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A modified regularization method for finding zeros of monotone operators in Hilbert spaces

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Abstract
We study the regularization method for solving the variational inclusion problem of the sum of two monotone operators in Hilbert spaces. The strong convergence theorem is then established under some relaxed conditions which mainly improves and recovers that of Qin *et al.* (Fixed Point Theory Appl. 2014:75, 2014). We also apply our main result to the convex minimization problem, the fixed point problem and the variational inequality problem. Finally we provide numerical examples for supporting the main result.

MSC: 47H09; 47H10

Keywords: convex minimization problem; maximal monotone operator; inverse strongly monotone operator; regularization method; variational inequality problem

1 Introduction

Let C be a nonempty subset of a real Hilbert space H. Define the domain and the range of an operator $B : H \to 2^H$ by $D(B) = \{x \in H : Bx \neq \emptyset\}$ and $R(B) = \bigcup\{Bx : x \in D(B)\}$, respectively. The inverse of B, denoted by B^{-1}, is defined by $x \in B^{-1}y$ if and only if $y \in Bx$. We study the problem of finding $\hat{x}$ such that

$$0 \in A\hat{x} + B\hat{x},$$

where $A : C \to H$ is an operator and $B : D(B) \subset H \to 2^H$ is a set-valued operator. This problem is called the variational inclusion problem. Some typical problems arising in various branches of science, applied sciences, economics, and engineering such as machine learning, image restoration, and signal recovery can be viewed as this form. To be more precise, it includes, as special cases, the variational inequality problem, the split feasibility problem, the linear inverse problem, and the following convex minimization problem:

$$\min_{x\in H} F(x) + G(x),$$

where $F : H \to \mathbb{R}$ is a smooth convex function, and $G : H \to \mathbb{R}$ is a non-smooth convex function. That is,

$$F(\hat{x}) + G(\hat{x}) = \min_{x\in H} F(x) + G(x) \quad \Leftrightarrow \quad 0 \in \nabla F(\hat{x}) + \partial G(\hat{x}),$$

where ∇F is the gradient of F and ∂G is the subdifferential of G defined by

$$\partial G(x) = \{z \in H : \langle y - x, z\rangle + G(x) \le G(y), \forall y \in H\}.$$

For $r > 0$, define the mapping $T_r : C \to D(B)$ as follows:

$$T_r = (I + rB)^{-1}(I - rA). \tag{1.1}$$

We see that

$$T_r x = x \quad \Leftrightarrow \quad x = (I + rB)^{-1}(x - rAx) \quad \Leftrightarrow \quad x - rAx \in x + rBx \quad \Leftrightarrow \quad 0 \in Ax + Bx,$$

which shows that the fixed points set of T_r coincides with the solutions set of $(A + B)^{-1}(0)$. This suggests the following iteration process: $x_0 \in C$ and

$$x_{n+1} = (I + r_n B)^{-1}(x_n - r_n A x_n) = T_{r_n} x_n, \quad n \ge 0,$$

where $\{r_n\} \subset (0,\infty)$ and $D(B) \subset C$. This method is called a forward-backward splitting algorithm [1, 2]. If $A \equiv 0$, then we obtain the proximal point algorithm [3–6] and if $B \equiv 0$, then we obtain the gradient method [7]. However, it is noted that the sequences generated by these schemes converge weakly in general. In the literature, many methods have been suggested to solve the variational inclusion problem for maximal monotone operators; see, *e.g.*, [8–12].

Very recently, Qin *et al.* [13] proved the following theorem in Hilbert spaces.

Theorem Q *Let $A : C \to H$ be an α-inverse strongly monotone mapping and let B be a maximal monotone operator on H. Assume that $D(B) \subset C$ and $(A + B)^{-1}(0)$ is nonempty. Let $f : C \to C$ be a fixed k-contraction and let $J_{r_n} = (I + r_n B)^{-1}$. Let $\{z_n\}$ be a sequence in C in the following process*: $z_0 \in C$ *and*

$$\begin{aligned} w_n &= \alpha_n f(z_n) + (1 - \alpha_n) z_n, \\ z_{n+1} &= J_{r_n}(w_n - r_n A w_n + e_n), \quad n \ge 0, \end{aligned} \tag{1.2}$$

where $\{\alpha_n\} \subset (0,1)$, $\{e_n\} \subset H$, and $\{r_n\} \subset (0,2\alpha)$. If the control sequences satisfy the following restrictions:

(a) $\alpha_n \to 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(b) $0 < a \le r_n \le b < 2\alpha$ *and* $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$;

(c) $\sum_{n=0}^{\infty} \|e_n\| < \infty$.

