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# Approximation properties of $q$-Kantorovich-Stancu operator

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**Abstract**

In this paper we study some properties of Kantorovich-type generalizations of the $q$-Stancu operators. We obtain some approximation properties for these operators, estimating the rate of convergence by using the first and second modulus of continuity. Also, we investigate the statistical approximation properties of the $q$-Kantorovich-Stancu operators using the Korovkin-type statistical approximation theorem.

**MSC:** $q$-Stancu-Kantorovich operators; modulus of continuity; rate of convergence; Voronovskaja theorem

**Keywords:** 41A10; 41A25; 41A36

## 1 Introduction

In recent years, many researches focused their attention on the study of a generalized version in $q$-calculus of the well-known linear and positive operators [1–6]. Lupaş [7] initiated in 1987 the convergence of Bernstein operators based on $q$-integers and in 1996 another generalization of these operators was introduced by Philips [8]. Also, in [9], Agratini introduced a new class of $q$-Bernstein-type operators, which fix certain polynomials. More results on $q$-Bernstein polynomials were obtained by Ostrovska [10]. Muraru [11] proposed and studied some approximation properties of the $q$-Bernstein-Schurer operators. In [12], Ren and Zeng introduced a modified version of the $q$-Bernstein-Schurer operators and investigated the statistical approximation properties. The Kantorovich-type generalization of these operators was given in [13] by Özarslan and Vedi. In [14], Agrawal *et al.* introduced a Stancu-type generalization of the Bernstein-Schurer operators based on $q$-integer. They obtained the rate of convergence of these operators in terms of the modulus of continuity and by a Voronovskaja-type theorem. Many generalizations and applications of the Stancu operators were considered in the last years [15–17]. The goal of the present paper is to study some approximation properties of the $q$-analog of the Stancu-Kantorovich operators.

Before proceeding, we mention some basic definitions and notations from $q$-calculus. Let $q > 0$. For each nonnegative integer $k$, the $q$-integer $[k]_q$ and $q$-factorial $[k]_q!$ are defined by

$$[k]_q := \begin{cases} \frac{1-q^k}{1-q}, & q \neq 1, \\ k, & q = 1, \end{cases}$$

$$[k]_q! := \begin{cases} [k]_q[k-1]_q \cdots [1]_q, & k \geq 1, \\ 1, & k = 0, \end{cases}$$

respectively.

For the integers $n$, $k$ satisfying $n \geq k \geq 0$, the $q$-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

We denote $(a+b)_q^k = \prod_{j=0}^{k-1}(a + bq^j)$.

The $q$-Jackson integral on the interval $[0,b]$ is defined as

$$\int_0^b f(t)\,d_qt = (1-q)b\sum_{j=0}^{\infty} f\bigl(q^j b\bigr)q^j, \quad 0 < q < 1,$$

provided that the sums converge absolutely. Suppose that $0 < a < b$. The $q$-Jackson integral on the interval $[a,b]$ is defined as

$$\int_a^b f(t)\,d_qt = \int_0^b f(t)\,d_qt - \int_0^a f(t)\,d_qt, \quad 0 < q < 1.$$

The Riemann-type $q$-integral (see [18]) is defined by

$$\int_a^b f(t)\,d_q^R t = (1-q)(b-a)\sum_{j=0}^{\infty} f\bigl(a + (b-a)q^j\bigr)q^j.$$

The classical Stancu-Kantorovich operators $S_n^{(\alpha,\beta)}$, $n = 1, 2, \ldots$, are defined by

$$\begin{aligned} S_n^{(\alpha,\beta)}(f,x) &:= (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_{\frac{k+\alpha}{n+1+\beta}}^{\frac{k+1+\alpha}{n+1+\beta}} f(t)\,dt \\ &= \sum_{k=0}^{n} p_{n,k}(x)\int_0^1 f\left(\frac{t+k+\alpha}{n+1+\beta}\right)dt, \quad f:[0,1]\to\mathbb{R}, \end{aligned} \tag{1.1}$$

where $p_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}$ and $0 \leq \alpha \leq \beta$.

In [19], Ren and Zeng introduced two kinds of Kantorovich-type $q$-Bernstein-Stancu operators. The first version is defined using the $q$-Jackson integral as follows:

$$S_{n,q}^{(\alpha,\beta)}(f,x) = \bigl([n+1]_q + \beta\bigr)\sum_{k=0}^{n} q^{-k}p_{n,k}(q;x)\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}} f(t)\,d_qt, \tag{1.2}$$

where $f \in C[0,1]$ and $p_{n,k}(q;x) = \begin{bmatrix} n \\ k \end{bmatrix}_q x^k(1-x)_q^{n-k}$.

To guarantee the positivity of the $q$-Bernstein-Stancu-Kantorovich operators, in [19] $S_{n,q}^{(\alpha,\beta)}(f;x)$ is redefined by putting the Riemann-type $q$-integral into the operators instead of the $q$-Jackson integral as

$$\tilde{S}_{n,q}^{(\alpha,\beta)}(f,x) = \left([n+1]_q + \beta\right) \sum_{k=0}^{n} q^{-k} p_{n,k}(q;x) \int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}} f(t)\, d_q^R t. \tag{1.3}$$

These operators verify the following.

**Lemma 1.1** ([19]) *For* $\tilde{S}_{n,q}^{(\alpha,\beta)}$, $0 < q < 1$, *and* $0 \leq \alpha \leq \beta$, *we have*

(i)
$$\tilde{S}_{n,q}^{(\alpha,\beta)}(1;x) = 1,$$

(ii)
$$\tilde{S}_{n,q}^{(\alpha,\beta)}(t;x) = \frac{2q[n]_q}{[2]_q([n+1]_q+\beta)}x + \frac{1+[2]_q\alpha}{[2]_q([n+1]_q+\beta)},$$

(iii)
$$\begin{aligned}\tilde{S}_{n,q}^{\alpha,\beta)}\left(t^2;x\right) = {} & \frac{q[n]_q[n-1]_q}{([n+1]_q+\beta)^2}\left(1+\frac{2(q-1)}{[2]_q}+\frac{(q-1)^2}{[3]_q}\right)x^2 + \frac{[n]_q}{([n+1]_q+\beta)^2} \\ & \cdot \left(1+2\alpha+\frac{2(q-1)(1+\alpha)}{[2]_q}+\frac{2}{[2]_q}+\frac{2(q-1)}{[3]_q}+\frac{(q-1)^2}{[3]_q}\right)x \\ & + \frac{1}{([n+1]_q+\beta)^2}\left(\frac{1}{[3]_q}+\frac{2\alpha}{[2]_q}+\alpha^2\right).\end{aligned}$$

In [20], Mahmudov and Sabancigil introduced a $q$-type generalization of the Bernstein-Kantorovich operators as follows:

$$B_{n,q}^{*}(f,x) := \sum_{k=0}^{n} p_{n,k}(q;x) \int_0^1 f\left(\frac{[k]_q + q^k t}{[n+1]_q}\right) d_q t, \tag{1.4}$$

where $f \in C[0,1]$ and $0 < q \leq 1$. In [21], inspired by (1.4) we introduced a $q$-type generalization of the Stancu-Kantorovich operators as follows:

$$S_{n,q}^{*(\alpha,\beta)}(f,x) = \sum_{k=0}^{n} p_{n,k}(q;x) \int_0^1 f\left(\frac{[k]_q + q^k t + \alpha}{[n+1]_q + \beta}\right) d_q t, \tag{1.5}$$

where $0 \leq \alpha \leq \beta$ and $f \in C[0,1]$.

