\sum_{n=1}^{\infty}nl(n)\int_{n^\alpha}^{\infty}x^{-q}\big\{E|Y|^qI\big[|Y|\le x\big]+x^qP\big(|Y|>x\big)\big\}\,dx \\
&\le C\sum_{n=1}^{\infty}nl(n)\sum_{m=n}^{\infty}\int_{m^\alpha}^{(m+1)^\alpha}\big\{x^{-q}E|Y|^qI\big[|Y|\le x\big]+P\big(|Y|>x\big)\big\}\,dx \\
&\le C\sum_{n=1}^{\infty}nl(n)\sum_{m=n}^{\infty}\big\{m^{\alpha(1-q)-1}E|Y|^qI\big[|Y|\le(m+1)^\alpha\big]+m^{\alpha-1}P\big(|Y|>m^\alpha\big)\big\} \\
&= C\sum_{m=1}^{\infty}\big\{m^{\alpha(1-q)-1}E|Y|^qI\big[|Y|\le(m+1)^\alpha\big]+m^{\alpha-1}P\big(|Y|>m^\alpha\big)\big\}\sum_{n=1}^{m}nl(n) \\
&\le C\sum_{m=1}^{\infty}m^{\alpha(p-q)-1}h(m)\sum_{k=1}^{m}E|Y|^qI\big[k^\alpha<|Y|\le(k+1)^\alpha\big] \\
&\quad+C\sum_{m=1}^{\infty}m^{\alpha p-1}h(m)\sum_{k=m}^{\infty}EI\big[k^\alpha<|Y|\le(k+1)^\alpha\big] \\
&= C\sum_{k=1}^{\infty}E|Y|^qI\big[k^\alpha<|Y|\le(k+1)^\alpha\big]\sum_{m=k}^{\infty}m^{\alpha(p-q)-1}h(m) \\
&\quad+C\sum_{k=1}^{\infty}EI\big[k^\alpha<|Y|\le(k+1)^\alpha\big]\sum_{m=1}^{k}m^{\alpha p-1}h(m) \\
&\le C\sum_{k=1}^{\infty}k^{\alpha(p-q)}h(k)E|Y|^qI\big[k^\alpha<|Y|\le(k+1)^\alpha\big] \\
&\quad+C\sum_{k=1}^{\infty}k^{\alpha p}h(k)EI\big[k^\alpha<|Y|\le(k+1)^\alpha\big] \\
&\le CE|Y|^ph\big(|Y|^{\frac{1}{\alpha}}\big)<\infty.
\end{aligned}
\tag{3.12}
$$

For I_{21}, if $p = 1$, take $q > \max\{1+\delta, 2\}$ by the same argument as above one gets for any $\delta > 0$

$$\begin{aligned}
I_{21} &\le C\sum_{m=1}^{\infty}\bigl\{m^{\alpha(1-q)-1}E|Y|^q I\bigl[|Y| \le (m+1)^\alpha\bigr] + m^{\alpha-1}P\bigl(|Y| > m^\alpha\bigr)\bigr\}\sum_{n=1}^{m} nl(n) \\
&= C\sum_{m=1}^{\infty}\bigl\{m^{\alpha(1-q)-1}E|Y|^q I\bigl[|Y| \le (m+1)^\alpha\bigr] + m^{\alpha-1}P\bigl(|Y| > m^\alpha\bigr)\bigr\}\sum_{n=1}^{m} n^{-1}l(n) \\
&\le C\sum_{m=1}^{\infty}\bigl\{m^{\alpha(1-q)-1}E|Y|^q I\bigl[|Y| \le (m+1)^\alpha\bigr] + m^{\alpha-1}P\bigl(|Y| > m^\alpha\bigr)\bigr\}\sum_{n=1}^{m} n^{-1+\alpha\delta}h(n) \\
&\le C\sum_{m=1}^{\infty}\bigl\{m^{\alpha(1-q+\delta)-1}h(n)E|Y|^q I\bigl[|Y| \le (m+1)^\alpha\bigr] + m^{\alpha(1+\delta)-1}h(x)EI\bigl[|Y| > m^\alpha\bigr]\bigr\} \\
&\le CE|Y|^{1+\delta}h\bigl(|Y|^{\frac{1}{\alpha}}\bigr) < \infty.
\end{aligned} \tag{3.13}$$

It follows from (3.12) and (3.13) that, for $p \ge 1$,

$$I_{21} < \infty. \tag{3.14}$$

It remains to estimate $I_{22} < \infty$.