Then $\{z_n\}$ converges strongly to a point $\bar{x} \in (A + B)^{-1}(0)$, where $\bar{x} = P_{(A+B)^{-1}(0)} f(\bar{x})$.

In this paper, motivated by Qin *et al.* [13], we prove that the above theorem still holds even if the additional conditions that $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ and $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$ are removed. As a direct consequence, we obtain some results concerning the fixed point problem of strict pseudocontractions, the convex minimization problem and the variational inequality problem. We also provide examples as well as numerical results.

2 Preliminaries and lemmas

We now provide some basic concepts, definitions and lemmas which will be used in the sequel.

Let C be a nonempty, closed, and convex subset of a real Hilbert space H with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. For each $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that $\|x - P_C x\| = \min_{y\in C}\|x - y\|$. Then P_C is called the metric projection of H on to C. For $x \in H$, we know that

$$\langle x - P_C x, y - P_C x\rangle \le 0 \tag{2.1}$$

for all $y \in C$. Recall that the mapping $T : C \to C$ is said to be

(i) nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$;

(ii) k-contractive if there exists $0 < k < 1$ such that

$$\|Tx - Ty\| \le k\|x - y\|$$

for all $x, y \in C$;

(iii) firmly nonexpansive if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 - \big\|(I - T)x - (I - T)y\big\|^2$$

for all $x, y \in C$.

(iv) monotone if $\langle Tx - Ty, x - y\rangle \ge 0$ for all $x, y \in C$;

(v) α-inverse strongly monotone if there exists $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \ge \alpha\|Tx - Ty\|^2$$

for all $x, y \in C$. We denote by $F(T)$ the fixed points set of T, that is, $F(T) = \{x \in C : x = Tx\}$.

A set-valued operator B is said to be *monotone* if, for each $x, y \in D(B)$,

$$\langle u - v, x - y\rangle \ge 0, \quad u \in Bx, v \in By.$$

A monotone operator A is said to be *maximal* if $R(I + rB) = H$ for all $r > 0$ (see Minty [14]). For a maximal monotone operator B on H, and $r > 0$, we define the single-valued resolvent $J_r : H \to D(B)$ by $J_r = (I + rB)^{-1}$. It is well known that J_r is firmly nonexpansive, and $F(J_r) = B^{-1}(0)$.

We now collect some crucial lemmas.

Lemma 2.1 [15] *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let the mapping $A : C \to H$ be α-inverse strongly monotone and $r > 0$ be a constant. Then we have*

$$\big\|(I - rA)x - (I - rA)y\big\|^2 \le \|x - y\|^2 + r(r - 2\alpha)\|Ax - Ay\|^2$$

for all $x, y \in C$. In particular, if $0 < r \le 2\alpha$, then $I - rA$ is nonexpansive.

Lemma 2.2 [9] *Let $A : C \to H$ be a mapping and $B : D(B) \subset H \to 2^H$ a monotone operator. Then $\|x - T_s x\| \le 2\|x - T_r x\|$ for all $0 < s \le r$ and $x \in C$.*

Lemma 2.3 [16] *Let C be a nonempty, closed, and convex subset of a Hilbert space H, and $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. If $x_n \rightharpoonup x$ and $\|x_n - Tx_n\| \to 0$, then $x \in F(T)$.*

Lemma 2.4 [17] *Let $\{a_n\}$ and $\{c_n\}$ are sequences of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \delta_n)a_n + b_n + c_n, \quad n \geq 0,$$

where $\{\delta_n\}$ is a sequence in $(0,1)$ and $\{b_n\}$ is a real sequence. Assume $\sum_{n=0}^{\infty} c_n < \infty$. Then the following results hold:

(i) *If $b_n \leq \delta_n M$ for some $M \geq 0$, then $\{a_n\}$ is a bounded sequence.*

(ii) *If $\sum_{n=0}^{\infty} \delta_n = \infty$ and $\limsup_{n\to\infty} b_n/\delta_n \leq 0$, then $\lim_{n\to\infty} a_n = 0$.*

We need the following crucial lemma proved by He-Yang [18].

