Chen et al. *Journal of Inequalities and Applications* (2015) 2015:210
DOI 10.1186/s13660-015-0731-3

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH

Open Access

Hybrid iterative algorithm for finite families of countable Bregman quasi-Lipschitz mappings with applications in Banach spaces

Minjiang Chen[1], Jianzhi Bi[1] and Yongfu Su[2*]

*Correspondence:
tjsuyongfu@163.com
[2]Department of Mathematics, Tianjin Polytechnic University, Tianjin, 300387, China
Full list of author information is available at the end of the article

Abstract

The purpose of this paper is to introduce and consider a new hybrid shrinking projection method for finding a common element of the set EP of solutions of a generalized equilibrium problem, the common fixed point set F of finite uniformly closed families of countable Bregman quasi-Lipschitz mappings in reflexive Banach spaces. It is proved that under appropriate conditions, the sequence generated by the hybrid shrinking projection method converges strongly to some point in $EP \cap F$. Relative examples are given. Strong convergence theorems are proved. The application for Bregman asymptotically quasi-nonexpansive mappings is also given. The main innovative points in this paper are as follows: (1) the notion of the uniformly closed family of countable Bregman quasi-Lipschitz mappings is presented and the useful conclusions are given; (2) the relative examples of the uniformly closed family of countable Bregman quasi-Lipschitz mappings are given in classical Banach spaces l^2 and L^2; (3) the application for Bregman asymptotically quasi-nonexpansive mappings is also given; (4) because the main theorems do not need the boundedness of the domain of mappings, so a corresponding technique for the proof is given. This new results improve and extend the previously known ones in the literature.

MSC: 47H05; 47H09; 47H10

Keywords: Bregman distance; Bregman quasi-Lipschitz mapping; generalized projection; hybrid algorithm; Bregman asymptotically quasi-nonexpansive mappings; finite families; equilibrium problem

1 Introduction

Let C be a nonempty subset of a real Banach space and T be a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T. Recall that T is said to be asymptotically nonexpansive [1] if there exists a sequence $\{k_n\} \subset [1, +\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \le k_n \|x - y\|, \quad \forall x, y \in C, n \ge 1.$$

In the framework of Hilbert spaces, Takahashi *et al.* [2] have introduced a new hybrid iterative scheme called a shrinking projection method for nonexpansive mappings. It is an advantage of projection methods that the strong convergence of iterative sequences is guaranteed without any compact assumption. Moreover, Schu [3] has introduced a modified Mann iteration to approximate fixed points of asymptotically nonexpansive mappings in uniformly convex Banach spaces. Motivated by [2, 3], Inchan [4] has introduced a new hybrid iterative scheme by using the shrinking projection method with the modified Mann iteration for asymptotically nonexpansive mappings. The mapping T is said to be asymptotically nonexpansive in the intermediate sense (*cf.* [5]) if

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(\left\| T^n x - T^n y \right\| - \|x - y\| \right) \leq 0. \tag{1.1}$$

If $F(T)$ is nonempty and (1.1) holds for all $x \in C$ and $y \in F(T)$, then T is said to be asymptotically quasi-nonexpansive in the intermediate sense. It is worth mentioning that the class of asymptotically nonexpansive mappings in the intermediate sense contains properly the class of asymptotically nonexpansive mappings since the mappings in the intermediate sense are not Lipschitz continuous in general.

Recently, many authors have studied further new hybrid iterative schemes in the framework of real Banach spaces; for instance, see [6–8]. Qin and Wang [9] have introduced a new class of mappings which are asymptotically quasi-nonexpansive with respect to the Lyapunov functional (*cf.* [10]) in the intermediate sense. By using the shrinking projection method, Hao [11] has proved a strong convergence theorem for an asymptotically quasi-nonexpansive mapping with respect to the Lyapunov functional in the intermediate sense.

In 1967, Bregman [12] discovered an elegant and effective technique for using of the so-called Bregman distance function (see Section 2) in the process of designing and analyzing feasibility and optimization algorithms. This opened a growing area of research in which Bregman's technique is applied in various ways in order to design and analyze not only iterative algorithms for solving feasibility and optimization problems, but also algorithms for solving variational inequalities, for approximating equilibria, and for computing fixed points of nonlinear mappings.

Many authors have studied iterative methods for approximating fixed points of mappings of nonexpansive type with respect to the Bregman distance; see [13–17]. In [18], the authors has introduced a new class of nonlinear mappings which is an extension of asymptotically quasi-nonexpansive mappings with respect to the Bregman distance in the intermediate sense and has proved the strong convergence theorems for asymptotically quasi-nonexpansive mappings with respect to Bregman distances in the intermediate sense by using the shrinking projection method.

The purpose of this paper is to introduce and consider a new hybrid shrinking projection method for finding a common element of the set EP of solutions of a generalized equilibrium problem, the common fixed point set F of finite uniformly closed families of countable Bregman quasi-Lipschitz mappings in reflexive Banach spaces. It is proved that under appropriate conditions, the sequence generated by the hybrid shrinking projection method converges strongly to some point in $EP \cap F$. Relative examples are given. Strong convergence theorems are proved. The application for Bregman asymptotically quasi-nonexpansive mappings is also given. The main innovative points in this paper are as follows: (1) the notion of the uniformly closed family of countable Bregman quasi-Lipschitz mappings is presented and the useful conclusions are given; (2) the relative examples of the uniformly closed family of countable Bregman quasi-Lipschitz mappings are given in classical Banach spaces l^2 and L^2; (3) the application for Bregman asymptotically quasi-nonexpansive mappings is also given; (4) because the main theorems do not need the boundedness of the domain of mappings, so a corresponding technique for the proof is given. This new results improve and extend the previously known ones in the literature.

2 Preliminaries

Throughout this paper, we assume that E is a real reflexive Banach space with the dual space of E^* and $\langle\cdot,\cdot\rangle$ the pairing between E and E^*.

Let $f : E \to (-\infty,+\infty]$ be a function. The effective domain of f is defined by

$$\operatorname{dom} f := \{x \in E : f(x) < +\infty\}.$$

When $\operatorname{dom} f \neq \emptyset$, we say that f is proper. We denote by $\operatorname{int}\operatorname{dom} f$ the interior of the effective domain of f. We denote by $\operatorname{ran} f$ the range of f.

The function f is said to be strongly coercive if

$$\lim_{\|x\|\to\infty} \frac{f(x)}{\|x\|} = +\infty.$$

Given a proper and convex function $f : E \to (-\infty,+\infty]$, the subdifferential of f is a mapping $\partial f : E \to E^*$ defined by

$$\partial f(x) = \{x^* \in E^* : f(y) \geq f(x) + \langle x^*, y-x\rangle, \forall y \in E\}$$

for all $x \in E$.

The Fenchel conjugate function of f is the convex function $f^* : E \to (-\infty,+\infty)$ defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x), x \in E\}.$$

We know that $x^* \in \partial f(x)$ if and only if

$$f(x) + f^*(x^*) = \langle x^*, x\rangle$$

for all $x \in E$ (see [18]).

Proposition 2.1 ([19]) *Let $f : E \to (-\infty,+\infty]$ be a proper, convex, and lower semicontinuous function. Then the following conditions are equivalent*:

(i) $\operatorname{ran}\partial f = E^*$ *and* $\partial f^* = (\partial f)^{-1}$ *is bounded on bounded subsets of* E^*;

(ii) f *is strongly coercive.*

Let $f : E \to (-\infty,+\infty]$ be a convex function and $x \in \operatorname{int}\operatorname{dom} f$. For any $y \in E$, we define the right-hand derivative of f at x in the direction y by

$$f^\circ(x,y) = \lim_{t\downarrow 0} \frac{f(x+ty) - f(x)}{t}. \tag{2.1}$$

The function f is said to be Gâteaux differentiable at x if the limit (2.1) exists for any y. In this case, the gradient of f at x is the function $\nabla f(x) : E \to E^*$ defined by $\langle \nabla f(x), y\rangle = f^\circ(x,y)$

for all $y \in E$. The function f is said to be Gâteaux differentiable if it is Gâteaux differentiable at each $x \in \operatorname{int}\operatorname{dom} f$. If the limit (2.1) is attained uniformly in $\|y\| = 1$, then the function f is said to be Fréchet differentiable at x. The function f is said to be uniformly Fréchet differentiable on a subset C of E if the limit (2.1) is attained uniformly for $x \in C$ and $\|y\| = 1$. We know that if f is uniformly Fréchet differentiable on bounded subsets of E, then f is uniformly continuous on bounded subsets of E (*cf.* [19]). We will need the following results.