For I_{22}, we consider the following two cases. If $1 \le p < 2$, take $q > 2$, note that $\alpha p + \frac{q}{2} - \frac{\alpha pq}{2} - 1 = (\alpha p - 1)(1 - \frac{q}{2}) < 0$. Then by C_r inequality and Lemma 2.2, we obtain

$$\begin{aligned}
I_{22} &\le C\sum_{n=1}^{\infty} n^{\frac{q}{2}}l(n)\int_{n^\alpha}^{\infty} x^{-q}\bigl\{\bigl(E|Y|^2 I[|Y| \le x]\bigr)^{\frac{q}{2}} + x^q\bigl(P(|Y| > x)\bigr)^{\frac{q}{2}}\bigr\}\,dx \\
&\le C\sum_{n=1}^{\infty} n^{\frac{q}{2}}l(n)\sum_{m=n}^{\infty}\int_{m^\alpha}^{(m+1)^\alpha} \bigl\{x^{-q}\bigl(E|Y|^2 I[|Y| \le x]\bigr)^{\frac{q}{2}} + \bigl(P(|Y| > x)\bigr)^{\frac{q}{2}}\bigr\}\,dx \\
&\le C\sum_{n=1}^{\infty} n^{\frac{q}{2}}l(n)\sum_{m=n}^{\infty}\bigl\{m^{\alpha(1-q)-1}\bigl(E|Y|^2 I\bigl[|Y| \le (m+1)^2\bigr]\bigr)^{\frac{q}{2}} + m^{\alpha-1}\bigl(P\bigl(|Y| > m^\alpha\bigr)\bigr)^{\frac{q}{2}}\bigr\} \\
&= C\sum_{m=1}^{\infty}\bigl\{m^{\alpha(1-q)-1}\bigl(E|Y|^2 I\bigl[|Y| \le (m+1)^\alpha\bigr]\bigr)^{\frac{q}{2}} + m^{\alpha-1}\bigl(P\bigl(|Y| > m^\alpha\bigr)\bigr)^{\frac{q}{2}}\bigr\}\sum_{n=1}^{m} n^{\frac{q}{2}}l(n) \\
&\le C\sum_{m=1}^{\infty} m^{\alpha(p-q)+\frac{q}{2}-2}h(m)\bigl(E|Y|^2 I\bigl[|Y| \le (m+1)^\alpha\bigr]\bigr)^{\frac{q}{2}} \\
&\quad + C\sum_{m=1}^{\infty} m^{\alpha p+\frac{q}{2}-2}h(m)\bigl(EI\bigl[|Y| > m^\alpha\bigr]\bigr)^{\frac{q}{2}} \\
&\le C\sum_{m=1}^{\infty} m^{\alpha p+\frac{q}{2}-\frac{\alpha pq}{2}-2}h(m)\bigl(E|Y|^p\bigr)^{\frac{q}{2}} < \infty.
\end{aligned} \tag{3.15}$$

If $p \ge 2$, take $q > \frac{p\alpha-1}{\alpha-\frac{1}{2}} > 2$, which yields $\alpha(p-q) + \frac{q}{2} - 2 < -1$. Then we get

$$\begin{aligned}
I_{22} &\le C\sum_{m=1}^{\infty}\bigl\{m^{\alpha(1-q)-1}\bigl(E|Y|^2 I\bigl[|Y| \le (m+1)^\alpha\bigr]\bigr)^{\frac{q}{2}} \\
&\quad + m^{\alpha-1}\bigl(P\bigl(|Y| > m^\alpha\bigr)\bigr)^{\frac{q}{2}}\bigr\}\sum_{n=1}^{m} n^{\frac{q}{2}}l(n)
\end{aligned}$$

$$\begin{aligned}
&\le C\sum_{m=1}^{\infty} m^{\alpha(p-q)+\frac{q}{2}-2}h(m)\bigl(E|Y|^2 I\bigl[|Y|\le (m+1)^{\alpha}\bigr]\bigr)^{\frac{q}{2}} \\
&\quad + C\sum_{m=1}^{\infty} m^{\alpha p+\frac{q}{2}-2}h(m)\bigl(EI\bigl[|Y|>m^{\alpha}\bigr]\bigr)^{\frac{q}{2}} \\
&\le C\sum_{m=1}^{\infty} m^{\alpha(p-q)+\frac{q}{2}-2}h(m)\bigl(E|Y|^2\bigr)^{\frac{q}{2}} < \infty. \qquad (3.16)
\end{aligned}$$