Lemma 2.5 [18] *Assume $\{s_n\}$ is a sequence of nonnegative real numbers such that*

$$s_{n+1} \leq (1 - \gamma_n)s_n + \gamma_n\delta_n, \quad n \geq 0$$

and

$$s_{n+1} \leq s_n - \eta_n + \rho_n, \quad n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$, $\{\eta_n\}$ is a sequence of nonnegative real numbers and $\{\delta_n\}$, and $\{\rho_n\}$ are real sequences such that

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$,

(ii) $\lim_{n\to\infty} \rho_n = 0$,

(iii) $\lim_{k\to\infty} \eta_{n_k} = 0$ *implies* $\limsup_{n\to\infty} \delta_{n_k} \leq 0$ *for any subsequence $\{n_k\}$ of $\{n\}$.*

Then $\lim_{n\to\infty} s_n = 0$.

3 Main results

In this section, we present the main theorem of this paper.

Theorem 3.1 *Let $A : C \to H$ be an α-inverse strongly monotone mapping and let B be a maximal monotone operator on H such that $D(B) \subset C$ and $(A + B)^{-1}(0)$ is nonempty. Let $f : C \to C$ be a k-contraction. Assume that $\{\alpha_n\} \subset (0,1)$, $\{e_n\} \subset H$, and $\{r_n\} \subset (0, 2\alpha)$ with the following restrictions*:

(a) $\alpha_n \to 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(b) $0 < a \leq r_n \leq b < 2\alpha$;

(c) $\sum_{n=0}^{\infty} \|e_n\| < \infty$ *or* $\|e_n\|/\alpha_n \to 0$.

Then the sequence $\{z_n\}$ generated by (1.2) *converges strongly to a point $\overline{x} \in (A + B)^{-1}(0)$, where $\overline{x} = P_{(A+B)^{-1}(0)} f(\overline{x})$.*

Proof Let $\{x_n\}$ be a sequence generated by $x_0 \in C$ and

$$y_n = \alpha_n f(x_n) + (1 - \alpha_n)x_n,$$

$$x_{n+1} = J_{r_n}(y_n - r_n Ay_n), \quad n \geq 0.$$

Firstly, we shall show that $\{x_n\}$ and $\{z_n\}$ are equivalent. Indeed,

$$\begin{aligned}\|y_n - w_n\| &= \big\|\alpha_n\big(f(x_n) - f(z_n)\big) + (1-\alpha_n)(x_n - z_n)\big\| \\ &\leq \alpha_n k\|x_n - z_n\| + (1-\alpha_n)\|x_n - z_n\| \\ &= \big(1 - \alpha_n(1-k)\big)\|x_n - z_n\|.\end{aligned}$$

Using Lemma 2.1, condition (b), and the fact that J_{r_n} is nonexpansive, we obtain

$$\begin{aligned}\|x_{n+1} - z_{n+1}\| &= \big\|J_{r_n}(y_n - r_nAy_n) - J_{r_n}(w_n - r_nAw_n + e_n)\big\| \\ &\leq \big\|(y_n - r_nAy_n) - (w_n - r_nAw_n + e_n)\big\| \\ &\leq \|y_n - w_n\| + \|e_n\| \\ &\leq \big(1 - \alpha_n(1-k)\big)\|x_n - z_n\| + \|e_n\|.\end{aligned}$$

Applying Lemma 2.4(ii) with conditions (a) and (c), we conclude that $\|x_n - z_n\| \to 0$.

On the other hand, it can be checked that $P_{(A+B)^{-1}(0)}f$ is a contraction. So there exists a unique point $\overline{x} \in C$ such that

$$\overline{x} = P_{(A+B)^{-1}(0)}f(\overline{x}). \tag{3.1}$$

To finish our proof, it suffices to show that $x_n \to \overline{x}$ as $n \to \infty$.