Proposition 2.2 ([20]) *If a function $f : E \to R$ is convex, uniformly Fréchet differentiable, and bounded on bounded subsets of E, then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

Proposition 2.3 ([20]) *Let $f : E \to R$ be a convex function which is bounded on bounded subsets of E. Then the following assertions are equivalent*:

(i) *f is strongly coercive and uniformly convex on bounded subsets of E*;
(ii) *f^* is Fréchet differentiable and ∇f^* is uniformly norm-to-norm continuous on bounded subsets of* $\operatorname{dom} f^* = E^*$.

A function $f : E \to (-\infty, +\infty]$ is said to be admissible if it is proper, convex, and lower semicontinuous on E and Gâteaux differentiable on $\operatorname{int}\operatorname{dom} f$. Under these conditions we know that f is continuous in $\operatorname{int}\operatorname{dom} f$, ∂f is single-valued and $\partial f = \nabla f$; see [17, 21]. An admissible function $f : E \to (-\infty, +\infty]$ is called Legendre (*cf.* [17]) if it satisfies the following two conditions:

(L1) the interior of the domain of f, $\operatorname{int}\operatorname{dom} f$, is nonempty, f is Gâteaux differentiable, and $\operatorname{dom} \nabla f = \operatorname{int}\operatorname{dom} f$;
(L2) the interior of the domain of f^*, $\operatorname{int}\operatorname{dom} f^*$ is nonempty, f^* is Gâteaux differentiable, and $\operatorname{dom} \nabla f^* = \operatorname{int}\operatorname{dom} f^*$.

Let f be a Legendre function on E. Since E is reflexive, we always have $\nabla f = (\nabla f^*)^{-1}$. This fact, when combined with conditions (L1) and (L2), implies the following equalities:

$$\operatorname{ran} \nabla f = \operatorname{dom} f^* = \operatorname{int}\operatorname{dom} f^* \quad \text{and} \quad \operatorname{ran} \nabla f^* = \operatorname{dom} f = \operatorname{int}\operatorname{dom} f.$$

Conditions (L1) and (L2) imply that the functions f and f^* are strictly convex on the interior of their respective domains. In [22], author gave an example of the Legendre function.

Let $f : E \to (-\infty, +\infty]$ be a convex function on E which is Gâteaux differentiable on $\operatorname{int}\operatorname{dom} f$. The bifunction $D_f : \operatorname{dom} f \times \operatorname{int}\operatorname{dom} f \to [0, +\infty)$ given by

$$D_f(x, y) = f(x) - f(y) - \langle x - y, \nabla f(y)\rangle$$

is called the Bregman distance with respect to f (*cf.* [23]). In general, the Bregman distance is not a metric since it is not symmetric and does not satisfy the triangle inequality. However, it has the following important property, which is called the three point identity (*cf.* [24]): for any $x \in \operatorname{dom} f$ and $y, z \in \operatorname{int}\operatorname{dom} f$,

$$D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle x - y, \nabla f(z) - \nabla f(y)\rangle. \tag{2.2}$$

With a Legendre function $f : E \to (-\infty, +\infty]$, we associate the bifunction $W_f : \operatorname{dom} f^* \times \operatorname{dom} f \to [0, +\infty)$ defined by

$$W^f(w, x) = f(x) - \langle w, x\rangle + f^*(w).$$

Proposition 2.4 ([14]) *Let $f : E \to (-\infty, +\infty]$ be a Legendre function such that ∇f^* is bounded on bounded subsets of* $\operatorname{int}\operatorname{dom} f^*$. *Let $x \in \operatorname{int}\operatorname{dom} f$. If the sequence $\{D_f(x, x_n)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

Proposition 2.5 ([14]) *Let $f : E \to (-\infty, +\infty]$ be a Legendre function. Then the following statements hold*:

(i) *the function $W^f(\cdot, x)$ is convex for all $x \in \operatorname{dom} f$*;
(ii) *$W^f(\nabla f(x), y) = D_f(y, x)$ for all $x \in \operatorname{int}\operatorname{dom} f$ and $y \in \operatorname{dom} f$.*

Let $f : E \to (-\infty, +\infty]$ be a convex function on E which is Gâteaux differentiable on $\operatorname{int}\operatorname{dom} f$. The function f is said to be totally convex at a point $x \in \operatorname{int}\operatorname{dom} f$ if its modulus of total convexity at x, $\nu_f(x, \cdot) : [0, +\infty) \to [0, +\infty]$, defined by

$$\nu_f(x, t) = \inf\bigl\{D_f(y, x) : y \in \operatorname{dom} f, \|y - x\| = t\bigr\},$$

is positive whenever $t > 0$. The function f is said to be totally convex when it is totally convex at every point of $\operatorname{int}\operatorname{dom} f$. The function f is said to be totally convex on bounded sets if, for any nonempty bounded set $B \subset E$, the modulus of total convexity of f on B, $\nu_f(B, t)$ is positive for any $t > 0$, where $\nu_f(B, \cdot) : [0, +\infty) \to [0, +\infty]$ is defined by

$$\nu_f(B, t) = \inf\bigl\{\nu_f(x, t) : x \in B \cap \operatorname{int}\operatorname{dom} f\bigr\}.$$

We remark in passing that f is totally convex on bounded sets if and only if f is uniformly convex on bounded sets; see [25, 26].

Proposition 2.6 ([25]) *Let $f : E \to (-\infty, +\infty]$ be a convex function whose domain contains at least two points. If f is lower semi-continuous, then f is totally convex on bounded sets if and only if f is uniformly convex on bounded sets.*

Proposition 2.7 ([27]) *Let $f : E \to R$ be a totally convex function. If $x \in E$ and the sequence $\{D_f(x_n, x)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

Let $f : E \to [0, +\infty)$ be a convex function on E which is Gâteaux differentiable on $\operatorname{int}\operatorname{dom} f$. The function f is said to be sequentially consistent (*cf.* [26]) if for any two sequences $\{x_n\}$ and $\{y_n\}$ in $\operatorname{int}\operatorname{dom} f$ and $\operatorname{dom} f$, respectively, such that the first one is bounded,

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \quad \Rightarrow \quad \lim_{n\to\infty} \|y_n - x_n\| = 0.$$

Proposition 2.8 ([28]) *A function $f : E \to [0, +\infty)$ is totally convex on bounded subsets of E if and only if it is sequentially consistent.*

Let C be a nonempty, closed, and convex subset of E. Let $f : E \to (-\infty, +\infty]$ be a convex function on E which is Gâteaux differentiable on $\operatorname{int}\operatorname{dom} f$. The Bregman projection $\operatorname{proj}_C^f(x)$ with respect to f (*cf.* [28]) of $x \in \operatorname{int}\operatorname{dom} f$ onto C is the minimizer over C of the functional $D_f(\cdot, x) :\to [0, +\infty]$, that is,

$$\operatorname{proj}_C^f(x) = \operatorname{argmin}\bigl\{D_f(y, x) : y \in C\bigr\}.$$

Let E be a Banach space with dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{f \in E^* : \langle x,f\rangle = \|x\|^2 = \|f\|^2\},$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. It is well known that if E is smooth, then J is single-valued.

