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Several closed expressions for the Euler numbers

Chun-Fu Wei[1,2*] and Feng Qi[3,4]

*Correspondence:
mathwcf@163.com
[1]School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo, Henan 454010, China
[2]School of Energy Science and Engineering, Henan Polytechnic University, Jiaozuo, Henan 454010, China
Full list of author information is available at the end of the article

Abstract
In the paper, the authors establish several closed expressions for the Euler numbers in the form of a determinant or double sums and in terms of, for example, the Stirling numbers of the second kind.

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Keywords: Euler number; closed expression; determinant; Stirling number of the second kind; double sum

1 Introduction and main results

It is well known ([1], p.75, item 4.3.69) that the secant function $\sec z$ may be expanded at $z = 0$ into the power series

$$\sec z = \sum_{k=0}^{\infty} (-1)^k E_{2k} \frac{z^{2k}}{(2k)!}, \quad |z| < \frac{\pi}{2}, \tag{1.1}$$

where E_k are called in number theory the Euler numbers which may also be defined ([2], p.15) by

$$\frac{1}{\cosh z} = \frac{2e^z}{e^{2z}+1} = \sum_{k=0}^{\infty} E_k \frac{z^k}{k!} = \sum_{k=0}^{\infty} E_{2k} \frac{z^{2k}}{(2k)!}, \quad |z| < \frac{\pi}{2}. \tag{1.2}$$

In number theory, the numbers

$$S_k = (-1)^k E_{2k} \tag{1.3}$$

are called in [3], p.128, for example, the secant numbers or the zig numbers.

These numbers also occur in combinatorics, specifically when counting the number of alternating permutations of a set with an even number of elements.

The first few secant numbers S_k for $k = 1, 2, 3, 4$ are 1, 5, 61, 1,385. The first few Euler numbers E_{2k} for $0 \le k \le 9$ are

$$\begin{aligned} &E_0 = 1, \quad E_2 = -1, \quad E_4 = 5, \quad E_6 = -61, \quad E_8 = 1{,}385, \\ &E_{10} = -50{,}521, \quad E_{12} = 2{,}702{,}765, \quad E_{14} = -199{,}360{,}981, \\ &E_{16} = 19{,}391{,}512{,}145, \quad E_{18} = -2{,}404{,}879{,}675{,}441. \end{aligned}$$

It is a classical topic to find closed expressions for the Euler numbers E_{2k} and the tangent number S_k. These numbers are closely connected with many other numbers and functions, such as the Bernoulli numbers, the Genocchi numbers, the tangent numbers, the Euler polynomials, the Stirling numbers of two kinds, and the Riemann zeta function, in number theory and combinatorics. There has been a plenty of literature such as [1, 2, 4–10] and closely related references therein.

In mathematics, a closed expression is a mathematical expression that can be evaluated in a finite number of operations. It may contain constants, variables, four arithmetic operations, and elementary functions, but usually no limit.

In this paper, we establish several closed expressions for the Euler numbers E_{2k} in the form of a determinant of order $2k$ or double sums and in terms of, for example, the Stirling numbers of the second kind $S(n,k)$ which may be generated ([2], p.20) by

$$\frac{(e^x-1)^k}{k!}=\sum_{n=k}^{\infty}S(n,k)\frac{x^n}{n!},\quad k\in\mathbb{N},$$

may be computed ([2], p.21) by the closed expression

$$S(k,m)=\frac{1}{m!}\sum_{\ell=1}^{m}(-1)^{m-\ell}\binom{m}{\ell}\ell^k,\quad 1\le m\le k,$$

and may be interpreted combinatorially as the number of ways of partitioning a set of n elements into k nonempty subsets.

Our main results may be formulated as the following theorems.

Theorem 1.1 *For $k\in\mathbb{N}$,*

$$E_{2k}=(-1)^k\left|\binom{i}{j-1}\cos\left((i-j+1)\frac{\pi}{2}\right)\right|_{(2k)\times(2k)},$$

where $|c_{ij}|_{k\times k}$ is the determinant of a matrix $[c_{ij}]_{k\times k}$ of elements c_{ij} and order k.

