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A new type of multivalued contraction in partial Hausdorff metric spaces endowed with a graph

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Abstract

A generalization, namely the K-comparison function, of a comparison function is introduced. Using K-comparison functions we introduce K_G-contractive mappings. We obtain a fixed point theorem for such mappings on partial Hausdorff metric spaces endowed with a graph. We also construct examples in support of our results.

MSC: 47H10; 54H25

Keywords: K-comparison function; K_G-contractive mapping; partial metric space; partial Hausdorff metric space

1 Introduction

Development in metric fixed point theory is based on two things: the first is to modify contraction condition and the second is to modify the structure of a metric space. Matthews [1] introduced the notion of a partial metric space and extended Banach contraction principle in the setting of partial metric space. The work of Matthews [1] has been extended by many authors; see for example [2–17]. Using the notion of a partial metric on a set X, Aydi *et al.* [18] defined a partial Hausdorff metric on the set of closed and bounded subsets of the set X. Moreover, they [18] extended Nadler's fixed point theorem in the setting of a partial Hausdorff metric spaces. Jachymaski [19] generalized the Banach fixed point theorem for mappings of a complete metric space endowed with a graph. He introduced the notion of Banach G-contractions. Here G stands for a directed graph in a metric space whose vertex set coincides with the metric space. Many authors extended the Banach G-contraction in different ways; see, for example, [19–28].

In this paper, we introduce the notions of a K-comparison function and a K_G-contractive mapping. We establish a fixed point theorem for K_G-contractive mappings, in the setting of partial Hausdorff metric spaces endowed with a graph.

2 Preliminaries

In this section we recollect some definitions from partial (Hausdorff) metric spaces and comparison functions. We also present some results from partial (Hausdorff) metric spaces for ready reference. Throughout this paper, $\mathbb{R}^+ = [0, \infty)$.

Definition 2.1 [1] Let X be a nonempty set. A mapping $p : X \times X \to \mathbb{R}^+$ is a partial metric on X, if for all $x, y, z \in X$. We have

(P1) $p(x,x) = p(y,y) = p(x,y)$ if and only if $x = y$;
(P2) $p(x,x) \leq p(x,y)$;
(P3) $p(x,y) = p(y,x)$;
(P4) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

Remark 2.2 [18] If $p(x,y) = 0$, then (P1) and (P2) implies $x = y$ but the converse is not true in general.

Example 2.3 [1] Let X be the set of all closed intervals of real line $\mathbb{R}$, that is, $X = \{[a,b] : a, b \in \mathbb{R}, a \leq b\}$ and define a function $p : X \times X \to \mathbb{R}^+$ by $p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$, then (X,p) is a partial metric space.

Lemma 2.4 [1] *Every metric space is a partial metric space.*

Remark 2.5 [1] Every partial metric p on X generates a T_0 topology τ_p on X with as a base the family of the open balls (p balls) $\{B_p(x,\epsilon) : x \in X, \epsilon > 0\}$, where

$$B_p(x,\epsilon) = \{y \in X : p(x,y) < p(x,x) + \epsilon\}.$$

Definition 2.6 [1] Let (X,p) be a partial metric space. Then:

(a) A sequence $\{x_n\}$ in (X,p) is said to be convergent to a point $x \in X$ with respect to τ_p if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$.
(b) A sequence $\{x_n\}$ in X will be a Cauchy sequence if and only if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists and is finite.
(c) A partial metric space (X,p) is called a complete partial metric space if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p to a point $x \in X$.

Remark 2.7 [1] Let (X,p) be a partial metric space, then the function $d_p : X \times X \to [0,\infty)$ defined by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

is a metric on X.

Lemma 2.8 [1] *Let (X,p) be a partial metric space, a sequence $\{x_n\}$ in (X,d_p) is said to be convergent to a point $x \in X$ if and only if*

$$p(x,x) = \lim_{n\to\infty} p(x,x_n) = \lim_{n,m\to\infty} p(x_n,x_m).$$

Lemma 2.9 [18] *Let (X,p) be a partial metric space. Then*:

(a) *A sequence $\{x_n\}$ in X is Cauchy with respect to p if and only if it is Cauchy with respect to d_p.*
(b) *A partial metric space (X,p) is complete if and only if the metric space (X,d_p) is complete.*

A subset A of a partial metric space (X,p) is a bounded [18], if there exists $x_0 \in A$ such that $p(x_0,a) < p(x_0,x_0) + M$. A subset A of a partial metric space is closed if it is closed with respect to the topology τ_p on X. Let $\mathrm{CB}_p(X)$ be the family of all nonempty closed and bounded subsets of a partial metric space (X,p). We use the following notions and terminologies [18]. For $A,B \in \mathrm{CB}_p(X)$, $p(x,A) = \inf\{p(x,a) : a \in A\}$, $p(A,B) = \inf\{p(x,y) : x \in A, y \in B\}$. The functions $\delta_p : \mathrm{CB}_p(X) \times \mathrm{CB}_p(X) \to \mathbb{R}^+$ and $H_p : \mathrm{CB}_p(X) \times \mathrm{CB}_p(X) \to \mathbb{R}^+$ are defined by $\delta_p(A,B) = \sup\{p(a,B) : a \in A\}$ and $H_p(A,B) = \max\{\delta_p(A,B), \delta_p(B,A)\}$, respectively.