Proposition 2.9 ([29]) *Let $f : E \to R$ be an admissible, strongly coercive, and strictly convex function. Let C be a nonempty, closed, and convex subset of* $\operatorname{dom} f$. *Then* $\operatorname{proj}_C^f(x)$ *exists uniquely for all* $x \in \operatorname{int}\operatorname{dom} f$.

Let $f(x) = \frac{1}{2}\|x\|^2$.

(i) If E is a Hilbert space, then the Bregman projection is reduced to the metric projection onto C.

(ii) If E is a smooth Banach space, then the Bregman projection is reduced to the generalized projection $\Pi_C(x)$ which is defined by

$$\Pi_C(x) = \operatorname{argmin}\{\phi(y,x) : y \in C\},$$

where ϕ is the Lyapunov functional (*cf.* [10]) defined by

$$\phi(y,x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2$$

for all $y, x \in E$.

Proposition 2.10 ([26]) *Let $f : E \to (-\infty,+\infty]$ be a totally convex function. Let C be a nonempty, closed, and convex subset of* $\operatorname{int}\operatorname{dom} f$ *and* $x \in \operatorname{int}\operatorname{dom} f$. *If $x^* \in C$, then the following statements are equivalent*:

(i) *The vector x^* is the Bregman projection of x onto C.*

(ii) *The vector x^* is the unique solution z of the variational inequality*

$$\langle z - y, \nabla f(x) - \nabla f(z)\rangle \geq 0, \quad \forall y \in C.$$

(iii) *The vector x^* is the unique solution z of the inequality*

$$D_f(y,z) + D_f(z,x) \leq D_f(y,x), \quad \forall y \in C.$$

In this paper, we present the following definition.

Definition 2.11 Let C be a nonempty, closed, and convex subset of E and $f : E \to (-\infty,+\infty]$ be an admissible function. Let T be a mapping from C into itself with a nonempty fixed point set $F(T)$. The mapping T is said to be Bregman quasi-Lipschitz if there exists a constant $L \geq 1$ such that

$$D_f(p,Tx) \leq LD_f(p,x), \quad \forall p \in F(T), \forall x \in C.$$

The mapping T is said to be Bregman quasi-nonexpansive if

$$D_f(p, Tx) \leq D_f(p,x), \quad \forall p \in F(T), \forall x \in C.$$

Bregman quasi-Lipschitz mappings are a more generalized class than the class of Bregman quasi-mappings. On the other hand, this class also contains the relatively quasi-Lipschitz mappings and quasi-Lipschitz mappings. Therefore, Bregman quasi-Lipschitz mappings are very important in the nonlinear analysis and fixed point theory and applications.

Definition 2.12 Let C be a nonempty, closed, and convex subset of E. Let $\{T_n\}$ be sequence of mappings from C into itself with a nonempty common fixed point set $F = \bigcap_{n=1}^{\infty} F(T_n)$. $\{T_n\}$ is said to be uniformly closed if for any convergent sequence $\{z_n\} \subset C$ such that $\|T_n z_n - z_n\| \to 0$ as $n \to \infty$, the limit of $\{z_n\}$ belongs to F.

In Section 4, we will give two examples of a uniformly closed family of countable Bregman quasi-Lipschitz mappings.

Let E be a real Banach space with the dual E^* and C be a nonempty closed convex subset of E. Let $A : C \to E^*$ be a nonlinear mapping and $F : C \times C \to R$ be a bifunction. Then consider the following generalized equilibrium problem of finding $u \in C$ such that:

$$F(u,y) + \langle Au, y - u\rangle \geq 0, \quad \forall y \in C. \tag{2.3}$$

The set of solutions of (2.3) is denoted by EP, *i.e.*,

$$EP = \{u \in C : F(u,y) + \langle Au, y - u\rangle \geq 0, \forall y \in C\}.$$

Whenever $E = H$ a Hilbert space, problem (2.3) was introduced and studied by Takahashi and Takahashi [30].

Whenever $A \equiv 0$, problem (2.3) is equivalent to finding $u \in C$ such that

$$F(u,y) \geq 0, \quad \forall y \in C, \tag{2.4}$$

which is called the equilibrium problem. The set of its solutions is denoted by $EP(F)$.

Whenever $F \equiv 0$, problem (1.1) is equivalent to finding $u \in C$ such that

$$\langle Au, y - u\rangle \geq 0, \quad \forall y \in C,$$

which is called the variational inequality of Browder type. The set of its solutions is denoted by $VI(C,A)$.

Problem (2.3) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, the Nash equilibrium problem in noncooperative games and others; see, *e.g.*, [31, 32].

In order to solve the equilibrium problem, let us assume that $F : C \times C \to (-\infty, +\infty)$ satisfies the following conditions [33]:

(A1) $F(x,x) = 0$ for all $x \in C$,

(A2) F is monotone, *i.e.*, $F(x,y) + F(y,x) \leq 0$, for all $x, y \in C$, (A3) for all $x, y, z \in C$, $\limsup_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x, y)$,

(A4) for all $x \in C$, $F(x, \cdot)$ is convex and lower semi-continuous.

For $r > 0$, we define a mapping $K_r : E \to C$ as follows:

$$T_r(x) = \left\{ z \in C : F(z, y) + \frac{1}{r} \langle y - z, \nabla f(z) - \nabla f(x) \rangle \geq 0, \forall y \in C \right\} \tag{2.5}$$

for all $x \in E$. The following two lemmas were proved in [14].

Lemma 2.13 *Let E be a reflexive Banach space and let $f : E \to R$ be a Legendre function. Let C be a nonempty, closed, and convex subset of E and let $F : C \times C \to R$ be a bifunction satisfying* (A1)-(A4). *For $r > 0$, let $T_r : E \to C$ be the mapping defined by* (2.5). *Then* $\operatorname{dom} T_r = E$.

Lemma 2.14 *Let E be a reflexive Banach space and let $f : E \to R$ be a convex, continuous, and strongly coercive function which is bounded on bounded subsets and uniformly convex on bounded subsets of E. Let C be a nonempty, closed, and convex subset of E and let $F : C \times C \to R$ be a bifunction satisfying* (A1)-(A4). *For $r > 0$, let $T_r : E \to C$ be the mapping defined by* (2.5). *Then the following statements hold*:

(i) *T_r is single-valued.*

(ii) *T_r is a firmly nonexpansive-type mapping, i.e., for all $x, y \in E$,*

$$\langle T_r x - T_r y, \nabla f(T_r x) - \nabla f(T_r y) \rangle \leq \langle T_r x - T_r y, \nabla f(x) - \nabla f(y) \rangle.$$

(iii) *$F(T_r) = \hat{F}(T_r) = EP(F)$.*

(iv) *$EP(F)$ is closed and convex.*

(v) *$D_f(p, T_r x) + D_f(T_r x, x) \leq D_f(p, x)$, $\forall p \in EP(F)$, $\forall x \in E$.*

Lemma 2.15 *Let E be a reflexive Banach space and let $f : E \to R$ be a convex, continuous, and strongly coercive function which is bounded on bounded subsets and uniformly convex on bounded subsets of E. Let C be a nonempty, closed, and convex subset of E and let $F : C \times C \to R$ be a bifunction satisfying* (A1)-(A4). *Let $A : C \to E^*$ be a monotone mapping, i.e.,*

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C.$$

For $r > 0$, let $K_r : E \to C$ be the mapping defined by

$$K_r(x) = \left\{ z \in C : F(z, y) + \langle Az, y - z \rangle + \frac{1}{r} \langle y - z, \nabla f(z) - \nabla f(x) \rangle \geq 0, \forall y \in C \right\}.$$

Then the following statements hold:

(i) *K_r is single-valued.*

(ii) *K_r is a firmly nonexpansive-type mapping, i.e., for all $x, y \in E$,*

$$\langle K_r x - K_r y, \nabla f(K_r x) - \nabla f(K_r y) \rangle \leq \langle K_r x - K_r y, \nabla f(x) - \nabla f(y) \rangle.$$

(iii) *$F(K_r) = \hat{F}(K_r) = EP$.*

(iv) *EP is closed and convex.*
(v) $D_f(p, K_r x) + D_f(K_r x, x) \le D_f(p, x)$, $\forall p \in EP(F)$, $\forall x \in E$.