We next show that $\{x_n\}$ is bounded. Fixing $p \in (A+B)^{-1}(0)$, we obtain

$$\begin{aligned}\|y_n - p\| &= \big\|\alpha_n\big(f(x_n) - f(p)\big) + \alpha_n\big(f(p) - p\big) + (1-\alpha_n)(x_n - p)\big\| \\ &\leq \alpha_n k\|x_n - p\| + \alpha_n\big\|f(p) - p\big\| + (1-\alpha_n)\|x_n - p\| \\ &= \big(1 - \alpha_n(1-k)\big)\|x_n - p\| + \alpha_n\big\|f(p) - p\big\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1} - p\| &= \big\|J_{r_n}(y_n - r_nAy_n) - J_{r_n}(p - r_nAp)\big\| \\ &\leq \|y_n - p\| \\ &\leq \big(1 - \alpha_n(1-k)\big)\|x_n - p\| + \alpha_n\big\|f(p) - p\big\|.\end{aligned}$$

Hence $\{x_n\}$ is bounded by Lemma 2.4(i). So are $\{f(x_n)\}$ and $\{y_n\}$. Observing

$$\begin{aligned}\|y_n - \overline{x}\|^2 &= \alpha_n\big\langle f(x_n) - f(\overline{x}), y_n - \overline{x}\big\rangle + \alpha_n\big\langle f(\overline{x}) - \overline{x}, y_n - \overline{x}\big\rangle + (1-\alpha_n)\langle x_n - \overline{x}, y_n - \overline{x}\rangle \\ &\leq \alpha_n k\|x_n - \overline{x}\|\|y_n - \overline{x}\| + \alpha_n\big\langle f(\overline{x}) - \overline{x}, y_n - \overline{x}\big\rangle + (1-\alpha_n)\|x_n - \overline{x}\|\|y_n - \overline{x}\| \\ &= \big(1 - \alpha_n(1-k)\big)\|x_n - \overline{x}\|\|y_n - \overline{x}\| + \alpha_n\big\langle f(\overline{x}) - \overline{x}, y_n - \overline{x}\big\rangle \\ &\leq \frac{1}{2}\big(1 - \alpha_n(1-k)\big)\big(\|x_n - \overline{x}\|^2 + \|y_n - \overline{x}\|^2\big) + \alpha_n\big\langle f(\overline{x}) - \overline{x}, y_n - \overline{x}\big\rangle,\end{aligned}$$

we have

$$\|y_n - \overline{x}\|^2 \leq \left(1 - \frac{2\alpha_n(1-k)}{1+\alpha_n(1-k)}\right)\|x_n - \overline{x}\|^2 + \frac{2\alpha_n}{1+\alpha_n(1-k)}\big\langle f(\overline{x}) - \overline{x}, y_n - \overline{x}\big\rangle.$$

So, by Lemma 2.1 and the firm nonexpansiveness of J_{r_n}, we have

$$\begin{aligned}\|x_{n+1}-\overline{x}\|^2 &= \big\|J_{r_n}(y_n-r_nAy_n)-J_{r_n}(\overline{x}-r_nA\overline{x})\big\|^2\\ &\le \big\|(y_n-r_nAy_n)-(\overline{x}-r_nA\overline{x})\big\|^2\\ &\quad-\big\|(I-J_{r_n})(y_n-r_nAy_n)-(I-J_{r_n})(\overline{x}-r_nA\overline{x})\big\|^2\\ &\le \|y_n-\overline{x}\|^2-r_n(2\alpha-r_n)\|Ay_n-A\overline{x}\|^2\\ &\quad-\big\|(I-J_{r_n})(y_n-r_nAy_n)-(I-J_{r_n})(\overline{x}-r_nA\overline{x})\big\|^2\\ &\le \left(1-\frac{2\alpha_n(1-k)}{1+\alpha_n(1-k)}\right)\|x_n-\overline{x}\|^2+\frac{2\alpha_n}{1+\alpha_n(1-k)}\big\langle f(\overline{x})-\overline{x},y_n-\overline{x}\big\rangle\\ &\quad-r_n(2\alpha-r_n)\|Ay_n-A\overline{x}\|^2\\ &\quad-\big\|(I-J_{r_n})(y_n-r_nAy_n)-(I-J_{r_n})(\overline{x}-r_nA\overline{x})\big\|^2.\end{aligned}$$