Theorem 1.2 *For $k\in\mathbb{N}$,*

$$E_{2k}=(2k+1)\sum_{\ell=1}^{2k}(-1)^{\ell}\frac{1}{2^{\ell}(\ell+1)}\binom{2k}{\ell}\sum_{q=0}^{\ell}\binom{\ell}{q}(2q-\ell)^{2k}. \tag{1.4}$$

Theorem 1.3 *For $n\in\mathbb{N}$,*

$$E_n=1+\sum_{k=1}^{n}\frac{(k+1)!}{2^k}S(n,k)\sum_{\ell=1}^{k}(-1)^{\ell}\frac{2^{\ell}}{\ell+1}\binom{\ell+1}{k-\ell} \tag{1.5}$$

and

$$E_n=1+\sum_{\ell=1}^{n}(-1)^{\ell}\frac{1}{\ell+1}\sum_{k=0}^{n-\ell}\frac{(k+\ell+1)!}{2^k}\binom{\ell+1}{k}S(n,k+\ell). \tag{1.6}$$

Theorem 1.4 *For $k \in \mathbb{N}$,*

$$E_{2k} = \sum_{i=1}^{2k} (-1)^i \frac{1}{2^i} \sum_{\ell=0}^{2i} (-1)^\ell \binom{2i}{\ell} (i-\ell)^{2k}. \tag{1.7}$$

2 Lemmas

In order to prove our main results, we need the following lemmas.

Lemma 2.1 *Let $u = u(x)$ and $v = v(x) \neq 0$ be differentiable functions, let $U_{n+1,1}$ be an $(n+1) \times 1$ matrix whose elements $u_{k,1} = u^{(k-1)}(x)$ for $1 \le k \le n+1$, let $V_{n+1,n}$ be an $(n+1) \times n$ matrix whose elements $v_{i,j} = \binom{i-1}{j-1} v^{(i-j)}(x)$ for $1 \le i \le n+1$ and $1 \le j \le n$, and let $|W_{n+1,n+1}|$ denote the determinant of the $(n+1) \times (n+1)$ matrix $W_{n+1,n+1} = [U_{n+1,1} \; V_{n+1,n}]$. Then the nth derivative of the ratio $\frac{u(x)}{v(x)}$ may be computed by*

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\frac{u}{v}\right) = (-1)^n \frac{|W_{n+1,n+1}|}{v^{n+1}}.$$

Proof This is a reformulation of [11], p.40, Exercise 5. □

The Bell polynomials of the second kind $\mathrm{B}_{n,k}$, or say, the partial Bell polynomials $\mathrm{B}_{n,k}$, may be defined ([12], p.134, Theorem A) by

$$\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{1 \le q \le n, \ell_q \in \{0\} \cup \mathbb{N} \\ \sum_{q=1}^n i\ell_q = n \\ \sum_{q=1}^n \ell_q = k}} \frac{n!}{\prod_{q=1}^{n-k+1} \ell_q!} \prod_{q=1}^{n-k+1} \left(\frac{x_q}{q!}\right)^{\ell_q}$$

for $n \ge k \ge 0$.

Lemma 2.2 (Faà di Bruno formula [12], p.139, Theorem C) *For $n \in \mathbb{N}$, the nth derivative of a composite function $f(g(x))$ may be computed in terms of the Bell polynomials of the second kind $\mathrm{B}_{n,k}$ by*

$$\frac{\mathrm{d}^n f(g(x))}{\mathrm{d}x^n} = \sum_{k=1}^n f^{(k)}\big(g(x)\big) \mathrm{B}_{n,k}\big(g'(x), g''(x), \dots, g^{(n-k+1)}(x)\big). \tag{2.1}$$

Lemma 2.3 ([12], p.135) *For $n \ge k \ge 0$, we have*

$$\mathrm{B}_{n,k}(1, 1, \dots, 1) = S(n,k) \tag{2.2}$$

and

$$\mathrm{B}_{n,k}\big(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}\big) = a^k b^n \mathrm{B}_{n,k}(x_1, x_n, \dots, x_{n-k+1}), \tag{2.3}$$

where a and b are any complex numbers.