**Lemma 1.2** ([21]) *For all* $n \in \mathbb{N}$, $x \in [0,1]$, *and* $0 < q \leq 1$, *we have*

$$\begin{aligned} & S_{n,q}^{*(\alpha,\beta)}(1,x) = 1, \qquad S_{n,q}^{*(\alpha,\beta)}(t,x) = \frac{2q}{[2]_q}\frac{[n]_q}{[n+1]_q+\beta}x + \frac{\alpha}{[n+1]_q+\beta} + \frac{1}{[2]_q([n+1]_q+\beta)}, \\ & S_{n,q}^{*(\alpha,\beta)}\left(t^2,x\right) = \frac{1}{([n+1]_q+\beta)^2}\left\{\frac{q^2(q+2)}{[3]_q}[n]_q[n-1]_q x^2 + \frac{q[n]_q}{[2]_q}\left(4\alpha+\frac{4+7q+q^2}{[3]_q}\right)x \right. \\ & \qquad \left. + \frac{2\alpha}{[2]_q} + \frac{1}{[3]_q} + \alpha^2\right\}. \end{aligned}$$

**Lemma 1.3** ([21]) *For all* $n \in \mathbb{N}$, $x \in [0,1]$, *and* $0 < q \leq 1$, *we have*

$$\begin{aligned} & S_{n,q}^{*(\alpha,\beta)}\left((t-x)^2,x\right) \\ & \quad \leq \frac{2[n+1]_q^2}{([n+1]_q+\beta)^2}\left\{\frac{4}{[n]_q}\left(x(1-x)+\frac{1}{[n]_q}\right) + \left(\frac{\alpha}{[n+1]_q} - \frac{\beta}{[n+1]_q}x\right)^2\right\}, \end{aligned}$$

$$\begin{aligned} &S_{n,q}^{*(\alpha,\beta)}\big((t-x)^4,x\big) \\ &\quad\le \frac{8[n+1]_q^2}{([n+1]_q+\beta)^2}\left\{\frac{C}{[n]_q^2}\left(x(1-x)+\frac{1}{[n]_q^2}\right)+\left(\frac{\alpha}{[n+1]_q}-\frac{\beta}{[n+1]_q}x\right)^4\right\}, \end{aligned}$$

*where $C$ is a positive absolute constant.*

Also, in [21] a Voronovskaja-type theorem for the $S_{n,q}^{*(\alpha,\beta)}$ was established.

**Theorem 1.4** ([21]) *Let $f'' \in C[0,1]$, $q_n \in (0,1)$, $q_n \to 1$, and $q_n^n \to a$, $a \in [0,1)$ as $n \to \infty$. Then we have*

$$\lim_{n\to\infty}[n]_q\big(S_{n,q_n}^{*(\alpha,\beta)}(f,x)-f(x)\big)=\left(-\frac{1+a+2\beta}{2}x+\alpha+\frac{1}{2}\right)f'(x)+\frac{1}{2}\left(-\frac{2a+1}{3}x^2+x\right)f''(x).$$

The paper is organized as follows. In Section 2 we prove a Voronovskaja-type asymptotic formula for $\tilde{S}_{n,q}^{\alpha,\beta}$. In Section 3 we establish some approximation properties of the $q$-Stancu-Kantorovich operators $\tilde{S}_{n,q}^{\alpha,\beta}$ and $S_{n,q}^{*\alpha,\beta}$. In the final section we give statistical approximation results for the $q$-Stancu-Kantorovich operators.

## 2 A Voronovskaya theorem for $q$-Stancu-Kantorovich operators

In this section we shall establish a Voronovskaja-type theorem for the $q$-Stancu-Kantorovich operators $\tilde{S}_{n,q}^{(\alpha,\beta)}$. First, we need the auxiliary result contained in the following lemma.

**Lemma 2.1** *Assume that $0 < q_n < 1$, $q_n \to 1$, and $q_n^n \to a$, $a \in [0,1)$ as $n \to \infty$. Then we have*

$$\begin{aligned} &\lim_{n\to\infty}[n]_{q_n}\tilde{S}_{n,q_n}^{(\alpha,\beta)}(t-x,x)=-\frac{1+a+2\beta}{2}x+\alpha+\frac{1}{2}, \\ &\lim_{n\to\infty}[n]_{q_n}\tilde{S}_{n,q_n}^{(\alpha,\beta)}\big((t-x)^2,x\big)=x(1-x). \end{aligned}$$

*Proof* To prove the lemma we use the formulas for $\tilde{S}_{n,q_n}^{(\alpha,\beta)}(t^i,x)$, $i=0,1,2$, given in Lemma 1.1. We have