Proof Let

$$G(x,y) = F(x,y) + \langle Ax, y-x\rangle, \quad \forall x,y \in C.$$

It is easy to show that, $G(x,y)$ satisfies conditions (A1)-(A4). Replacing $F(x,y)$ by $G(x,y)$ in Lemma 2.14, we can get the conclusions. □

3 Main results

Theorem 3.1 *Let $f : E \to (-\infty, +\infty]$ be a Legendre function which is totally convex on bounded subsets of E. Suppose that ∇f^* is bounded on bounded subsets of* $\operatorname{int}\operatorname{dom} f^*$. *Let C be a nonempty, closed, and convex subset of* $\operatorname{int}\operatorname{dom} f$. *Let $\{T_n\} : C \to C$ be a uniformly closed family of countable Bregman quasi-Lipschitz mappings with the condition* $\lim_{n\to\infty} L_n = 1$, *where*

$$D_f(p, T_n x) \le L_n D_f(p,x), \quad \forall p \in F, \forall x \in C. \tag{3.1}$$

Let F be a common fixed point set of $\{T_n\}$. Then F is closed and convex.

Proof Firstly, we prove that F is closed. Let $\{p_n\} \subset F$, $p_n \to p$ as $n \to \infty$, then $\|T_n p_n - p_n\| = 0 \to 0$ as $n \to \infty$. Since $\{T_n\}$ is uniformly closed, we know that $p \in F$. Hence F is closed.

Next we prove that F is convex. Let $p_1, p_2 \in F$, $p = tp_1 + (1-t)p_2$, where $t \in (0,1)$. We prove that $p \in F$. By the three point identity (2.2), we know that

$$D_f(x,y) = D_f(x,z) + D_f(z,y) + \big\langle x - z, \nabla f(z) - \nabla f(y)\big\rangle.$$

This implies

$$D_f(p_i, T_n p) = D_f(p_i, p) + D_f(p, T_n p) + \big\langle p_i - p, \nabla f(p) - \nabla f(T_n p)\big\rangle \tag{3.2}$$

for $i = 1,2$. Combining (3.1) and (3.2) yields

$$D_f(p, T_n p) \le (L_n - 1) D_f(p_i, p) - \big\langle p_i - p, \nabla f(p) - \nabla f(T_n p)\big\rangle \tag{3.3}$$

for $i = 1,2$. Multiplying t and $(1-t)$ on both sides of (3.3) with $i = 1$ and $i = 2$, respectively, yields

$$\lim_{n\to\infty} D_f(p, T_n p) \le \lim_{n\to\infty} \big(\xi_n - \big\langle tp_1 + (1-t)p_2 - p, \nabla f(p) - \nabla f(T_n p)\big\rangle\big) = 0,$$

where

$$\xi_n = (L_n - 1)\big[t D_f(p_1, p) + (1-t) D_f(p_2, p)\big].$$

This implies that $\{D_f(p, T_n p)\}$ is bounded. By Propositions 2.4 and 2.8, we see that the sequence $\{T_n p_n\}$ is bounded and $\|p - T_n p\| \to 0$ as $n \to \infty$. Since $p \to p$, and $\{T_n\}$ is uniformly closed, then $p \in F$. Therefore F is convex. This completes the proof. □

Next we will prove the main strong convergence theorem for the finite families of countable Bregman quasi-Lipschitz mappings by using a new hybrid projection scheme. In this scheme, we will use some detailed technology.

Theorem 3.2 *Let $f : E \to (-\infty, +\infty]$ be a Legendre function which is bounded, strongly coercive, uniformly Fréchet differentiable and totally convex on bounded subsets on E. Let C be a nonempty, closed, and convex subset of $\operatorname{int}\operatorname{dom} f$. Let $\{T_n^{(i)}\}_{n=1}^{\infty} : C \to C$ be N uniformly closed families of countable Bregman quasi-Lipschitz mappings with the condition $\lim_{n\to\infty} L_n^{(i)} = 1$ for $i = 1, 2, 3, \ldots, N$. Let $F = \bigcap_{n=1}^{\infty} \bigcap_{i=1}^{N} F(T_n^{(i)})$ and $F \cap EP$ be nonempty. Let $\{x_n\}$ be a sequence of C generated by*

$$\begin{cases} x_0 \in \operatorname{int}\operatorname{dom} f, \quad \text{arbitrarily}, \\ y_{i,n} = \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(T_n^{(i)} x_n)), \quad i = 1, 2, 3, \ldots, N, \\ F(u_{i,n}, y) + \langle A u_{i,n}, y - u_{i,n}\rangle + \frac{1}{r_n}\langle \nabla f(u_{i,n}) - \nabla f(y_{i,n}), y - u_{i,n}\rangle \geq 0, \quad \forall y \in C, \\ C_{i,n+1} = \{z \in C_n : D_f(z, u_{i,n}) \leq D_f(z, y_{i,n}) \leq D_f(z, x_n) + \xi_n\}, \quad n \geq 1, \\ C_{i,1} = C, \qquad C_{n+1} = \bigcap_{i=1}^{N} C_{i,n+1}, \\ x_n = P_{C_n}^f x_0, \end{cases}$$

where

$$\begin{aligned} &\xi_n = (L_n - 1) \sup_{p \in F\cap EP \cap B(P_{F\cap EP}^f x_0, 1)} D_f(p, x_0), \\ &B(x, 1) = \{y \in E : D_f(y, x) \leq 1\}, \\ &L_n = \max\{L_n^{(1)}, L_n^{(2)}, L_n^{(3)}, \ldots, L_n^{(N)}\} \end{aligned}$$

and $\{\alpha_n\}$ is a sequence satisfying $\limsup_{n\to\infty} \alpha_n < 1$. Then $\{x_n\}$ converges to $q = P_{F\cap EP}^f x_0$.

Proof We divide the proof into six steps.

Step 1. We show that C_n is closed and convex for all $n \geq 1$. It is obvious that $C_{i,1} = C$ is closed and convex. Suppose that $C_{i,k}$ is closed and convex for some $k \geq 1$. We see for each $i = 1, 2, 3, \ldots, N$ that

$$C_{i,k+1} = \{z \in C : D_f(z, u_{i,k}) \leq D_f(z, y_{i,k}) \leq D_f(z, x_k) + \xi_k\} \cap C_{i,k}$$

and

$$D_f(z, u_{i,k}) \leq D_f(z, y_{i,k}) \leq D_f(z, x_k) + \xi_k$$

is equivalent to

$$\begin{cases} \langle \nabla f(x_k) - \nabla f(y_{i,k}), z\rangle \leq \langle f^*(\nabla f(x_k)) - f^*(\nabla f(y_{i,k}))\rangle + \xi_k, \\ \langle \nabla f(y_{i,k}) - \nabla f(u_{i,k}), z\rangle \leq \langle f^*(\nabla f(y_{i,k})) - f^*(\nabla f(u_{i,k}))\rangle. \end{cases} \tag{3.4}$$

Therefore

$$C_{i,k+1} = \{z \in C : z \text{ satisfies (3.4)}\} \cap C_{i,k}.$$

It is easy to see that if z_1, z_2 satisfy (3.4), the element $z = tz_1 + (1 - t)z_2$ satisfies also (3.4) for all $t \in (0,1)$, so that the set

$$\{z \in C : z \text{ satisfies (3.4)}\}$$

is convex and closed, and hence $C_{i,k+1}$ is closed and convex for all $n \geq 1$. Therefore $C_{n+1} = \bigcap_{i=1}^{N} C_{i,n+1}$ is closed and convex.