This implies that

$$\|x_{n+1}-\overline{x}\|^2\le\left(1-\frac{2\alpha_n(1-k)}{1+\alpha_n(1-k)}\right)\|x_n-\overline{x}\|^2+\frac{2\alpha_n}{1+\alpha_n(1-k)}\big\langle f(\overline{x})-\overline{x},y_n-\overline{x}\big\rangle \tag{3.2}$$

and

$$\begin{aligned}\|x_{n+1}-\overline{x}\|^2 &\le \|x_n-\overline{x}\|^2-r_n(2\alpha-r_n)\|Ay_n-A\overline{x}\|^2\\ &\quad-\big\|(I-J_{r_n})(y_n-r_nAy_n)-(I-J_{r_n})(\overline{x}-r_nA\overline{x})\big\|^2\\ &\quad+\frac{2\alpha_n}{1+\alpha_n(1-k)}\big\|f(\overline{x})-\overline{x}\big\|\,\|y_n-\overline{x}\|.\end{aligned} \tag{3.3}$$

We set, for all $n\ge1$, $s_n=\|x_n-\overline{x}\|^2$, $\gamma_n=\frac{2\alpha_n(1-k)}{1+\alpha_n(1-k)}$, $\delta_n=\frac{1}{1-k}\langle f(\overline{x})-\overline{x},y_n-\overline{x}\rangle$, $\rho_n=\frac{2\alpha_n}{1+\alpha_n(1-k)}\|f(\overline{x})-\overline{x}\|\|y_n-\overline{x}\|$, and $\eta_n=r_n(2\alpha-r_n)\|Ay_n-A\overline{x}\|^2+\|(I-J_{r_n})(y_n-r_nAy_n)-(I-J_{r_n})(\overline{x}-r_nA\overline{x})\|^2$. We can check that all sequences satisfies conditions (i) and (ii) in Lemma 2.5. Then (3.2) and (3.3) can be rewritten as the following inequalities:

$$s_{n+1}\le(1-\gamma_n)s_n+\gamma_n\delta_n,\quad n\ge0$$

and

$$s_{n+1}\le s_n-\eta_n+\rho_n,\quad n\ge0.$$

To complete the proof, we verify that the condition (iii) in Lemma 2.5 is satisfied. Let $\{n_k\}\subset\{n\}$ be such that $\eta_{n_k}\to0$. Then, by condition (b), we have

$$\lim_{n\to\infty}\|Ay_{n_k}-A\overline{x}\|=0$$

and

$$\lim_{n\to\infty}\big\|(I-J_{r_{n_k}})(y_{n_k}-r_{n_k}Ay_{n_k})-(I-J_{r_{n_k}})(\overline{x}-r_{n_k}A\overline{x})\big\|=0.$$

Hence we obtain

$$\lim_{n\to\infty}\|y_{n_k}-J_{r_{n_k}}(y_{n_k}-r_{n_k}Ay_{n_k})\|=0.$$

By Lemma 2.2(ii) and (b), we have

$$\|J_a(y_{n_k}-aAy_{n_k})-y_{n_k}\|\le 2\|J_{r_{n_k}}(y_{n_k}-r_{n_k}Ay_{n_k})-y_{n_k}\|\to 0,$$

where $J_a=(I+aB)^{-1}$. Since $\{y_n\}$ is bounded, by Lemma 2.3, we have $\omega_w(y_{n_k})\subset(A+B)^{-1}(0)$. Hence

$$\limsup_{k\to\infty}\langle f(\overline{x})-\overline{x},y_{n_k}-\overline{x}\rangle=\langle f(\overline{x})-\overline{x},y-\overline{x}\rangle\le 0,$$

where $y\in\omega_w(y_{n_k})$. It follows that $\limsup_{n\to\infty}\delta_{n_k}\le 0$. So, by Lemma 2.5, we conclude that $x_n\to\overline{x}$ as $n\to\infty$. We thus complete the proof. □

Remark 3.2 We remove the additionally required conditions: $\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$ and $\sum_{n=0}^{\infty}|r_{n+1}-r_n|<\infty$ proposed in the main theorem of Qin *et al.* [13].

4 Applications and numerical examples

In this section, we give some applications of our result to the variational inequality problem, the fixed point problem of strict pseudocontractions and the convex minimization problem.