$$\begin{aligned} &\lim_{n\to\infty}[n]_{q_n}\tilde{S}_{n,q_n}^{(\alpha,\beta)}(t-x;x) \\ &\quad=\lim_{n\to\infty}[n]_{q_n}\left\{\left(\frac{2q_n}{[2]_{q_n}}\frac{[n]_{q_n}}{[n+1]_{q_n}+\beta}-1\right)x+\frac{\alpha}{[n+1]_{q_n}+\beta}+\frac{1}{[2]_{q_n}([n+1]_{q_n}+\beta}\right\} \\ &\quad=\lim_{n\to\infty}\left\{\frac{[n]_{q_n}}{[2]_{q_n}([n+1]_{q_n}+\beta)}\big(-1-q_n^{n+1}-[2]_{q_n}\beta\big)x\right. \\ &\qquad\left.+\frac{\alpha[n]_{q_n}}{[n+1]_{q_n}+\beta}+\frac{[n]_{q_n}}{[2]_{q_n}([n+1]_{q_n}+\beta)}\right\} \\ &\quad=-\frac{1+a+2\beta}{2}x+\alpha+\frac{1}{2}, \\ &\lim_{n\to\infty}[n]_{q_n}\tilde{S}_{n,q_n}^{(\alpha,\beta)}\big((t-x)^2;x\big) \\ &\quad=\lim_{n\to\infty}[n]_{q_n}\big\{\tilde{S}_{n,q_n}^{(\alpha,\beta)}\big(t^2,x\big)-x^2-2x\tilde{S}_{n,q_n}^{(\alpha,\beta)}(t-x,x)\big\} \end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} [n]_{q_n} \left( \frac{q_n [n]_{q_n} [n-1]_{q_n}}{([n+1]_{q_n} + \beta)^2} \cdot \frac{4q_n^3 + q_n^2 + q_n}{[2]_{q_n}[3]_{q_n}} - 1 \right) x^2 \\
&\quad + \lim_{n\to\infty} \frac{[n]_{q_n}^2}{([n+1]_{q_n} + \beta)^2} \left( 1 + 2\alpha + \frac{2(q_n-1)(1+\alpha)}{[2]_{q_n}} \right. \\
&\quad \left. + \frac{2}{[2]_{q_n}} + \frac{2(q_n-1)}{[3]_{q_n}} + \frac{(q_n-1)^2}{[3]_{q_n}} \right) x \\
&\quad + \lim_{n\to\infty} \frac{[n]_{q_n}}{([n+1]_{q_n} + \beta)^2} \left( \frac{1}{[3]_{q_n}} + \frac{2\alpha}{[2]_{q_n}} + \alpha^2 \right) - \lim_{n\to\infty} 2x [n]_{q_n} \tilde{S}_{n,q_n}^{(\alpha,\beta)}(t-x,x) \\
&= \lim_{n\to\infty} [n]_{q_n} \left( \frac{(4q_n^3 + q_n^2 + q_n) \cdot ([n]_{q_n}^2 - [n]_{q_n})}{[2]_{q_n}[3]_{q_n}([n+1]_{q_n} + \beta)^2} - 1 \right) x^2 \\
&\quad + (2\alpha + 2)x - 2x \left( -\frac{1 + a + 2\beta}{2} x + \alpha + \frac{1}{2} \right) \\
&= \lim_{n\to\infty} \frac{[n]_{q_n}}{[2]_{q_n}[3]_{q_n}([n+1]_{q_n} + \beta)^2} \bigl\{ \bigl(4q_n^3 + q_n^2 + q_n\bigr)[n]_{q_n}^2 - \bigl(4q_n^3 + q_n^2 + q_n\bigr)[n]_{q_n} \\
&\quad - [2]_{q_n}[3]_{q_n} \bigl([n+1]_{q_n} + \beta\bigr)^2 \bigr\} x^2 + x + (1 + a + 2\beta)x^2 \\
&= \lim_{n\to\infty} \frac{[n]_{q_n}}{[2]_{q_n}[3]_{q_n}([n+1]_{q_n} + \beta)^2} \bigl\{ \bigl(4q_n^3 + q_n^2 + q_n\bigr)[n]_{q_n}^2 - \bigl(4q_n^3 + q_n^2 + q_n\bigr)[n]_{q_n} \\
&\quad - [2]_{q_n}[3]_{q_n} \bigl(1 + q_n[n]_{q_n} + \beta\bigr)^2 \bigr\} x^2 + x + (1 + a + 2\beta)x^2 \\
&= \lim_{n\to\infty} \frac{[n]_{q_n}}{[2]_{q_n}[3]_{q_n}([n+1]_{q_n} + \beta)^2} \bigl\{ \bigl(4q_n^3 + q_n^2 + q_n - [2]_{q_n}[3]_{q_n} q_n^2\bigr)[n]_{q_n}^2 \\
&\quad - \bigl(4q_n^3 + q_n^2 + q_n + 2q_n(1+\beta)[2]_{q_n}[3]_{q_n}\bigr)[n]_{q_n} - [2]_{q_n}[3]_{q_n}(1+\beta)^2 \bigr\} x^2 \\
&\quad + x + (1 + a + 2\beta)x^2 \\
&= x(1-x).
\end{aligned}
$$

□

The main result of this section is the following Voronovskaja-type theorem.

**Theorem 2.2** *Let $f'' \in C[0,1]$, $q_n \in (0,1)$, $q_n \to 1$, and $q_n^n \to a$, $a \in [0,1)$ as $n \to \infty$. Then we have*

$$
\lim_{n\to\infty} [n]_{q_n} \bigl( \tilde{S}_{n,q_n}^{(\alpha,\beta)}(f,x) - f(x) \bigr) = \left( -\frac{1 + a + 2\beta}{2} x + \alpha + \frac{1}{2} \right) f'(x) + \frac{1}{2} x(1-x) f''(x).
$$

*Proof* From the Taylor theorem, we have

$$
f(t) = f(x) + (t-x)f'(x) + \frac{1}{2}(t-x)^2 f''(x) + \frac{1}{2}(t-x)^2 \bigl(f''(\xi) - f''(x)\bigr), \tag{2.1}
$$

where $\xi$ lies between $t$ and $x$.

Applying $\tilde{S}_{n,q_n}^{(\alpha,\beta)}$ on both sides of (2.1), we obtain

$$
\begin{aligned}
[n]_{q_n} \bigl( \tilde{S}_{n,q_n}^{(\alpha,\beta)}(f,x) - f(x) \bigr) &= [n]_{q_n} f'(x) \tilde{S}_{n,q_n}^{(\alpha,\beta)}(t-x,x) + \frac{1}{2} [n]_{q_n} f''(x) \tilde{S}_{n,q_n}^{(\alpha,\beta)}\bigl((t-x)^2, x\bigr) \\
&\quad + [n]_{q_n} \tilde{S}_{n,q_n}^{(\alpha,\beta)} \left( \frac{(t-x)^2}{2} \bigl(f''(\xi) - f''(x)\bigr), x \right).
\end{aligned} \tag{2.2}
$$

For all $x,t \in [0,1]$, $|f''(\xi)-f''(x)| \le \omega_{f''}(\delta)(1+\frac{(t-x)^2}{\delta^2})$ for any $\delta > 0$. Therefore, it follows that

$$\left|[n]_{q_n}\tilde{S}_{n,q_n}^{(\alpha,\beta)}\big((t-x)^2\big(f''(\xi)-f''(x)\big),x\big)\right| \le \omega_{f''}(\delta)[n]_{q_n}\tilde{S}_{n,q_n}^{(\alpha,\beta)}\left((t-x)^2+\frac{(t-x)^4}{\delta^2},x\right). \quad (2.3)$$

Let $B_{n,q}^{(\alpha,\beta)}(f,x)=\sum_{k=0}^{n} p_{n,k}(q;x)f(\frac{[k]_q+\alpha}{[n]_q+\beta})$ be $q$-Bernstein-Stancu operators. It follows that