Step 2. We show that $F \cap EP \cap B(P^f_{F\cap EP}x_0, 1) \subset C_n$ for all $n \geq 1$. It is obvious that $F \cap EP \cap B(P^f_{F\cap EP}x_0, 1) \subset C_{i,1} = C$ for all $1 \leq i \leq N$. Suppose that $F \cap EP \cap B(P^f_{F\cap EP}x_0, 1) \subset C_n$ for some $n \geq 1$. Let $p \in F \cap EP \cap B(P^f_{F\cap EP}x_0, 1)$. By Proposition 2.7, we have

$$\begin{aligned} D_f(p, y_{i,n}) &= D_f\bigl(p, \nabla f^*\bigl(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f\bigl(T_n^{(i)}x_n\bigr)\bigr)\bigr) \\ &= W^f\bigl(p, \nabla f^*\bigl(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f\bigl(T_n^{(i)}x_n\bigr)\bigr)\bigr) \\ &\leq \alpha_n W^f\bigl(p, \nabla f(x_n)\bigr) + (1-\alpha_n) W^f\bigl(p, \nabla f\bigl(T_n^{(i)}x_n\bigr)\bigr) \\ &= \alpha_n D_f(p, x_n) + (1-\alpha_n) D_f\bigl(p, T_n^{(i)}x_n\bigr) \\ &\leq \alpha_n D_f(p, x_n) + (1-\alpha_n) D_f(p, x_n) + \xi_n \\ &\leq D_f(p, x_n) + \xi_n. \end{aligned} \tag{3.5}$$

On the other hand, by Lemma 2.15, we have $p = K_r(p)$ and

$$D_f(p, K_r y_{i,n}) + D_f(K_n y_{i,n}, y_{i,n}) \leq D_f(p, y_{i,n}),$$

that is,

$$D_f(p, u_{i,n}) + D_f(K_n y_{i,n}, y_{i,n}) \leq D_f(p, y_{i,n}). \tag{3.6}$$

Combining (3.5) and (3.6) we know that $p \in C_{i,n+1}$ for all $1 \leq i \leq N$, which implies that $F \cap EP \cap B(P^f_{F\cap EP}x_0, 1) \subset C_{i,n+1}$. Therefore $F \cap EP \cap B(P^f_{F\cap EP}x_0, 1) \subset C_{n+1}$. By induction we know that $F \cap EP \cap B(P^f_{F\cap EP}x_0, 1) \subset C_n$ for all $n \geq 1$.

Step 3. We show that $\{x_n\}$ converges to a point $p \in C$.

Since $x_n = P^f_{C_n}x_0$ and $C_{n+1} \subset C_n$, then we get

$$D_f(x_n, x_0) \leq D_f(x_{n+1}, x_0) \quad \text{for all } n \geq 1. \tag{3.7}$$

Therefore $\{D_f(x_n, x_0)\}$ is nondecreasing. On the other hand, by Proposition 2.10, we have

$$D_f(x_n, x_0) = D_f\bigl(P^f_{C_n}x_0, x_0\bigr) \leq D_f(p, x_0) - D_f(p, x_n) \leq D_f(p, x_0)$$

for all $p \in F \subset C_n$ and for all $n \geq 1$. Therefore, $D_f(x_n, x_0)$ is also bounded. This together with (3.7) implies that the limit of $\{D_f(x_n, x_0)\}$ exists. Put

$$\lim_{n\to\infty} D_f(x_n, x_0) = d. \tag{3.8}$$

From Proposition 2.10, we have, for any positive integer m, that

$$\begin{aligned} D_f(x_{n+m}, x_n) &= D_f\bigl(x_{n+m}, P^f_{C_n}x_0\bigr) \leq D_f(x_{n+m}, x_0) - D_f\bigl(P^f_{C_n}x_0, x_0\bigr) \\ &= D_f(x_{n+m}, x_0) - D_f(x_n, x_0) \end{aligned}$$

for all $n \geq 1$. This together with (3.8) implies that

$$\lim_{n\to\infty} D_f(x_{n+m}, x_n) = 0$$

holds uniformly for all m. Therefore, we get that

$$\lim_{n\to\infty} \|x_{n+m} - x_n\| = 0$$

holds uniformly for all m. Then $\{x_n\}$ is a Cauchy sequence, therefore there exists a point $p \in C$ such that $x_n \to p$.

Step 4. We show that the limit of $\{x_n\}$ belongs to F.

Since $x_{n+1} \in C_{n+1}$, we have for all $1 \leq i \leq N$ that

$$D_f(x_{n+1}, u_{i,n}) \leq D_f(x_{n+1}, y_{i,n}) \leq D_f(x_{n+1}, x_n) + \xi_n \to 0$$

as $n \to \infty$. By Proposition 2.8, we obtain that

$$\lim_{n\to\infty} \|x_{n+1} - y_{i,n}\| = 0, \lim_{n\to\infty} \|x_{n+1} - u_{i,n}\| = 0. \tag{3.9}$$

From

$$y_{i,n} = \nabla f^*\big(\alpha_n \nabla f(x_n) + (1 - \alpha_n)\nabla f\big(T_n^{(i)} x_n\big)\big),$$

we get

$$\nabla f(y_{i,n}) = \alpha_n \nabla f(x_n) + (1 - \alpha_n)\nabla f\big(T_n^{(i)} x_n\big),$$

which implies that

$$\nabla f(y_{i,n}) - \nabla f(x_n) = (1 - \alpha_n)\big(\nabla f\big(T_n^{(i)} x_n\big) - \nabla f(x_n)\big).$$

By Proposition 2.2, we have

$$\lim_{n\to\infty} \big\|\nabla f(y_{i,n}) - \nabla f(x_n)\big\| = 0,$$

so that

$$\lim_{n\to\infty} \big\|\nabla f\big(T_n^{(i)} x_n\big) - \nabla f(x_n)\big\| = 0.$$

By Propositions 2.3 and 2.8, ∇f^* is uniformly continuous on bounded subsets of E and thus

$$\lim_{n\to\infty} \big\|T_n^{(i)} x_n - x_n\big\| = 0.$$

Since $\{T_n^{(i)}\}$ is an asymptotically countable family of Bregman weak relatively nonexpansive mappings and $x_n \to p$, so that $p \in \bigcap_{n=1}^{\infty} F(T_n^{(i)})$ for each $1 \leq i \leq N$. Therefore $p \in F = \bigcap_{n=1}^{\infty} \bigcap_{i=1}^{N} F(T_n^{(i)})$.

Step 5. We show that the limit of $\{x_n\}$ belongs to EP.

We have proved that $x_n \to p$ as $n \to \infty$. Now let us show that $p \in EP$. Since ∇f is uniformly norm-to-norm continuous on bounded subsets of E, from (3.9) we have $\lim_{n\to\infty} \|\nabla f(u_{i,n}) - \nabla f(y_{i,n})\| = 0$. From $\liminf_{n\to\infty} r_n > 0$ it follows that

$$\lim_{n\to\infty} \frac{\|\nabla f(u_{i,n}) - \nabla f(y_{i,n})\|}{r_n} = 0.$$

By the definition of $u_n := K_{r_n} y_n$, we have

$$G(u_{i,n}, y) + \frac{1}{r_n}\bigl\langle y - u_{i,n}, \nabla f(u_{i,n}) - \nabla f(y_{i,n})\bigr\rangle \geq 0, \quad \forall y \in C,$$

where

$$G(u_{i,n}, y) = F(u_{i,n}, y) + \langle Au_{i,n}, y - u_{i,n}\rangle.$$

We have from (A2) that

$$\frac{1}{r_n}\bigl\langle y - u_{i,n}, \nabla f(u_{i,n}) - \nabla f(y_{i,n})\bigr\rangle \geq -G(u_{i,n}, y) \geq G(y, u_{i,n}), \quad \forall y \in C.$$

Since $y \mapsto f(x,y) + \langle Ax, y - x\rangle$ is convex and lower semi-continuous, letting $n \to \infty$ in the last inequality, from (A4) we have

$$G(y,p) \leq 0, \quad \forall y \in C.$$

For t, with $0 < t < 1$, and $y \in C$, let $y_t = ty + (1-t)p$. Since $y \in C$ and $p \in C$, then $y_t \in C$ and hence $G(y_t, p) \leq 0$. So, from (A1) we have

$$0 = G(y_t, y_t) \leq tG(y_t, y) + (1-t)G(y_t, p) \leq tG(y_t, y).$$

Dividing by t, we have

$$G(y_t, y) \geq 0, \quad \forall y \in C.$$

Letting $t \to 0$, from (A3) we can get

$$G(p, y) \geq 0, \quad \forall y \in C.$$

So, $p \in EP$.