4.1 Variational inequality problem

Let C be a nonempty subset of a Hilbert space H. The variational inequality problem is to find $x\in C$ such that

$$\langle Ax,y-x\rangle\ge 0,\quad \forall y\in C. \tag{4.1}$$

The solution set of (4.1) is denoted by $\mathrm{VI}(A,C)$. It is well known that $F(P_C(I-rA))=\mathrm{VI}(A,C)$ for all $r>0$. Define the indicator function of C, denoted by i_C, as $i_C(x)=0$ if $x\in C$ and $i_C(x)=\infty$ if $x\notin C$. We see that ∂i_C is maximal monotone. So, for $r>0$, we can define $J_r=(I+r\partial i_C)^{-1}$. Moreover, $x=J_ry$ if and only if $x=P_Cy$. Hence we obtain the following result.

Theorem 4.1 *Let $A:C\to H$ be an α-inverse strongly monotone mapping such that $\mathrm{VI}(A,C)$ is nonempty. Let $f:C\to C$ be a k-contraction. Let $\{z_n\}$ be a sequence in C defined by $z_0\in C$ and*

$$\begin{aligned} w_n&=\alpha_nf(z_n)+(1-\alpha_n)z_n,\\ z_{n+1}&=P_C(w_n-r_nAw_n+e_n),\quad n\ge 0, \end{aligned} \tag{4.2}$$

where $\{\alpha_n\}\subset(0,1)$, $\{e_n\}\subset H$, and $\{r_n\}\subset(0,2\alpha)$. Assume that

(a) $\alpha_n\to 0$ *and* $\sum_{n=0}^{\infty}\alpha_n=\infty$;
(b) $0<a\le r_n\le b<2\alpha$;
(c) $\sum_{n=0}^{\infty}\|e_n\|<\infty$ *or* $\|e_n\|/\alpha_n\to 0$.

Then $\{z_n\}$ converges strongly to a point $\overline{x}\in\mathrm{VI}(A,C)$, where $\overline{x}=P_{\mathrm{VI}(A,C)}f(\overline{x})$.

4.2 Fixed point problem of strict pseudocontractions

A mapping $T : C \to C$ is called β-strictly pseudocontractive if there exists $\beta \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \beta \big\|(I - T)x - (I - T)y\big\|^2$$

for all $x, y \in C$. It is well known that if T is β-strictly pseudocontractive, then $I - T$ is $\frac{1-\beta}{2}$-inverse strongly monotone. Moreover, by putting $A = I - T$, we have $F(T) = \mathrm{VI}(A, C)$. So we immediately obtain the following result.

Theorem 4.2 *Let $T : C \to C$ be a β-strict pseudocontraction such that $F(T) \neq \emptyset$ and let $f : C \to C$ be a k-contraction. Let $\{z_n\}$ be a sequence in C defined by $z_0 \in C$ and*

$$\begin{aligned} w_n &= \alpha_n f(z_n) + (1 - \alpha_n) z_n, \\ z_{n+1} &= P_C\big((1 - r_n) w_n + r_n T w_n + e_n\big), \quad n \geq 0, \end{aligned} \tag{4.3}$$

where $\{\alpha_n\} \subset (0,1)$, $\{e_n\} \subset H$, and $\{r_n\} \subset (0, 1 - \beta)$. Assume that

(a) $\alpha_n \to 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(b) $0 < a \leq r_n \leq b < 1 - \beta$;

(c) $\sum_{n=0}^{\infty} \|e_n\| < \infty$ *or* $\|e_n\|/\alpha_n \to 0$.

Then $\{z_n\}$ converges strongly to a point $\overline{x} \in F(T)$, where $\overline{x} = P_{F(T)} f(\overline{x})$.

4.3 Convex minimization problem

We next consider the following convex minimization problem:

$$\min_{x \in H} F(x) + G(x),$$

where $F : H \to \mathbb{R}$ is a convex and differentiable function and $G : H \to \mathbb{R}$ is a convex function. It is well known that if ∇F is $(1/L)$-Lipschitz continuous, then it is L-inverse strongly monotone [19]. Moreover, ∂G is maximal monotone [20]. Putting $A = \nabla F$ and $B = \partial G$, we then obtain the following result.