$$\begin{aligned}
&\tilde{S}_{n,q_n}^{(\alpha,\beta)}\big((t-x)^4,x\big)\\
&\quad=\big([n+1]_{q_n}+\beta\big)\sum_{k=0}^{n} q_n^{-k}p_{n,k}(q_n;x)\int_{\frac{[k]_{q_n}+\alpha}{[n+1]_{q_n}+\beta}}^{\frac{[k+1]_{q_n}+\alpha}{[n+1]_{q_n}+\beta}}(t-x)^4 d_{q_n}^R t\\
&\quad=\big([n+1]_{q_n}+\beta\big)\sum_{k=0}^{n} q_n^{-k}p_{n,k}(q_n;x)(1-q_n)\frac{[k+1]_{q_n}-[k]_{q_n}}{[n+1]_{q_n}+\beta}\\
&\qquad\times\sum_{j=0}^{\infty}\left(\frac{[k]_{q_n}+\alpha}{[n+1]_{q_n}+\beta}+\frac{[k+1]_{q_n}-[k]_{q_n}}{[n+1]_{q_n}+\beta}q_n^j-x\right)^4 q_n^j\\
&\quad=(1-q_n)\sum_{k=0}^{n} p_{n,k}(q_n;x)\sum_{j=0}^{\infty}\left(\frac{[k]_{q_n}+\alpha}{[n+1]_{q_n}+\beta}+\frac{q_n^k}{[n+1]_{q_n}+\beta}q_n^j-x\right)^4 q_n^j\\
&\quad\le 8(1-q_n)\sum_{k=0}^{n} p_{n,k}(q_n;x)\sum_{j=0}^{\infty}\left(\frac{[k]_{q_n}+\alpha}{[n+1]_{q_n}+\beta}-x\right)^4 q_n^j\\
&\qquad+8(1-q_n)\sum_{k=0}^{n} p_{n,k}(q_n;x)\sum_{j=0}^{\infty}\left(\frac{q_n^k}{[n+1]_{q_n}+\beta}\right)^4 q_n^{5j}\\
&\quad=8\sum_{k=0}^{n} p_{n,k}(q_n;x)\left(\frac{[k]_{q_n}+\alpha}{[n+1]_{q_n}+\beta}-x\right)^4\\
&\qquad+\frac{8}{1+q_n+q_n^2+q_n^3+q_n^4}\sum_{k=0}^{n} p_{n,k}(q_n;x)\left(\frac{q_n^k}{[n+1]_{q_n}+\beta}\right)^4\\
&\quad\le 8\sum_{k=0}^{n} p_{n,k}(q_n;x)\left(\frac{[k]_{q_n}+\alpha}{[n+1]_{q_n}+\beta}-\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}+\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}-x\right)^4\\
&\qquad+\frac{8}{1+q_n+q_n^2+q_n^3+q_n^4}\sum_{k=0}^{n} p_{n,k}(q_n;x)\left(\frac{q_n^k}{[n]_{q_n}+\beta}\right)^4\\
&\quad\le 64\sum_{k=0}^{n} p_{n,k}(q_n;x)\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}\right)^4\left(\frac{q_n^n}{[n+1]_{q_n}+\beta}\right)^4+64B_{n,q_n}^{(\alpha,\beta)}\big((t-x)^4,x\big)\\
&\qquad+8\sum_{k=0}^{n} p_{n,k}(q_n;x)\left(\frac{q_n^k}{[n]_{q_n}+\beta}\right)^4\\
&\quad\le 64\left(\frac{q_n^n}{[n+1]_{q_n}+\beta}\right)^4+64B_{n,q_n}^{(\alpha,\beta)}\big((t-x)^4,x\big)+8\sum_{k=0}^{n} p_{n,k}(q_n;x)\left(\frac{q_n^k}{[n]_{q_n}+\beta}\right)^4.
\end{aligned}$$

From [14], Corrollary 1, we have $B_{n,q}^{(\alpha,\beta)}((t-x)^r,x)=O(\frac{1}{[n]_q^{[\frac{r+1}{2}]}})$, where $x\in[0,1]$ and $[\alpha]$ denotes the integer part of $\alpha$. Also, we have

$$\begin{aligned}
&\sum_{k=0}^{n} p_{n,k}(q_n;x)\left(\frac{q_n^k}{[n]_{q_n}+\beta}\right)^4 \\
&= \frac{1}{([n]_{q_n}+\beta)^2}\sum_{k=0}^{n} p_{n,k}(q_n;x)\left(\frac{1-(1-q_n)[k]_{q_n}-(1-q_n)\alpha+(1-q_n)\alpha}{[n]_{q_n}+\beta}\right)^2 \\
&\leq \frac{2}{([n]_{q_n}+\beta)^2}\left\{\frac{(1+(1-q_n)\alpha)^2}{([n]_{q_n}+\beta)^2}+(1-q_n)^2\sum_{k=0}^{n} p_{n,k}(q_n;x)\left(\frac{[k]+\alpha}{[n]_{q_n}+\beta}\right)^2\right\} \\
&\leq \frac{2}{([n]_{q_n}+\beta)^2}\left\{\frac{(1+(1-q_n)\alpha)^2}{([n]_{q_n}+\beta)^2}+(1-q_n)^2 B_{n,q_n}^{(\alpha,\beta)}\left(t^2,x\right)\right\} \\
&= \frac{2}{([n]_{q_n}+\beta)^4}\left\{\left(1+(1-q_n)\alpha\right)^2+(1-q_n)^2\left[[n]_{q_n}^2x^2+[n]_{q_n}x(1-x)+2\alpha[n]_{q_n}x+\alpha^2\right]\right\} \\
&= O\left(\frac{1}{[n]_{q_n}^2}\right).
\end{aligned}$$

Therefore

$$\tilde{S}_{n,q_n}^{(\alpha,\beta)}\left((t-x)^4,x\right) = O\left(\frac{1}{[n]_{q_n}^2}\right). \tag{2.4}$$

In view of the Lemma 2.1 and the relation (2.4), we have

$$\left|[n]_{q_n}\tilde{S}_{n,q_n}^{(\alpha,\beta)}\left((t-x)^2\left(f''(\xi)-f''(x)\right),x\right)\right| \leq \omega_{f''}(\delta)\left(O(1)+\frac{1}{\delta^2}O\left(\frac{1}{[n]_{q_n}}\right)\right).$$

Choosing $\delta = [n]_{q_n}^{-1/2}$, we get

$$\left|[n]_{q_n}\tilde{S}_{n,q_n}^{(\alpha,\beta)}\left((t-x)^2\left(f''(\xi)-f''(x)\right),x\right)\right| = \omega_{f''}\left([n]_{q_n}^{-1/2}\right)O(1).$$

Hence

$$\lim_{n\to\infty}\left|[n]_{q_n}\tilde{S}_{n,q_n}^{(\alpha,\beta)}\left((t-x)^2\left(f''(\xi)-f''(x)\right),x\right)\right| = 0.$$

In view of Lemma 2.1, we obtain

$$\lim_{n\to\infty}[n]_{q_n}\left(\tilde{S}_{n,q_n}^{(\alpha,\beta)}(f,x)-f(x)\right) = \left(-\frac{1+a+2\beta}{2}x+\alpha+\frac{1}{2}\right)f'(x)+\frac{1}{2}\left(x(1-x)\right)f''(x). \qquad \square$$

## 3 Approximation properties of $q$-Stancu-Kantorovich operators

Recall that the first and second modulus of continuity of $f \in C[0,1]$ are defined, respectively, by

$$\omega(f,\delta) := \sup_{0<h<\delta;x,x+h\in[0,1]}\left|f(x+h)-f(x)\right|$$

and

$$\omega_2(f,\delta) := \sup_{0<h<\delta;x,x+2h\in[0,1]}\left|f(x+2h)-2f(x+h)+f(x)\right|, \quad \text{where } \delta > 0.$$

Let us consider the following $K$-functional:

$$K_2(f,\delta) := \inf\{\|f-g\| + \delta\|g''\|, g \in C^2[0,1]\}, \quad \text{where } \delta \geq 0.$$

It is well known (see [22]) that there exists an absolute constant $C > 0$ such that

$$K_2(f,\delta) \leq C\omega_2(f,\sqrt{\delta}). \tag{3.1}$$

Denote $a_n = \frac{2q}{[2]_q}\frac{[n]_q}{[n+1]_q+\beta}$, $b_n = \frac{\alpha}{[n+1]_q+\beta} + \frac{1}{[2]_q([n+1]_q+\beta)}$, $\delta_n(x) = x(1-x) + \frac{1}{[n]_q}$, $\nu(\alpha,\beta,x) = (2\alpha^2 + 2\beta^2 + 2\alpha + 4)\frac{[n+1]_q^2}{([n+1]_q+\beta)^2}\frac{1}{[n]_q}\delta_n(x)$, $\tilde{\nu}(\alpha,\beta,x) = (2\alpha^2 + \beta^2 + 4\alpha + 4)\frac{[n]_q}{([n]_q+\beta)^2}\delta_n(x)$.