Step 6. Finally, we prove that $p = P^f_{F\cap EP} x_0$, from Proposition 2.10, we have

$$D_f\bigl(p, P^f_{F\cap EP} x_0\bigr) + D_f\bigl(P^f_{F\cap EP} x_0, x_0\bigr) \leq D_f(p, x_0). \tag{3.10}$$

On the other hand, since $x_n = P^f_{C_n} x_0$ and $F \cdot EP \subset C_n$ for all n, also from Proposition 2.10, we have

$$D_f\bigl(P^f_{F\cap EP} x_0, x_{n+1}\bigr) + D_f(x_{n+1}, x_0) \leq D_f\bigl(P^f_{F\cap EP} x_0, x_0\bigr). \tag{3.11}$$

By the definition of $D_f(x,y)$, we know that

$$\lim_{n\to\infty} D_f(x_{n+1},x_0) = D_f(p,x_0). \tag{3.12}$$

Combining (3.10), (3.11), and (3.12), we know that $D_f(p,x_0) = D_f(P^f_{F\cap EP}x_0,x_0)$. Therefore, it follows from the uniqueness of $P^f_{F\cap EP}x_0$ that $p = P^f_{F\cap EP}x_0$. This completes the proof. □

Definition 3.3 Let C be a nonempty, closed, and convex subset of E. Let T be a mapping from C into itself with a nonempty fixed point set $F(T)$. The mapping T is said to be Lyapunov quasi-Lipschitz if there exists a constant $L \geq 1$ such that

$$\phi(p,Tx) \leq L\phi(p,x), \quad \forall p \in F(T), \forall x \in C.$$

The mapping T is said to be Lyapunov quasi-nonexpansive if

$$\phi(p,Tx) \leq \phi(p,x), \quad \forall p \in F(T), \forall x \in C.$$

If we choose $f(x) = \frac{1}{2}\|x\|^2$ for all $x \in E$, then Theorem 3.2 reduces to the following corollary.

Corollary 3.4 *Let E be a smooth Banach space and C be a closed convex subset of E. Let $\{T_n^{(i)}\}_{n=1}^\infty : C \to C$ be N uniformly closed families of countable Lyapunov quasi-Lipschitz mappings with the condition $\lim_{n\to\infty} L_n^{(i)} = 1$ for $i = 1,2,3,\ldots,N$. Let $F = \bigcap_{n=1}^\infty \bigcap_{i=1}^N F(T_n^{(i)})$ and $F\cap EP$ be nonempty. Let $\{x_n\}$ be a sequence of C generated by*

$$\begin{cases} x_0 \in \operatorname{int}\operatorname{dom} f, \quad \textit{arbitrarily}, \\ y_{i,n} = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT_n^{(i)}x_n), \quad i = 1,2,3,\ldots,N, \\ F(u_{i,n},y) + \langle Au_{i,n}, y-u_{i,n}\rangle + \frac{1}{r_n}\langle J(u_{i,n}) - J(y_{i,n}), y-u_{i,n}\rangle \geq 0, \quad \forall y \in C, \\ C_{i,n+1} = \{z \in C_n : \phi(z,u_{i,n}) \leq \phi(z,y_{i,n}) \leq \phi(z,x_n) + \xi_n\}, \quad n \geq 1, \\ C_{i,1} = C, \qquad C_{n+1} = \bigcap_{i=1}^N C_{i,n+1}, \\ x_n = P^f_{C_n}x_0, \end{cases}$$

where

$$\xi_n = (L_n - 1)\sup_{p\in F\cap EP\cap B(P^f_{F\cap EP}x_0,1)} \phi(p,x_0),$$

$$B(x,1) = \bigl\{y \in E : \phi(y,x) \leq 1\bigr\},$$

$$L_n = \max\bigl\{L_n^{(1)}, L_n^{(2)}, L_n^{(3)}, \ldots, L_n^{(N)}\bigr\}$$

and $\{\alpha_n\}$ is a sequence satisfying $\limsup_{n\to\infty}\alpha_n < 1$. Then $\{x_n\}$ converges to $q = P^f_{F\cap EP}x_0$.

4 Example

Let E be a smooth Banach space and C be a nonempty closed convex and balanced subset of E. Let $\{x_n\}$ be a sequence in C such that $\|x_n\| = r > 0$, $\{x_n\}$ converges weakly to $x_0 \neq 0$

and $\|x_n - x_m\| \geq r > 0$ for all $n \neq m$. Define a countable family of mappings $\{T_n\}: C \to C$ as follows:

$$T_n(x) = \begin{cases} \frac{n+1}{n} x_n & \text{if } x = x_n\ (\exists n \geq 1), \\ -x & \text{if } x \neq x_n\ (\forall n \geq 1). \end{cases}$$

Conclusion 4.1 *$\{T_n\}$ has a unique common fixed point* 0, *that is,* $F = \bigcap_{n=1}^{\infty} F(T_n) = \{0\}$ *for all* $n \geq 0$.

Proof The conclusion is obvious. □

Conclusion 4.2 *$\{T_n\}$ is a uniformly closed family of countable Bregman quasi-Lipschitz mappings with the condition* $\lim_{n\to\infty} L_n = \lim_{n\to\infty} \frac{n+1}{n} = 1$.

Proof Take $f(x) = \frac{\|x\|^2}{2}$, then

$$D_f(x,y) = \phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for all $x, y \in C$ and

$$D_f(0, T_n x) = \|T_n x\|^2 = \begin{cases} \frac{n+1}{n}\|x_n\|^2 & \text{if } x = x_n, \\ \|x\|^2 & \text{if } x \neq x_n. \end{cases}$$

Therefore

$$D_f(0, T_n x) \leq \frac{n+1}{n}\|x\|^2 = \frac{n+1}{n} D_f(0, x)$$

for all $x \in C$. On the other hand, for any strong convergent sequence $\{z_n\} \subset E$ such that $z_n \to z_0$ and $\|z_n - T_n z_n\| \to 0$ as $n \to \infty$, it is easy to see that there exists a sufficiently large nature number N such that $z_n \neq x_m$ for any $n, m > N$. Then $Tz_n = -z_n$ for $n > N$, it follows from $\|z_n - T_n z_n\| \to 0$ that $2z_n \to 0$ and hence $z_n \to z_0 = 0$. That is, $z_0 \in F$. □

Example 4.3 Let $E = l^2$, where

$$l^2 = \left\{ \xi = (\xi_1, \xi_2, \xi_3, \ldots, \xi_n, \ldots) : \sum_{n=1}^{\infty} |x_n|^2 < \infty \right\},$$

$$\|\xi\| = \left(\sum_{n=1}^{\infty} |\xi_n|^2 \right)^{\frac{1}{2}}, \quad \forall \xi \in l^2,$$

$$\langle \xi, \eta \rangle = \sum_{n=1}^{\infty} \xi_n \eta_n, \quad \forall \xi = (\xi_1, \xi_2, \xi_3, \ldots, \xi_n, \ldots), \eta = (\eta_1, \eta_2, \eta_3, \ldots, \eta_n, \ldots) \in l^2.$$