Lemma 2.4 ([5], Theorem 2.1) *For $n \ge k \ge 1$, the Bell polynomials of the second kind $\mathrm{B}_{n,k}$ satisfy*

$$\mathrm{B}_{n,k}\left(0, 1, 0, \dots, \frac{1+(-1)^{n-k+1}}{2}\right) = \frac{1}{2^k k!} \sum_{\ell=0}^{2k} (-1)^\ell \binom{2k}{\ell} (k-\ell)^n. \tag{2.4}$$

Lemma 2.5 ([5], Theorem 4.1 and [13], Theorem 3.1) *For $n \ge k \ge 0$, the Bell polynomials of the second kind $\mathrm{B}_{n,k}$ satisfy*

$$\mathrm{B}_{n,k}(x,1,0,\dots,0) = \frac{(n-k)!}{2^{n-k}}\binom{n}{k}\binom{k}{n-k}x^{2k-n}. \tag{2.5}$$

Lemma 2.6 ([7], Theorem 1.2) *For $n \ge k \ge 1$, the Bell polynomials of the second kind $\mathrm{B}_{n,k}$ satisfy*

$$\begin{aligned}
&\mathrm{B}_{n,k}\left(-\sin x, -\cos x, \sin x, \cos x, \dots, -\sin\left[x+(n-k)\frac{\pi}{2}\right]\right)\\
&\quad = (-1)^{k+\frac{1}{2}[n+\frac{1-(-1)^n}{2}]}\frac{1}{k!}\cos^k x\sum_{\ell=1}^{k}(-1)^\ell\frac{1}{2^\ell}\binom{k}{\ell}\frac{1}{\cos^\ell x}\\
&\qquad \times\sum_{q=0}^{\ell}\binom{\ell}{q}(2q-\ell)^n\sin\left[(2q-\ell)x+\frac{1+(-1)^n}{2}\frac{\pi}{2}\right].
\end{aligned}$$

3 Proofs of main results

We now start out to prove our main results.

Proof of Theorem 1.1 Applying Lemma 2.1 to $u(z)=1$ and $v(z)=\cos z$ gives

$$\begin{aligned}
(\sec z)^{(n)} &= (-1)^n\frac{|[\delta_{i1}]_{1\le i\le n+1}\ A_{(n+1)\times n}|_{(n+1)\times(n+1)}}{\cos^{n+1}z}\\
&= (-1)^n\left|\binom{i-1}{j-1}\cos\left(z+(i-j)\frac{\pi}{2}\right)\right|_{2\le i\le n+1, 1\le j\le n}\\
&= (-1)^n\left|\binom{i}{j-1}\cos\left(z+(i-j+1)\frac{\pi}{2}\right)\right|_{n\times n},
\end{aligned}$$

where

$$A_{(n+1)\times n} = \left[\binom{i-1}{j-1}\cos\left(z+(i-j)\frac{\pi}{2}\right)\right]_{1\le i\le n+1, 1\le j\le n}$$

and

$$\delta_{ij} = \begin{cases} 1, & i=j,\\ 0 & i\ne j\end{cases}$$

is the Kronecker delta. Consequently, by taking the limit $z\to 0$, we find

$$\lim_{z\to 0}(\sec z)^{(n)} = (-1)^n\left|\binom{i}{j-1}\cos\left((i-j+1)\frac{\pi}{2}\right)\right|_{n\times n}$$

and, by (1.1) and (1.3),

$$(-1)^k E_{2k} = S_k = \lim_{z\to 0}(\sec z)^{(2k)} = \left|\binom{i}{j-1}\cos\left((i-j+1)\frac{\pi}{2}\right)\right|_{(2k)\times(2k)}.$$