**Theorem 3.1** *There exists an absolute constant $C > 0$ such that*

$$\left|S_{n,q}^{*(\alpha,\beta)}(f,x) - f(x)\right| \leq C\omega_2\left(f,\sqrt{\nu(\alpha,\beta,x)}\right) + \omega\left(f,\left|(a_n-1)x + b_n\right|\right),$$

*where $f \in C[0,1]$ and $0 < q < 1$.*

*Proof* Let

$$T_{n,q}^{*(\alpha,\beta)}(f,x) = S_{n,q}^{*(\alpha,\beta)}(f,x) + f(x) - f(a_nx + b_n), \quad \text{where } f \in C[0,1]. \tag{3.2}$$

Using the Taylor formula

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-s)g''(s)\,ds, \quad g \in C^2[0,1], \tag{3.3}$$

it follows that

$$T_{n,q}^{*(\alpha,\beta)}(g,x) = g(x) + S_{n,q}^{*(\alpha,\beta)}\left(\int_x^t (t-s)g''(s)\,ds, x\right) - \int_x^{a_nx+b_n} (a_nx + b_n - s)g''(s)\,ds,$$
$$g \in C^2[0,1].$$

Therefore

$$\begin{aligned}
&\left|T_{n,q}^{*(\alpha,\beta)}(g;x) - g(x)\right| \\
&\quad \leq S_{n,q}^{*(\alpha,\beta)}\left(\left|\int_x^t (t-s)g''(s)\,ds\right|, x\right) + \left|\int_x^{a_nx+b_n} |a_nx + b_n - s|\left|g''(s)\right|\,ds\right| \\
&\quad \leq \|g''\| S_{n,q}^{*(\alpha,\beta)}\left((t-x)^2, x\right) + \|g''\|(a_nx + b_n - x)^2 \\
&\quad \leq \|g''\| \frac{2[n+1]_q^2}{([n+1]_q+\beta)^2}\left\{\frac{4}{[n]_q}\left(x(1-x) + \frac{1}{[n]_q}\right) + \left(\frac{\alpha}{[n+1]_q} - \frac{\beta}{[n+1]_q}x\right)^2\right\} \\
&\qquad + \|g''\|\left(\frac{2q}{[2]_q}\frac{[n]_q}{[n+1]_q+\beta}x + \frac{\alpha}{[n+1]_q+\beta} + \frac{1}{[2]_q([n+1]_q+\beta)} - x\right)^2 \\
&\quad \leq \frac{[n+1]_q^2}{([n+1]_q+\beta)^2}\|g''\|\left\{\frac{8}{[n]_q}\left(x(1-x) + \frac{1}{[n]_q}\right) + \frac{4\alpha^2}{[n]_q^2} + \frac{4\beta^2}{[n]_q^2}x^2\right.
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{[n]_q^2}\left[2\left(\frac{2q}{[2]_q}[n]_q-[n+1]_q\right)^2x^2+2\left(\alpha+\frac{1}{[2]_q}-\beta x\right)^2\right]\bigg\}\\
&\le\frac{[n+1]_q^2}{([n+1]_q+\beta)^2}\frac{1}{[n]_q}\|g''\|\bigg\{8\left(x(1-x)+\frac{1}{[n]_q}\right)\\
&+\frac{4\alpha^2}{[n]_q}+\frac{4\beta^2}{[n]_q}x^2+\frac{2}{[n]_q}\left(\frac{1+q^{n+1}}{1+q}\right)^2x^2+\frac{4}{[n]_q}\left(\alpha+\frac{1}{[2]_q}\right)^2+\frac{4}{[n]_q}\beta^2x^2\bigg\}\\
&\le\frac{[n+1]_q^2}{([n+1]_q+\beta)^2}\frac{1}{[n]_q}\|g''\|\bigg\{8\left(x(1-x)+\frac{1}{[n]_q}\right)+\frac{4\alpha^2}{[n]_q}+\frac{4\beta^2}{[n]_q}x^2\\
&+\frac{2}{[n]_q}x^2+\frac{4}{[n]_q}(\alpha+1)^2+\frac{4}{[n]_q}\beta^2x^2\bigg\}\\
&\le 4\nu(\alpha,\beta,x)\|g''\|.
\end{aligned}
$$

Using the above relation we obtain

$$
\begin{aligned}
&\left|S_{n,q}^{*(\alpha,\beta)}(f;x)-f(x)\right|\\
&\quad\le\left|T_{n,q}^{*(\alpha,\beta)}(f-g;x)\right|+\left|T_{n,q}^{*(\alpha,\beta)}(g;x)-g(x)\right|+\left|f(x)-g(x)\right|+\left|f(a_nx+b_n)-f(x)\right|\\
&\quad\le 4\|f-g\|+4\nu(\alpha,\beta,x)\|g''\|+\omega\big(f,\left|(a_n-1)x+b_n\right|\big)\\
&\quad\le 4K_2\big(f,\nu(\alpha,\beta,x)\big)+\omega\big(f,\left|(a_n-1)x+b_n\right|\big)
\end{aligned}
$$

and using (3.1) the theorem is proved. □

**Theorem 3.2** *There exists an absolute constant $C>0$ such that*

$$
\left|\tilde{S}_{n,q}^{(\alpha,\beta)}(f,x)-f(x)\right|\le C\omega_2\big(f,\sqrt{\tilde{\nu}(\alpha,\beta,x)}\big)+\omega\big(f,\left|(a_n-1)x+b_n\right|\big),
$$