Theorem 4.3 *Let H be a Hilbert space. Let $F : H \to \mathbb{R}$ be a convex and differentiable function with $(1/L)$-Lipschitz continuous gradient ∇F and $G : H \to \mathbb{R}$ be a convex and lower semi-continuous function such that $\Omega := (\nabla F + \partial G)^{-1}(0) \neq \emptyset$. Let $f : H \to H$ be a k-contraction. Let $\{z_n\}$ be generated by $z_0 \in H$ and*

$$\begin{aligned} w_n &= \alpha_n f(z_n) + (1 - \alpha_n) z_n, \\ z_{n+1} &= J_{r_n}\big(w_n - r_n \nabla F(w_n) + e_n\big), \quad n \geq 0, \end{aligned} \tag{4.4}$$

where $J_{r_n} = (I + r_n \partial G)^{-1}$, $\{\alpha_n\} \subset (0,1)$, $\{e_n\} \subset H$, and $\{r_n\} \subset (0, 2L)$. Assume that

(a) $\alpha_n \to 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(b) $0 < a \leq r_n \leq b < 2L$;

(c) $\sum_{n=0}^{\infty} \|e_n\| < \infty$ *or* $\|e_n\|/\alpha_n \to 0$.

Then $\{z_n\}$ converges strongly to a minimizer $\overline{x}$ of $F + G$, where $\overline{x} = P_{\Omega} f(\overline{x})$.

Table 1 Numerical results of Example 4.4 for iteration process (4.4)

n	$z_n = (t_n, u_n, v_n)^T$	$F(z_n) + G(z_n)$	$\\|z_{n+1} - z_n\\|_2$
1	(−0.9999988000, 4.9999960000, 0.9999984000)	21.499964000010	3.6464868709E+00
2	(−0.8749992000, 1.6249970000, −0.3750010000)	−1.601574899991	1.8007236270E+00
3	(−0.9004624463, 0.0000000000, −1.1504634130)	−7.134189931344	4.1056538071E−01
4	(−0.9346061331, 0.0000000000, −1.5596065914)	−7.400888643932	2.1362893808E−01
5	(−0.9593029918, 0.0000000000, −1.7718031709)	−7.473134980367	1.1059145541E−01
6	(−0.9750218023, 0.0000000000, −1.8812718377)	−7.492639856561	5.7254600304E−02
7	(−0.9845953935, 0.0000000000, −1.9377203595)	−7.497941972237	2.9747437021E−02
8	(−0.9903445225, 0.0000000000, −1.9669069578)	−7.499405811158	1.5563253898E−02
9	(−0.9938004751, 0.0000000000, −1.9820816494)	−7.499820249302	8.2314060025E−03
10	(−0.9959001978, 0.0000000000, −1.9900407454)	−7.499942002435	4.4231489769E−03
⋮	⋮	⋮	⋮
50	(−0.9999828882, 0.0000000000, −1.9999828718)	−7.499999999707	1.4278683289E−06
51	(−0.9999838977, 0.0000000000, −1.9999838817)	−7.499999999740	1.3175686638E−06
52	(−0.9999848292, 0.0000000000, −1.9999848135)	−7.499999999770	1.2177300519E−06
53	(−0.9999856901, 0.0000000000, −1.9999856747)	−7.499999999795	1.1271827509E−06
54	(−0.9999864870, 0.0000000000, −1.9999864719)	−7.499999999817	1.0449069970E−06
55	(−0.9999872257, 0.0000000000, −1.9999872109)	−7.499999999837	9.7001140362E−07

We next provide the example as well as its numerical results.

Example 4.4 Let $H = \mathbb{R}^3$. Minimize the following ℓ_1-least square problem:

$$\min_{x\in\mathbb{R}^3} \|x\|_1 + \frac{1}{2}\|x\|_2^2 + (2,1,3)x - 5,$$

where $x = (t,u,v)^T$.

Let $F(x) = \frac{1}{2}\|x\|_2^2 + (2,1,3)x - 5$ and $G(x) = \|x\|_1$. Then $\nabla F(x) = (t+2, u+1, v+3)^T$. Moreover, ∇F is 1-Lipschitz continuous and hence it is 1-inverse strongly monotone.