Hence, by (3.15) and (3.16) we get

$$I_{22} < \infty \quad \text{for } p \ge 1. \qquad (3.17)$$

Moreover, by (3.14) and (3.17), we also get

$$I_2 < \infty \quad \text{for } p \ge 1. \qquad (3.18)$$

The proof of (3.2) is completed by (3.6), (3.10), and (3.18). □

Proof of (3.3) By Lemma 2.1 and (3.2), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2}h(n)E\left\{\sup_{i\ge n}\left|i^{-\alpha}\sum_{j=1}^{i}X_j\right|-\epsilon\right\}^{+} \\
&\quad = \sum_{n=1}^{\infty} n^{\alpha p-2}h(n)\int_0^{\infty} P\left(\sup_{i\ge n}\left|i^{-\alpha}\sum_{j=1}^{i}X_j\right|>\epsilon+x\right)dx \\
&\quad = \sum_{k=1}^{\infty}\sum_{n=2^{k-1}}^{2^k-1} n^{\alpha p-2}h(n)\int_0^{\infty} P\left(\sup_{i\ge n}\left|i^{-\alpha}\sum_{j=1}^{i}X_j\right|>\epsilon+x\right)dx \\
&\quad \le C\sum_{k=1}^{\infty}\int_0^{\infty} P\left(\sup_{i\ge 2^{k-1}}\left|i^{-\alpha}\sum_{j=1}^{i}X_j\right|>\epsilon+x\right)dx\sum_{n=2^{k-1}}^{2^k-1} n^{\alpha p-2}h(n) \\
&\quad \le C\sum_{k=1}^{\infty} 2^{k(\alpha p-1)}h\bigl(2^k\bigr)\int_0^{\infty} P\left(\sup_{i\ge 2^{k-1}}\left|i^{-\alpha}\sum_{j=1}^{i}X_j\right|>\epsilon+x\right)dx \\
&\quad \le C\sum_{k=1}^{\infty} 2^{k(\alpha p-1)}h\bigl(2^k\bigr)\sum_{m=k}^{\infty}\int_0^{\infty} P\left(\max_{2^{m-1}\le i<2^m}\left|i^{-\alpha}\sum_{j=1}^{i}X_j\right|>\epsilon+x\right)dx \\
&\quad \le C\sum_{m=1}^{\infty}\int_0^{\infty} P\left(\max_{2^{m-1}\le i<2^m}\left|i^{-\alpha}\sum_{j=1}^{i}X_j\right|>\epsilon+x\right)dx\sum_{k=1}^{m} 2^{k(\alpha p-1)}h\bigl(2^k\bigr) \\
&\quad \le C\sum_{m=1}^{\infty} 2^{m(\alpha p-1)}h\bigl(2^m\bigr)\int_0^{\infty} P\left(\max_{2^{m-1}\le i<2^m}\left|\sum_{j=1}^{i}X_j\right|>(\epsilon+x)2^{(m-1)\alpha}\right)dx \\
&\qquad \bigl(\text{letting } y=2^{(m-1)\alpha}x\bigr) \\
&\quad \le C\sum_{m=1}^{\infty} 2^{m(\alpha p-1-\alpha)}h\bigl(2^m\bigr)\int_0^{\infty} P\left(\max_{1\le i<2^m}\left|\sum_{j=1}^{i}X_j\right|>\epsilon 2^{(m-1)\alpha}+y\right)dy
\end{aligned}$$

$$\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} h(n) \int_0^{\infty} P\left(\max_{1\leq i<n}\left|\sum_{j=1}^{i} X_j\right| > \epsilon n^{\alpha}2^{-\alpha} + y\right) dy$$

$$= C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} h(n) E\left(\max_{1\leq i<n}\left|\sum_{j=1}^{i} X_j\right| - \epsilon' n^{\alpha}\right)^{+} < \infty,$$

where $\epsilon' = \epsilon 2^{-\alpha}$. Hence the proof of (3.3) is completed. □

Remark There are many sequences of dependent random variables satisfying (3.1) for all $q \geq 2$.

Examples include sequences of NA random variables (see Shao [11]), ρ^*-mixing random variables (see Utev and Peligrad [12]), φ-mixing random variables (see Zhou [4]), and ρ-mixing random variables (see Zhou and Lin [5]).

Corollary 3.2 *Under the assumptions of Theorem* 3.1 *for any* $\epsilon > 0$

$$\sum_{n=1}^{\infty} n^{\alpha p-2} h(n) P\left(\max_{1\leq i\leq n}\left|\sum_{j=1}^{i} X_j\right| > \epsilon n^{\alpha}\right) < \infty. \tag{3.19}$$

Proof As in Remark 1.2 of Li and Zhang [3] we can obtain (3.19). □

Competing interests
The author declares that they have no competing interests.

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