*where $f\in C[0,1]$ and $0<q<1$.*

*Proof* We have

$$
\begin{aligned}
&\tilde{S}_{n,q}^{(\alpha,\beta)}\big((t-x)^2,x\big)\\
&\quad=\big([n+1]_q+\beta\big)\sum_{k=0}^{n}q^{-k}p_{n,k}(q;x)\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}}(t-x)^2d_q^Rt\\
&\quad=\big([n+1]_q+\beta\big)\sum_{k=0}^{n}q^{-k}p_{n,k}(q;x)(1-q)\frac{[k+1]_q-[k]_q}{[n+1]_q+\beta}\\
&\qquad\times\sum_{j=0}^{\infty}\left(\frac{[k]_q+\alpha}{[n+1]_q+\beta}+\frac{[k+1]_q-[k]_q}{[n+1]_q+\beta}q^j-x\right)^2q^j\\
&\quad=(1-q)\sum_{k=0}^{n}p_{n,k}(q;x)\sum_{j=0}^{\infty}\left(\frac{[k]_q+\alpha}{[n+1]_q+\beta}+\frac{q^k}{[n+1]_q+\beta}q^j-x\right)^2q^j\\
&\quad\le 2(1-q)\sum_{k=0}^{n}p_{n,k}(q;x)\sum_{j=0}^{\infty}\left(\frac{[k]_q+\alpha}{[n+1]_q+\beta}-x\right)^2q^j
\end{aligned}
$$

$$\begin{aligned}
&+ 2(1-q)\sum_{k=0}^{n} p_{n,k}(q;x)\sum_{j=0}^{\infty}\left(\frac{q^k}{[n+1]_q+\beta}\right)^2 q^{3j} \\
&= 2\sum_{k=0}^{n} p_{n,k}(q;x)\left(\frac{[k]_q+\alpha}{[n+1]_q+\beta}-x\right)^2 + \frac{2}{1+q+q^2}\sum_{k=0}^{n} p_{n,k}(q;x)\left(\frac{q^k}{[n+1]_q+\beta}\right)^2 \\
&\le 2\sum_{k=0}^{n} p_{n,k}(q;x)\left(\frac{[k]_q+\alpha}{[n]_q+\beta}-x+\frac{[k]_q+\alpha}{[n+1]_q+\beta}-\frac{[k]_q+\alpha}{[n]_q+\beta}\right)^2 \\
&+ 2\sum_{k=0}^{n} p_{n,k}(q;x)\frac{1}{([n+1]_q+\beta)^2}.
\end{aligned}$$

Let $B_{n,k}^{(\alpha,\beta)}$ be $q$-Bernstein-Stancu operators. From [14], Lemma 1 and Lemma 4 the following properties of the $q$-Bernstein-Stancu operators hold:

$$\begin{aligned}
&B_{n,q}^{(\alpha,\beta)}\left(t^2,x\right) = \frac{1}{([n]_q+\beta)^2}\left([n]_q^2x^2+[n]_qx(1-x)+2\alpha[n]_qx+\alpha^2\right), \\
&B_{n,q}^{(\alpha,\beta)}\left((t-x)^2,x\right) \le \frac{[n]_q}{([n]_q+\beta)^2}x(1-x).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\tilde{S}_{n,q}^{(\alpha,\beta)}\left((t-x)^2;x\right) \\
&\quad\le 4B_{n,q}^{(\alpha,\beta)}\left((t-x)^2;x\right) + 4\sum_{k=0}^{n} p_{n,k}(q;x)\left(\frac{[k]_q+\alpha}{[n]_q+\beta}\right)^2\left(\frac{q^n}{[n+1]_q+\beta}\right)^2 \\
&\qquad+ \frac{2}{([n+1]_q+\beta)^2} \\
&\quad\le 4\frac{[n]_q}{([n]_q+\beta)^2}x(1-x) + \frac{4}{([n+1]_q+\beta)^2}B_{n,q}^{(\alpha,\beta)}\left(t^2;x\right) + \frac{2}{([n+1]_q+\beta)^2} \\
&\quad= \frac{[n]_q}{([n]_q+\beta)^2}\left\{4x(1-x) + \frac{4}{[n]_q}B_{n,q}^{(\alpha,\beta)}\left(t^2;x\right) + \frac{2}{[n]_q}\right\} \\
&\quad= \frac{[n]_q}{([n]_q+\beta)^2}\left\{4x(1-x) + \frac{4}{[n]_q}\cdot\frac{1}{([n]_q+\beta)^2}\left([n]_q^2x^2+[n]_qx(1-x)\right.\right. \\
&\qquad\left.\left.+\alpha^2+2\alpha[n]_qx\right) + \frac{2}{[n]_q}\right\} \\
&\quad\le \frac{[n]_q}{([n]_q+\beta)^2}\left\{4x(1-x) + \frac{4}{[n]_q}x^2 + \frac{4}{[n]_q}x(1-x) + \frac{4\alpha^2}{[n]_q} + \frac{8\alpha}{[n]_q}x + \frac{2}{[n]_q}\right\} \\
&\quad= \frac{[n]_q}{([n]_q+\beta)^2}\left\{4\delta_n(x) + \frac{1}{[n]_q}\left(4x^2+4\alpha^2+8\alpha x+2\right)\right\} \\
&\quad\le \frac{[n]_q}{([n]_q+\beta)^2}\delta_n(x)\left(4\alpha^2+8\alpha+10\right).
\end{aligned}$$

Let

$$\tilde{T}_{n,q}^{(\alpha,\beta)}(f,x) = \tilde{S}_{n,q}^{(\alpha,\beta)}(f,x) + f(x) - f(a_nx+b_n), \quad \text{where } f \in C[0,1]. \tag{3.4}$$

Using the Taylor formula (3.3) it follows that

$$\begin{aligned}&\tilde{T}_{n,q}^{(\alpha,\beta)}(g,x)=g(x)+\tilde{S}_{n,q}^{(\alpha,\beta)}\left(\int_x^t (t-s)g''(s)\,ds,x\right)-\int_x^{a_nx+b_n}(a_nx+b_n-s)g''(s)\,ds,\\ &g\in C^2[0,1].\end{aligned}$$

Therefore

$$\begin{aligned}&\left|\tilde{T}_{n,q}^{(\alpha,\beta)}(g;x)-g(x)\right|\\ &\quad\le \tilde{S}_{n,q}^{(\alpha,\beta)}\left(\left|\int_x^t (t-s)g''(s)\,ds\right|,x\right)+\left|\int_x^{a_nx+b_n}|a_nx+b_n-s|\left|g''(s)\right|ds\right|\\ &\quad\le \left\|g''\right\|\tilde{S}_{n,q}^{(\alpha,\beta)}\left((t-x)^2,x\right)+\left\|g''\right\|(a_nx+b_n-x)^2\\ &\quad\le \frac{[n]_q}{([n]_q+\beta)^2}\delta_n(x)\left(4\alpha^2+8\alpha+10\right)\left\|g''\right\|\\ &\qquad+\left\|g''\right\|\left(\frac{2q}{[2]_q}\frac{[n]_q}{[n+1]_q+\beta}x+\frac{\alpha}{[n+1]_q+\beta}+\frac{1}{[2]_q([n+1]_q+\beta)}-x\right)^2\\ &\quad\le \frac{[n]_q}{([n]_q+\beta)^2}\delta_n(x)\left(4\alpha^2+8\alpha+10\right)\left\|g''\right\|\\ &\qquad+\left\|g''\right\|\frac{2}{([n]_q+\beta)^2}\left[\left(\frac{1+q^{n+1}}{1+q}\right)^2x^2+2\left(\alpha+\frac{1}{[2]_q}\right)^2+2\beta^2x^2\right]\\ &\quad\le \frac{[n]_q}{([n]_q+\beta)^2}\delta_n(x)\left(4\alpha^2+8\alpha+10\right)\left\|g''\right\|+\left\|g''\right\|\frac{2}{([n]_q+\beta)^2}\left[x^2+2(\alpha+1)^2+2\beta^2x^2\right]\\ &\quad\le \frac{[n]_q}{([n]_q+\beta)^2}\delta_n(x)\left(8\alpha^2+4\beta^2+16\alpha+16\right)\left\|g''\right\|=4\tilde{\nu}(\alpha,\beta,x)\left\|g''\right\|.\end{aligned}$$