From [21] we know that, for $r > 0$,

$$\begin{aligned}&(I + r\partial G)^{-1}(x)\\ &\quad= \bigl(\max\bigl\{|t| - r, 0\bigr\}\operatorname{sign}(t), \max\bigl\{|u| - r, 0\bigr\}\operatorname{sign}(u), \max\bigl\{|v| - r, 0\bigr\}\operatorname{sign}(v)\bigr)^T.\end{aligned}$$

Let $z_n = (t_n, u_n, v_n)^T$. Set $f(x) = \frac{x}{5}$ and choose $\alpha_n = \frac{10^{-6}}{n+1}$, $r_n = 0.5$, and $e_n = \frac{1}{(n+1)^3}(1,1,1)^T$. For the initial point $z_0 = (t_0, u_0, v_0)^T = (-3, 10, 4)^T$, computing $\{z_n\}$ by the algorithm (4.4), we obtain numerical results with an error 10^{-7} in Table 1.

From Table 1, we see that $z_\infty = (-1, 0, -2)$ is the minimizer of $F + G$ and its minimum value is −7.5.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References

1. Lions, PL, Mercier, B: Splitting algorithms for the sum of two nonlinear operators. SIAM J. Numer. Anal. **16**, 964-979 (1979)
2. Passty, GB: Ergodic convergence to a zero of the sum of monotone operators in Hilbert space. J. Math. Anal. Appl. **72**, 383-390 (1979)
3. Brézis, H, Lions, PL: Produits infinis de resolvantes. Isr. J. Math. **29**, 329-345 (1978)
4. Güler, O: On the convergence of the proximal point algorithm for convex minimization. SIAM J. Control Optim. **29**, 403-419 (1991)
5. Martinet, B: Régularisation d'inéquations variationnelles par approximations successives. Rev. Fr. Inform. Rech. Oper. **4**, 154-158 (1970)
6. Rockafellar, RT: Monotone operators and the proximal point algorithm. SIAM J. Control Optim. **14**, 877-898 (1976)
7. Dunn, JC: Convexity, monotonicity, and gradient processes in Hilbert space. J. Math. Anal. Appl. **53**, 145-158 (1976)
8. Combettes, PL: Iterative construction of the resolvent of a sum of maximal monotone operators. J. Convex Anal. **16**, 727-748 (2009)
9. López, G, Martín-Márquez, V, Wang, F, Xu, HK: Forward-backward splitting methods for accretive operators in Banach spaces. Abstr. Appl. Anal. **2012**, Article ID 109236 (2012)
10. Takahashi, W: Viscosity approximation methods for resolvents of accretive operators in Banach spaces. J. Fixed Point Theory Appl. **1**, 135-147 (2007)
11. Wang, F, Cui, H: On the contraction-proximal point algorithms with multi-parameters. J. Glob. Optim. **54**, 485-491 (2012)
12. Xu, HK: A regularization method for the proximal point algorithm. J. Glob. Optim. **36**, 115-125 (2006)
13. Qin, X, Cho, SY, Wang, L: A regularization method for treating zero points of the sum of two monotone operators. Fixed Point Theory Appl. **2014**, 75 (2014)
14. Minty, GJ: On the maximal domain of a monotone function. Mich. Math. J. **8**, 135-137 (1961)
15. Nadezhkina, N, Takahashi, W: Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings. J. Optim. Theory Appl. **128**, 191-201 (2006)
16. Browder, FE: Nonexpansive nonlinear operators in a Banach space. Proc. Natl. Acad. Sci. USA **54**, 1041-1044 (1965)
17. Xu, HK: Iterative algorithms for nonlinear operators. J. Lond. Math. Soc. **66**, 240-256 (2002)
18. He, S, Yang, C: Solving the variational inequality problem defined on intersection of finite level sets. Abstr. Appl. Anal. **2013**, Article ID 942315 (2013)
19. Baillon, JB, Haddad, G: Quelques proprietes des operateurs angle-bornes et cycliquement monotones. Isr. J. Math. **26**, 137-150 (1977)
20. Rockafellar, RT: On the maximal monotonicity of subdifferential mappings. Pac. J. Math. **33**, 209-216 (1970)
21. Hale, ET, Yin, W, Zhang, Y: A fixed-point continuation method for ℓ_1-regularized minimization with applications to compressed sensing. Tech. rep., CAAM TR07-07 (2007)