The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2 By (2.1) applied to $f(u)=\frac{1}{u}$ and $u=g(z)=\cos z$ and by Lemma 2.6, we have

$$\begin{aligned}
(\sec z)^{(n)} &= \sum_{k=1}^{n}\left(\frac{1}{u}\right)^{(k)}\mathrm{B}_{n,k}\left(-\sin z,-\cos z,\sin z,\dots,-\sin\left[z+(n-k)\frac{\pi}{2}\right]\right)\\
&= \sum_{k=1}^{n}\frac{(-1)^k k!}{u^{k+1}}\mathrm{B}_{n,k}\left(-\sin z,-\cos z,\sin z,\cos z,\dots,-\sin\left[z+(n-k)\frac{\pi}{2}\right]\right)\\
&= \sum_{k=1}^{n}\frac{(-1)^k k!}{(\cos z)^{k+1}}\mathrm{B}_{n,k}\left(-\sin z,-\cos z,\sin z,\cos z,\dots,-\sin\left[z+(n-k)\frac{\pi}{2}\right]\right)\\
&= \sum_{k=1}^{n}\frac{(-1)^k k!}{(\cos z)^{k+1}}(-1)^{k+\frac{1}{2}[n+\frac{1-(-1)^n}{2}]}\frac{1}{k!}\cos^k z\sum_{\ell=1}^{k}(-1)^\ell\frac{1}{2^\ell}\binom{k}{\ell}\frac{1}{\cos^\ell z}\\
&\quad\times\sum_{q=0}^{\ell}\binom{\ell}{q}(2q-\ell)^n\sin\left[(2q-\ell)z+\frac{1+(-1)^n}{2}\frac{\pi}{2}\right]\\
&\to (-1)^{\frac{1}{2}[n+\frac{1-(-1)^n}{2}]}\sum_{k=1}^{n}\sum_{\ell=1}^{k}(-1)^\ell\frac{1}{2^\ell}\binom{k}{\ell}\\
&\quad\times\sum_{q=0}^{\ell}\binom{\ell}{q}(2q-\ell)^n\sin\left[\frac{1+(-1)^n}{2}\frac{\pi}{2}\right]
\end{aligned}$$

as $z\to 0$. Hence,

$$\begin{aligned}
(-1)^m E_{2m} &= (-1)^{\frac{1}{2}[2m+\frac{1-(-1)^{2m}}{2}]}\sum_{k=1}^{2m}\sum_{\ell=1}^{k}(-1)^\ell\frac{1}{2^\ell}\binom{k}{\ell}\\
&\quad\times\sum_{q=0}^{\ell}\binom{\ell}{q}(2q-\ell)^{2m}\sin\left[\frac{1+(-1)^{2m}}{2}\frac{\pi}{2}\right]\\
&= (-1)^m\sum_{k=1}^{2m}\sum_{\ell=1}^{k}(-1)^\ell\frac{1}{2^\ell}\binom{k}{\ell}\sum_{q=0}^{\ell}\binom{\ell}{q}(2q-\ell)^{2m}.
\end{aligned}$$

Consequently, by interchanging the first two sums, we get

$$\begin{aligned}
E_{2m} &= \sum_{\ell=1}^{2m}\sum_{k=\ell}^{2m}(-1)^\ell\frac{1}{2^\ell}\binom{k}{\ell}\sum_{q=0}^{\ell}\binom{\ell}{q}(2q-\ell)^{2m}\\
&= \sum_{\ell=1}^{2m}(-1)^\ell\frac{1}{2^\ell}\left[\sum_{q=0}^{\ell}\binom{\ell}{q}(2q-\ell)^{2m}\right]\left[\sum_{k=\ell}^{2m}\binom{k}{\ell}\right]\\
&= \sum_{\ell=1}^{2m}(-1)^\ell\frac{1}{2^\ell}\left[\sum_{q=0}^{\ell}\binom{\ell}{q}(2q-\ell)^{2m}\right]\frac{2m-\ell+1}{\ell+1}\binom{2m+1}{\ell}\\
&= \sum_{\ell=1}^{2m}(-1)^\ell\frac{1}{2^\ell}\left[\sum_{q=0}^{\ell}\binom{\ell}{q}(2q-\ell)^{2m}\right]\frac{2m+1}{\ell+1}\binom{2m}{\ell}
\end{aligned}$$