Using the above relation we obtain

$$\begin{aligned}&\left|\tilde{S}_{n,q}^{(\alpha,\beta)}(f;x)-f(x)\right|\\ &\quad\le \left|\tilde{T}_{n,q}^{(\alpha,\beta)}(f-g;x)\right|+\left|\tilde{T}_{n,q}^{(\alpha,\beta)}(g;x)-g(x)\right|+\left|f(x)-g(x)\right|+\left|f(a_nx+b_n)-f(x)\right|\\ &\quad\le 4\|f-g\|+4\tilde{\nu}(\alpha,\beta,x)\left\|g''\right\|+\omega\left(f,\left|(a_n-1)x+b_n\right|\right)\\ &\quad\le 4K_2\left(f,\tilde{\nu}(\alpha,\beta,x)\right)+\omega\left(f,\left|(a_n-1)x+b_n\right|\right).\end{aligned}$$

and using (3.1) the theorem is proved. □

In order to start the next result we need the second order Ditzian-Totik modulus [22] defined by

$$\omega_{2,\Phi}(f,\delta):=\sup_{0<h\le\delta}\sup_{x\pm h\Phi(x)\in[0,1]}\left|f\left(x-\Phi(x)h\right)-2f(x)+f(x+\Phi(x)h\right|,\quad f\in C[0,1],$$

in which $\Phi:[0,1]\to\mathbb{R}$ is an admissible step-weight function.

The weighted $K$-functional of second order for $f\in C[0,1]$ is defined by

$$K_{2,\Phi}(f,\delta):=\inf\left\{\|f-g\|+\delta\left\|\Phi^2g''\right\|,g\in W^2(\Phi)\right\},\quad \delta\ge 0,$$

where

$$W^2(\Phi) := \{g \in C[0,1] \mid g' \in AC[0,1], \Phi^2 g'' \in C[0,1]\}$$

and

$$AC[0,1] := \{h \mid h \text{ is absolutely continuous in } [a,b], \text{ for every } 0 < a < b < 1\}.$$

It is well known that the $K$-functional $K_{2,\Phi}(f,\delta)$ and the Ditzian-Totik modulus $\omega_{2,\Phi}(f,\sqrt{\delta})$ are equivalent (see [22]).

Denote

$$\overrightarrow{\omega}_\psi(f,\delta) = \sup_{|h|\le\delta} \sup_{x,x+h\psi(x)\in[0,1]} \left|f\big(x+\psi(x)h\big) - f(x)\right|.$$

**Theorem 3.3** *Let $\Phi$ be an admissible step-weight function of the Ditzian-Totik modulus of smoothness such that $\Phi^2$ is concave and $\Phi \neq 0$. Then there exists an absolute constant $C > 0$ such that*

$$\left|S_{n,q}^{*(\alpha,\beta)}(f;x) - f(x)\right| \le C\omega_{2,\Phi}\left(f, \frac{\sqrt{\nu(\alpha,\beta,x)}}{\Phi(x)}\right) + \overrightarrow{\omega}_\psi\left(f, \frac{1}{[2]_q([n+1]_q+\beta)}\right),$$

*where $f \in C[0,1]$, $0 < q < 1$, and $\psi(x) = (2+[2]_q\beta)x + [2]_q\alpha + 1$, $x \in [0,1]$.*

*Proof* Applying the operators $T_{n,q}^{*(\alpha,\beta)}$ defined in (3.2) to Taylor's formula in a similar way to the proof of Theorem 3.1 we obtain

$$\begin{aligned}
&\left|T_{n,q}^{*(\alpha,\beta)}(g,x) - g(x)\right| \\
&\quad\le S_{n,q}^{*(\alpha,\beta)}\left(\left|\int_x^t (t-s)g''(s)\,ds\right|, x\right) + \left|\int_x^{a_nx+b_n} |a_nx+b_n-s|\left|g''(s)\right|ds\right| \\
&\quad\le \|\Phi^2 g''\| S_{n,q}^{*(\alpha,\beta)}\left(\left|\int_x^t \frac{|t-s|}{\Phi^2(s)}\,ds\right|, x\right) + \|\Phi^2 g''\|\left|\int_x^{a_nx+b_n} \frac{|a_nx+b_n-s|}{\Phi^2(s)}\,ds\right|.
\end{aligned}$$

Let $s = \tau x + (1-\tau)t$, $\tau \in [0,1]$. Since $\Phi^2$ is concave on $[0,1]$ it follows that

$$\Phi^2(s) \ge \tau\Phi^2(x) + (1-\tau)\Phi^2(t)$$

and

$$\frac{|t-s|}{\Phi^2(s)} = \frac{\tau|x-t|}{\Phi^2(s)} \le \frac{\tau|x-t|}{\tau\Phi^2(x) + (1-\tau)\Phi^2(t)} \le \frac{|x-t|}{\Phi^2(x)}.$$

Therefore

$$\begin{aligned}
\left|T_{n,q}^{*(\alpha,\beta)}(g,x) - g(x)\right| &\le \frac{\|\Phi^2 g''\|}{\Phi^2(x)}\left[S_{n,q}^{*(\alpha,\beta)}\big((t-x)^2;x\big) + (a_nx+b_n-x)^2\right] \\
&\le 4\frac{\|\Phi^2 g''\|}{\Phi^2(x)}\nu_n(\alpha,\beta,x).
\end{aligned}$$

Using the above relation we obtain

$$\begin{aligned}
&\left|S_{n,q}^{*(\alpha,\beta)}(f;x)-f(x)\right| \\
&\quad\leq \left|T_{n,q}^{*(\alpha,\beta)}(f-g;x)\right| + \left|T_{n,q}^{*(\alpha,\beta)}(g;x)-g(x)\right| + \left|f(x)-g(x)\right| + \left|f(a_nx+b_n)-f(x)\right| \\
&\quad\leq 4\|f-g\| + 4\frac{\|\Phi^2 g''\|}{\Phi^2(x)}\nu_n(\alpha,\beta,x) + \left|f(a_nx+b_n)-f(x)\right| \\
&\quad= 4K_{2,\Phi}\left(f,\frac{\nu(\alpha,\beta,x)}{\Phi^2(x)}\right) + \left|f(a_nx+b_n)-f(x)\right|.
\end{aligned}$$

Also, we have

$$\begin{aligned}
\left|f(a_nx+b_n)-f(x)\right| &= \left|f\left(x+\psi(x)\frac{(a_n-1)x+b_n}{\psi(x)}\right)-f(x)\right| \\
&\leq \sup\left|f\left(x+\psi(x)\frac{S_n^{*(\alpha,\beta)}((t-x),x)}{\psi(x)}\right)-f(x)\right| \\
&\leq \overrightarrow{\omega}_\psi\left(f,\frac{1}{\psi(x)}\left|\frac{-1-q^{n+1}-[2]_q\beta}{[2]_q([n+1]_q+\beta)}x+\frac{[2]_q\alpha+1}{[2]_q([n+1]_q+\beta)}\right|\right) \\
&\leq \overrightarrow{\omega}_\psi\left(f,\frac{1}{[2]_q([n+1]_q+\beta)}\right).
\end{aligned}$$

Therefore

$$\left|S_{n,q}^{*(\alpha,\beta)}(f,x)-f(x)\right| \leq 4K_{2,\Phi}\left(f,\frac{\nu(\alpha,\beta,x)}{\Phi^2(x)}\right) + \overrightarrow{\omega}_\psi\left(f,\frac{1}{[2]_q([n+1]_q+\beta)}\right).$$

Using the equivalence of the $K$-functional and the Ditzian-Totik modulus we get the desired estimate. □

In a similar way can be obtained the following result for the $q$-Stancu-Kantorovich operators $\tilde{S}_{n,q}^{(\alpha,\beta)}$.