Let $\{x_n\} \subset E$ be a sequence defined by

$$x_0 = (1, 0, 0, 0, \ldots),$$

$$x_1 = (1, 1, 0, 0, \ldots),$$

$$
\begin{aligned}
x_2 &= (1,0,1,0,0,\ldots), \\
x_3 &= (1,0,0,1,0,0,\ldots), \\
&\ldots, \\
x_n &= (\xi_{n,1},\xi_{n,2},\xi_{n,3},\ldots,\xi_{n,k},\ldots), \\
&\ldots,
\end{aligned}
$$

where

$$\xi_{n,k} = \begin{cases} 1 & \text{if } k = 1, n+1, \\ 0 & \text{if } k \neq 1, k \neq n+1 \end{cases}$$

for all $n \geq 1$. It is well known that $\|x_n\| = \sqrt{2}$, $\forall n \geq 1$ and $\{x_n\}$ converges weakly to x_0. Define a countable family of mappings $T_n : E \to E$ as follows:

$$T_n(x) = \begin{cases} \frac{n+1}{n} x_n & \text{if } x = x_n, \\ -x & \text{if } x \neq x_n \end{cases}$$

for all $n \geq 0$. By using Conclusions 4.1 and 4.2, $\{T_n\}$ is a uniformly closed family of countable Bregman quasi-Lipschitz mappings with the condition $\lim_{n\to\infty} L_n = \lim_{n\to\infty} \frac{n+1}{n} = 1$.

Example 4.4 Let $E = L^p[0,1]$ $(1 < p < +\infty)$ and

$$x_n = 1 - \frac{1}{2^n}, \quad n = 1,2,3,\ldots.$$

Define a sequence of functions in $L^p[0,1]$ by the following expression:

$$f_n(x) = \begin{cases} \frac{2}{x_{n+1}-x_n} & \text{if } x_n \leq x < \frac{x_{n+1}+x_n}{2}, \\ \frac{-2}{x_{n+1}-x_n} & \text{if } \frac{x_{n+1}+x_n}{2} \leq x < x_{n+1}, \\ 0 & \text{otherwise} \end{cases}$$

for all $n \geq 1$. Firstly, we can see, for any $x \in [0,1]$, that

$$\int_0^x f_n(t)\,dt \to 0 = \int_0^x f_0(t)\,dt, \tag{4.1}$$

where $f_0(x) \equiv 0$. It is well known that the above relation (4.1) is equivalent to $\{f_n(x)\}$ converges weakly to $f_0(x)$ in a uniformly smooth Banach space $L^p[0,1]$ $(1 < p < +\infty)$. On the other hand, for any $n \neq m$, we have

$$
\begin{aligned}
\|f_n - f_m\| &= \left(\int_0^1 \left|f_n(x) - f_m(x)\right|^p dx \right)^{\frac{1}{p}} \\
&= \left(\int_{x_n}^{x_{n+1}} \left|f_n(x) - f_m(x)\right|^p dx + \int_{x_m}^{x_{m+1}} \left|f_n(x) - f_m(x)\right|^p dx \right)^{\frac{1}{p}} \\
&= \left(\int_{x_n}^{x_{n+1}} \left|f_n(x)\right|^p dx + \int_{x_m}^{x_{m+1}} \left|f_m(x)\right|^p dx \right)^{\frac{1}{p}}
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\left(\frac{2}{x_{n+1} - x_n} \right)^p (x_{n+1} - x_n) + \left(\frac{2}{x_{m+1} - x_m} \right)^p (x_{m+1} - x_m) \right)^{\frac{1}{p}} \\
&= \left(\frac{2^p}{(x_{n+1} - x_n)^{p-1}} + \frac{2^p}{(x_{m+1} - x_m)^{p-1}} \right)^{\frac{1}{p}} \\
&\geq \left(2^p + 2^p\right)^{\frac{1}{p}} > 0.
\end{aligned}
$$

Let

$$u_n(x) = f_n(x) + 1, \quad \forall n \geq 1.$$

It is obvious that u_n converges weakly to $u_0(x) \equiv 1$ and

$$\|u_n - u_m\| = \|f_n - f_m\| \geq \left(2^p + 2^p\right)^{\frac{1}{p}} > 0, \quad \forall n \geq 1. \tag{4.2}$$

Define a mapping $T : E \to E$ as follows:

$$T_n(x) = \begin{cases} \frac{n+1}{n} u_n & \text{if } x = u_n \ (\exists n \geq 1), \\ -x & \text{if } x \neq u_n \ (\forall n \geq 1). \end{cases}$$

Since (4.2) holds, by using Conclusions 4.1 and 4.2, we know that $\{T_n\}$ is a uniformly closed family of countable Bregman quasi-Lipschitz mappings with the condition $\lim_{n\to\infty} L_n = \lim_{n\to\infty} \frac{n+1}{n} = 1$.

5 Application

The mapping T is said to be Bregman asymptotically quasi-nonexpansive (*cf.* [29]) if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, +\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$D_f\left(p, T^n x\right) \leq k_n D_f(p, x), \quad \forall p \in F(T), \forall x \in C.$$

Every Bregman quasi-nonexpansive mapping is Bregman asymptotically quasi-nonexpansive with $k_n \equiv 1$. Let $S_n = T^n$ for all $n \geq 1$, the above inequality becomes

$$D_f(p, S_n x) \leq k_n D_f(p, x), \quad \forall p \in F(T), \forall x \in C.$$

It is obvious that $\bigcap_{n=1}^{\infty} F(S_n) = \bigcap_{n=1}^{\infty} F(T^n) = F(T)$.

Lemma 5.1 *Assume that T is uniformly Lipschitz, that is, there exists a constant $L \geq 1$ such that*

$$\left\| T^n x - T^n y \right\| \leq L\|x - y\|, \quad \forall x, y \in C$$

for all $n \geq 1$. Then $\{S_n\} = \{T^n\}$ is uniformly closed.

Proof Assume $\|z_n - S_n z_n\| \to 0$, $z_n \to p$ as $n \to \infty$, we have $\|z_n - T^n z_n\| \to 0$, therefore

$$\left\|p - T^n p\right\| \leq \left\|p - T^n z_n\right\| + \left\|T^n z_n - T^n p\right\| \leq \left\|p - T^n z_n\right\| + L\|z_n - p\| \to 0$$

as $n \to \infty$. On the one hand, $T^n p \to p$, on the other hand, $T^{n+1} p \to Tp$, these imply that $p = Tp$. Hence $p \in \bigcap_{n=1}^{\infty} F(S_n)$. This completes the proof. □

Next we give an application of Theorem 3.2 to find the fixed point of Bregman asymptotically quasi-nonexpansive mappings.