which may be rearranged as (1.4). The proof of Theorem 1.2 is complete. □

Proof of Theorem 1.3 Let $f(u)=\frac{2u}{u^2+1}$ and $u=g(z)=e^z$. Then, by (2.1), (2.3), and (2.2) in sequence,

$$\begin{aligned}
\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left(\frac{2e^z}{e^{2z}+1}\right) &= \sum_{k=1}^{n}\frac{\mathrm{d}^k}{\mathrm{d}u^k}\left(\frac{2u}{u^2+1}\right)\mathrm{B}_{n,k}\bigl(e^z,e^z,\dots,e^z\bigr) \\
&= \sum_{k=1}^{n}\frac{\mathrm{d}^{k+1}\ln(u^2+1)}{\mathrm{d}u^{k+1}}e^{kz}\mathrm{B}_{n,k}(1,1,\dots,1) \\
&= \sum_{k=1}^{n}S(n,k)e^{kz}\frac{\mathrm{d}^{k+1}\ln(u^2+1)}{\mathrm{d}u^{k+1}},
\end{aligned}$$

where, by applying $f(v)=\ln v$ and $v=g(u)=u^2+1$ in (2.1) and making use of (2.3) and (2.5), we have

$$\begin{aligned}
\frac{\mathrm{d}^{k+1}\ln(u^2+1)}{\mathrm{d}u^{k+1}} &= \sum_{\ell=1}^{k+1}(\ln v)^{(\ell)}\mathrm{B}_{k+1,\ell}(2u,2,0,\dots,0) \\
&= \sum_{\ell=1}^{k+1}(-1)^{\ell-1}\frac{(\ell-1)!}{v^\ell}2^\ell\mathrm{B}_{k+1,\ell}(u,1,0,\dots,0) \\
&= \sum_{\ell=1}^{k+1}(-1)^{\ell-1}\frac{(\ell-1)!}{(u^2+1)^\ell}2^\ell\frac{(k+1-\ell)!}{2^{k+1-\ell}}\binom{k+1}{\ell}\binom{\ell}{k+1-\ell}u^{2\ell-k-1} \\
&= \sum_{\ell=1}^{k+1}(-1)^{\ell-1}\frac{(\ell-1)!(k+1-\ell)!}{2^{k+1-2\ell}}\binom{k+1}{\ell}\binom{\ell}{k+1-\ell}\frac{e^{(2\ell-k-1)z}}{(e^{2z}+1)^\ell}.
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left(\frac{2e^z}{e^{2z}+1}\right) &= \sum_{k=1}^{n}S(n,k)\sum_{\ell=1}^{k+1}(-1)^{\ell-1}\frac{(\ell-1)!(k+1-\ell)!}{2^{k+1-2\ell}} \\
&\quad\times\binom{k+1}{\ell}\binom{\ell}{k+1-\ell}\frac{e^{(2\ell-1)z}}{(e^{2z}+1)^\ell}.
\end{aligned}$$