**Theorem 3.4** *Let $\Phi$ be an admissible step-weight function of the Ditzian-Totik modulus of smoothness such that $\Phi^2$ is concave and $\Phi \neq 0$. Then there exists an absolute constant $C > 0$ such that*

$$\left|\tilde{S}_{n,q}^{(\alpha,\beta)}(f;x)-f(x)\right| \leq C\omega_{2,\Phi}\left(f,\frac{\sqrt{\tilde{\nu}(\alpha,\beta,x)}}{\Phi(x)}\right) + \overrightarrow{\omega}_\psi\left(f,\frac{1}{[2]_q([n+1]_q+\beta)}\right),$$

*where $f \in C[0,1]$, $0 < q < 1$, and $\psi(x) = (2+[2]_q\beta)x + [2]_q\alpha + 1$, $x \in [0,1]$.*

## 4 Statistical approximation of Korovkin type

The concept of statistical convergence was introduced by Fast [23] and Steinhaus [24] and recently has become an important area in approximation theory. The goal of this section is to obtain the statistical convergence properties of the Stancu-Kantorovich operators (1.3) and (1.5).

Let set $K \subseteq N$ and $K_n = \{k \leq n; k \in K\}$, the natural density of $K$ is defined by $\delta(K) := \lim_{n\to\infty}\frac{1}{n}|K_n|$ if the limit exists, where $|K_n|$ denote the cardinality of the set $K_n$.

A sequence $x = \{x_n\}$ is called statistically convergent to a number $L$, if for every $\epsilon > 0$, $\delta\{k \in N : |x_k - L| \geq \epsilon\} = 0$. This convergence is denoted as $\text{st-lim}_k x_k = L$.

In [25] Gadjiev and Orhan proved the following Bohman-Korovkin-type approximation theorem for statistical convergence.

**Theorem 4.1** ([25]) *If the sequence of positive linear operators $A_n : C[a,b] \to C[a,b]$ satisfies the conditions $st\text{-}\lim_n \|A_n(e_i) - e_i\|_{C[a,b]} = 0$ with $e_i(t) = t^i$, $i = 0,1,2$, then for any function $f \in C[a,b]$ we have*

$$st\text{-}\lim_n \bigl\|A_n(f) - f\bigr\|_{C[a,b]} = 0.$$

**Theorem 4.2** *Let $(q_n)$, $0 < q_n < 1$ be a sequence that satisfies $st\text{-}\lim_n q_n = 1$, $st\text{-}\lim_n q_n^n = a \in (0,1)$. Then for all $f \in C[0,1]$ we have*

$$st\text{-}\lim_n \bigl\|S_{n,q_n}^{*(\alpha,\beta)}(f,\cdot) - f\bigr\|_{C[0,1]} = 0.$$

*Proof* It is necessary to prove $\text{st-lim}_n \|S_{n,q_n}^{*(\alpha,\beta)}(e_i,\cdot) - e_i\|_{C[0,1]} = 0$, for $i = 0,1,2$, and the proof follows from Theorem 4.1. For the first equality it is clear from Lemma 1.2 that

$$\text{st-}\lim_n \bigl\|S_{n,q_n}^{*(\alpha,\beta)}(e_0,\cdot) - e_0\bigr\|_{C[0,1]} = 0. \tag{4.1}$$

For the second equality we have

$$\bigl\|S_{n,q_n}^{*(\alpha,\beta)}(e_1,\cdot) - e_1\bigr\|_{C[0,1]} \leq \left|\frac{2q_n}{[2]_q} \cdot \frac{[n]_q}{[n+1]_q + \beta} - 1\right| + \left|\frac{\alpha}{[n+1]_q + \beta} + \frac{1}{[2]_q([n+1]_q + \beta)}\right|.$$

We denote $\nu_n = |\frac{2q_n}{[2]_q} \cdot \frac{[n]_q}{[n+1]_q+\beta} - 1|$ and $\mu_n = \frac{\alpha}{[n+1]_q+\beta} + \frac{1}{[2]_q([n+1]_q+\beta)}$.

From $\text{st-lim}_n q_n = 1$ and $\text{st-lim}_n q_n^n = a \in (0,1)$ we have

$$\text{st-}\lim_n \nu_n = \text{st-}\lim_n \mu_n = 0. \tag{4.2}$$

Now, for a given $\epsilon > 0$, we define the following sets:

$$A := \bigl\{n \in \mathbb{N} \mid \bigl\|S_{n,q_n}^{*(\alpha,\beta)}(e_1,\cdot) - e_1\bigr\|_{C[0,1]} \geq \epsilon\bigr\},$$
$$A_1 := \left\{n \in \mathbb{N} \,\middle|\, \nu_n \geq \frac{\epsilon}{2}\right\} \quad \text{and} \quad A_2 := \left\{n \in \mathbb{N} \,\middle|\, \mu_n \geq \frac{\epsilon}{2}\right\}.$$

It is obvious that $A \subseteq A_1 \cup A_2$, which implies that $\delta(A) \leq \delta(A_1) + \delta(A_2)$. From (4.2) we find that the right hand side of the above inequality is zero and we get finally

$$\text{st-}\lim_n \bigl\|S_{n,q_n}^{*(\alpha,\beta)}(e_1,\cdot) - e_1\bigr\|_{C[0,1]} = 0. \tag{4.3}$$

In a similar way it can be proved that

$$\text{st-}\lim_n \bigl\|S_{n,q_n}^{*(\alpha,\beta)}(e_2,\cdot) - e_2\bigr\|_{C[0,1]} = 0. \tag{4.4}$$

From (4.1), (4.3), and (4.4), the statement of our theorem follows from the Korovkin-type statistical approximation theorem. □

A statistical approximation property of the $q$-Kantorovich-Stancu operators $\tilde{S}^{(\alpha,\beta)}$ is obtained in the following theorem.

**Theorem 4.3** *Let* $(q_n)$, $0 < q_n < 1$ *be a sequence that satisfies* $st\text{-}\lim_n q_n = 1$, $st\text{-}\lim_n q_n^n = a \in (0,1)$. *Then for all* $f \in C[0,1]$ *we have*

$$st\text{-}\lim_n \left\| \tilde{S}_{n,q_n}^{(\alpha,\beta)}(f,\cdot) - f \right\|_{C[0,1]} = 0.$$

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors read and approved the final manuscript.

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