Theorem 5.2 *Let $f : E \to (-\infty, +\infty]$ be a Legendre function which is bounded, strongly coercive, uniformly Fréchet differentiable, and totally convex on bounded subsets on E. Let C be a nonempty, closed, and convex subset of* $\operatorname{int}\operatorname{dom} f$. *Let $\{T_i\}_{i=1}^{N} : C \to C$ be an N uniformly Lipschitz Bregman asymptotically quasi-nonexpansive mapping with a nonempty common fixed point set $F = \bigcap_{i=1}^{N} F(T_i)$ and $F \cap EP$ be nonempty. Let $\{x_n\}$ be a sequence of C generated by*

$$\begin{cases} x_0 \in \operatorname{int}\operatorname{dom} f, \quad \textit{arbitrarily}, \\ y_{i,n} = \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(T_i^n x_n)), \quad i = 1, 2, 3, \ldots, N, \\ F(u_{i,n}, y) + \langle A u_{i,n}, y - u_{i,n} \rangle + \frac{1}{r_n} \langle \nabla f(u_{i,n}) - \nabla f(y_{i,n}), y - u_{i,n} \rangle \geq 0, \quad \forall y \in C, \\ C_{i,n+1} = \{z \in C_n : D_f(z, u_{i,n}) \leq D_f(z, y_{i,n}) \leq D_f(z, x_n) + \xi_n\}, \quad n \geq 1, \\ C_{i,1} = C, \qquad C_{n+1} = \bigcap_{i=1}^{N} C_{i,n+1}, \\ x_n = P_{C_n}^f x_0, \end{cases}$$

where

$$\xi_n = (k_n - 1) \sup_{p \in F \cap EP \cap B(P_{F \cap EP}^f x_0, 1)} D_f(p, x_0),$$

$$B(x, 1) = \bigl\{ y \in E : D_f(y, x) \leq 1 \bigr\}$$

and $\{\alpha_n\}$ is a sequence satisfying $\limsup_{n\to\infty} \alpha_n < 1$. *Then $\{x_n\}$ converges to $q = P_{F \cap EP}^f x_0$.*

Proof Let $S_n = T^n$ for all $n \geq 1$, by using Lemma 5.1 and Theorem 3.2 we can obtain the conclusion. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The main idea of this paper was proposed by the corresponding author YS, and YS prepared the manuscript initially for one family of countable Bregman quasi-Lipschitz mappings. MC performed all the steps of the proofs in this research for the finite families of countable Bregman quasi-Lipschitz mappings. JB performed the application to the Bregman asymptotically quasi-nonexpansive mappings. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics and Sciences, Shijiazhuang University of Economics, Shijiazhuang, 050031, China.
[2]Department of Mathematics, Tianjin Polytechnic University, Tianjin, 300387, China.

Acknowledgements
This project is supported by the National Natural Science Foundation of China under grant (11071279).

Received: 27 March 2015 Accepted: 9 June 2015 Published online: 26 June 2015

References
1. Goebel, K, Kirk, WA: A fixed point theorem for asymptotically nonexpansive mappings. Proc. Am. Math. Soc. **35**, 171-174 (1972)

2. Takahashi, W, Takeuchi, Y, Kubota, R: Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces. J. Math. Anal. Appl. **341**, 276-286 (2008)
3. Schu, J: Weak and strong convergence to fixed points of asymptotically nonexpansive mappings. Bull. Aust. Math. Soc. **43**, 153-159 (1991)
4. Inchan, I: Strong convergence theorems of modified Mann iteration methods for asymptotically nonexpansive mappings in Hilbert spaces. Int. J. Math. Anal. **2**, 1135-1145 (2008)
5. Bruck, RE, Kuczumow, T, Reich, S: Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform Opial property. Colloq. Math. **65**, 169-179 (1993)
6. Hecai, Y, Aichao, L: Projection algorithms for treating asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense. J. Inequal. Appl. (2013). doi:10.1186/1029-242X-2013-265
7. Qing, Y: Some results on asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense. J. Fixed Point Theory **2012**, Article ID 1 (2012)
8. Qin, X, Huang, S, Wang, T: On the convergence of hybrid projection algorithms for asymptotically quasi-ϕ-nonexpansive mappings. Comput. Math. Appl. **61**, 851-859 (2011)
9. Qin, X, Wang, L: On asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense. Abstr. Appl. Anal. **2012**, Article ID 636217 (2012). doi:10.1155/2012/636217
10. Alber, YI: Metric and generalized projection operators in Banach spaces: properties and applications. In: Kartsatos, AG (ed.) Theory and Applications of Nonlinear Operators of Accretive and Monotone Type. Lecture Notes Pure Appl. Math., vol. 178, pp. 15-50. Dekker, New York (1996)
11. Hao, Y: Some results on a modified Mann iterative scheme in a reflexive Banach space. Fixed Point Theory Appl. **2013**, Article ID 227 (2013). doi:10.1186/1687-1812-2013-227
12. Bregman, LM: The relaxation method of finding the common point of convex sets and its application to the solution of problems in convex programming. USSR Comput. Math. Math. Phys. **7**, 200-217 (1967)
13. Martin-Marquez, V, Reich, S, Sabach, S: Iterative methods for approximating fixed points of Bregman nonexpansive operators. Discrete Contin. Dyn. Syst. **6**, 1043-1063 (2013)
14. Reich, S, Sabach, S: Two strong convergence theorems for Bregman strongly nonexpansive operators in reflexive Banach spaces. Nonlinear Anal. **73**, 122-135 (2010)
15. Reich, S, Sabach, S: Existence and approximation of fixed points of Bregman firmly nonexpansive mappings in reflexive Banach spaces. In: Fixed-Point Algorithms for Inverse Problems in Science and Engineering. Springer Optim. Appl., vol. 49, pp. 301-316. Springer, New York (2011)
16. Suantai, S, Cho, YJ, Cholamjiak, P: Halpern's iteration for Bregman strongly nonexpansive mappings in reflexive Banach spaces. Comput. Math. Appl. **64**, 489-499 (2012)
17. Naraghirad, E, Yao, J-C: Bregman weak relatively nonexpansive mappings in Banach spaces. Fixed Point Theory Appl. **2013**, Article ID 141 (2013)
18. Bauschke, HH, Borwein, JM, Combettes, PL: Essential smoothness, essential strict convexity, and Legendre functions in Banach spaces. Commun. Contemp. Math. **3**, 615-647 (2001)
19. Barbu, V, Precupanu, T: Convexity and Optimization in Banach Spaces. Springer, Dordrecht (2012)
20. Reich, S, Sabach, S: A strong convergence theorem for a proximal-type algorithm in reflexive Banach spaces. J. Nonlinear Convex Anal. **10**, 471-485 (2009)
21. Zalinescu, C: Convex Analysis in General Vector Spaces. World Scientific, River Edge (2002)
22. Tomizawa, Y: A strong convergence theorem for Bregman asymptotically quasi-nonexpansive mappings in the intermediate sense. Fixed Point Theory Appl. **2014**, Article ID 154 (2014)
23. Censor, Y, Lent, A: An iterative row-action method for interval convex programming. J. Optim. Theory Appl. **34**, 321-353 (1981)
24. Chen, G, Teboulle, M: Convergence analysis of a proximal-like minimization algorithm using Bregman functions. SIAM J. Optim. **3**, 538-543 (1993)
25. Butnariu, D, Iusem, AN, Zalinescu, C: On uniform convexity, total convexity and convergence of the proximal points and outer Bregman projection algorithms in Banach spaces. J. Convex Anal. **10**, 35-61 (2003)
26. Butnariu, D, Resmerita, E: Bregman distances, totally convex functions, and a method for solving operator equations in Banach spaces. Abstr. Appl. Anal. **2006**, Article ID 84919 (2006)
27. Reich, S, Sabach, S: Two strong convergence theorems for a proximal method in reflexive Banach spaces. Numer. Funct. Anal. Optim. **31**, 22-44 (2010)
28. Butnariu, D, Iusem, AN: Totally Convex Functions for Fixed Points Computation and Infinite Dimensional Optimization. Kluwer Academic, Dordrecht (2000)
29. Alber, Y, Butnariu, D: Convergence of Bregman projection methods for solving consistent convex feasibility problems in reflexive Banach spaces. J. Optim. Theory Appl. **92**, 33-61 (1997)
30. Takahashi, S, Takahashi, W: Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space. Nonlinear Anal. **69**, 1025-1033 (2008)
31. Martinet, B: Regularisation d'inequations variationnelles par approximations successives. Rev. Fr. Autom. Inform. Rech. Opér. **4**, 154-159 (1970)
32. Kohsaka, F, Takahashi, W: Strong convergence of an iterative sequence for maximal monotone operators in a Banach space. Abstr. Appl. Anal. **2004**(3), 239-249 (2004)
33. Blum, E, Oettli, W: From optimization and variational inequalities to equilibrium problems. Math. Stud. **63**, 123-145 (1994)