Further taking the limit $z\to 0$ yields

$$\begin{aligned}
E_n &= \lim_{z\to 0}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left(\frac{2e^z}{e^{2z}+1}\right) \\
&= \sum_{k=1}^{n}S(n,k)\sum_{\ell=1}^{k+1}(-1)^{\ell-1}\frac{(\ell-1)!(k+1-\ell)!}{2^{k+1-\ell}}\binom{k+1}{\ell}\binom{\ell}{k+1-\ell} \\
&= S(n,1)+\sum_{k=1}^{n}S(n,k)\sum_{\ell=2}^{k+1}(-1)^{\ell-1}\frac{(\ell-1)!(k+1-\ell)!}{2^{k+1-\ell}}\binom{k+1}{\ell}\binom{\ell}{k+1-\ell} \\
&= 1+\sum_{k=1}^{n}S(n,k)\sum_{\ell=1}^{k}(-1)^{\ell}\frac{\ell!(k-\ell)!}{2^{k-\ell}}\binom{k+1}{\ell+1}\binom{\ell+1}{k-\ell} \\
&= 1+\sum_{k=1}^{n}\frac{1}{2^k}\left[\sum_{\ell=1}^{k}(-1)^\ell\ell!(k-\ell)!2^\ell\binom{k+1}{\ell+1}\binom{\ell+1}{k-\ell}\right]S(n,k)
\end{aligned}$$

and, by interchanging two sums in the above line, we get

$$\begin{aligned}
E_n &= 1 + \sum_{\ell=1}^{n} \sum_{k=\ell}^{n} (-1)^{\ell} \frac{\ell!(k-\ell)!}{2^{k-\ell}} \binom{k+1}{\ell+1} \binom{\ell+1}{k-\ell} S(n,k) \\
&= 1 + \sum_{\ell=1}^{n} (-1)^{\ell} \left[\sum_{k=0}^{n-\ell} \frac{k!}{2^k} \binom{k+\ell+1}{\ell+1} \binom{\ell+1}{k} S(n,k+\ell) \right] \ell!.
\end{aligned}$$

As a result of further simplifying, formulas (1.5) and (1.6) follow. The proof of Theorem 1.3 is complete. □

Proof of Theorem 1.4 Formula (1.7) was ever established in [5], Theorem 3.1. We now give a different proof for it.

Applying (2.1) to the functions $f(u) = \frac{1}{u}$ and $u = g(z) = \cosh z$ and making use of formula (2.4) yield

$$\begin{aligned}
\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left(\frac{1}{\cosh z}\right) &= \sum_{k=1}^{n} \left(\frac{1}{u}\right)^{(k)} \mathrm{B}_{n,k}(\sinh z, \cosh z, \sinh z, \ldots) \\
&= \sum_{k=1}^{n} \frac{(-1)^k k!}{u^{k+1}} \mathrm{B}_{n,k}(\sinh z, \cosh z, \sinh z, \ldots) \\
&= \sum_{k=1}^{n} \frac{(-1)^k k!}{(\cosh z)^{k+1}} \mathrm{B}_{n,k}(\sinh z, \cosh z, \sinh z, \ldots) \\
&\to \sum_{k=1}^{n} (-1)^k k! \mathrm{B}_{n,k}\left(0, 1, 0, \ldots, \frac{1+(-1)^{n-k+1}}{2}\right), \quad z \to 0 \\
&= \sum_{k=1}^{n} (-1)^k k! \frac{1}{2^k k!} \sum_{\ell=0}^{2k} (-1)^{\ell} \binom{2k}{\ell} (k-\ell)^n \\
&= \sum_{k=1}^{n} (-1)^k \frac{1}{2^k} \sum_{\ell=0}^{2k} (-1)^{\ell} \binom{2k}{\ell} (k-\ell)^n.
\end{aligned}$$

By virtue of (1.2), Theorem 1.4 follows immediately. □

Remark 3.1 This paper is a slightly corrected and revised version of the preprint [14].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

Author details
[1]School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo, Henan 454010, China. [2]School of Energy Science and Engineering, Henan Polytechnic University, Jiaozuo, Henan 454010, China. [3]College of Mathematics, Inner Mongolia University for Nationalities, Tongliao, Inner Mongolia 028043, China. [4]Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin, China.

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