Remark 2.10 [18] If $d_p(x,A) = \inf\{d_p(x,a) : a \in A\}$, then it is easy to prove that $p(x,A) = 0$ implies that $d_p(x,A) = 0$.

Lemma 2.11 [18] *Let (X,p) be a partial metric space and A be any nonempty subset of X, then $a \in \overline{A}$ if and only if $p(a,A) = p(a,a)$.*

Lemma 2.12 [18] *Let (X,p) be a partial metric space and A be any nonempty subset of X. If A is closed in (X,p), then A is closed in (X,d_p).*

Proposition 2.13 [18] *Let (X,p) be a partial metric space. For $A,B \in \mathrm{CB}_p(X)$, the following properties hold*:

(1) $H_p(A,A) \leq H_p(A,B)$;
(2) $H_p(A,B) = H_p(B,A)$;
(3) $H_p(A,C) \leq H_p(A,B) + H_p(B,C) - \inf_{c \in C} p(c,c)$;
(4) $H_p(A,B) = 0$ *implies that* $A = B$.

Lemma 2.14 [18] *Let (X,p) be a partial metric space, let $A,B \in \mathrm{CB}_p(X)$ and $h > 1$. For any $a \in A$, there exists $b \in B$ such that $p(a,b) \leq hH_p(A,B)$.*

Let $\xi : [0,\infty) \to [0,\infty)$ be a function. Consider the following conditions:

(i) ξ is an increasing function;
(ii) $\xi(t) < t$ for each $t > 0$;
(iii) $\xi(0) = 0$;
(iv) $\{\xi^n(t)\}$ converges to 0 for each $t \geq 0$;
(v) $\sum_{n=0}^{\infty} \xi^n(t)$ converges for each $t > 0$.

The function ξ satisfying (i) and (iv) is said to be a comparison function [29]. The function ξ satisfying (i) and (v) is known as a (c)-comparison function [29]. It is easily seen that (i) and (iv) imply (ii); and (i) and (ii) imply (iii) [29].

Property (A): ([19], Remark 3.1) For any sequence $\{x_n\}_{n\in\mathbb{N}}$ in X, if $x_n \to x$ and $(x_n,x_{n+1}) \in E$ for $n \in \mathbb{N}$, then $(x_n,x) \in E$.

3 Main results

We begin this section by introducing the notion of a K-comparison function.

Definition 3.1 A mapping $\zeta : [0,\infty) \to [0,\infty)$ is said to be a K-comparison if the following conditions hold:

(i) for each $t > 0$, we have $\zeta(t) < t$;
(ii) $\zeta(0) = 0$.

Note that any comparison or (c)-comparison function is K-comparison function but converse is not true in general.

Example 3.2 Let $\zeta : [0,\infty) \to [0,\infty)$ be a mapping such that

$$\zeta(t) = \begin{cases} \frac{t}{2} & \text{if } 0 \leq t \leq 2, \\ \sqrt{t} & \text{otherwise.} \end{cases}$$

Thus, ζ is a K-comparison function but neither a comparison nor a (c)-comparison function.

We denote the class of K-comparison functions by $\mathfrak{K}$. Throughout this section, (X,p) is a partial metric space, $G = (V,E)$ is a directed graph without parallel edges such that $V = X$ and $\triangle = \{(x,x) : x \in X\} \subset E$. For basic terminologies of graph theory we refer the reader to the excellent text by Chartrand *et al.* [30].

Definition 3.3 Let (X,p) be a partial metric space. A mapping $T : X \to \mathrm{CB}_p(X)$ is said to be K_G-contractive, if there exists $\zeta \in \mathfrak{K}$ with $\sup_{t>0} \frac{\zeta(t)}{t} < 1$ such that

(i) for each $(x,y) \in E$ with $x \neq y$, we have

$$H_p(Tx, Ty) \leq \zeta\left(\max\left\{p(x,y), \frac{p(x,Tx)+p(y,Ty)}{2}, \frac{p(x,Ty)+p(y,Tx)}{2}\right\}\right); \tag{3.1}$$

(ii) if $s \in Tx$ and $t \in Ty$ are such that

$$p(s,t) < p(x,y), \tag{3.2}$$

then we have $(s,t) \in E$, whenever $(x,y) \in E$ with $x \neq y$.

Theorem 3.4 *Let (X,p) be a complete partial metric space endowed with the graph G and Property* (A). *Let $T : X \to \mathrm{CB}_p(X)$ be a K_G-contractive mapping. Assume that there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0,x_1) \in E$. Then T has a fixed point.*

Proof By hypothesis, we have $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0,x_1) \in E$. If $x_0 = x_1$, then x_0 is a fixed point. Suppose that $x_0 \neq x_1$. Since T is a K_G-contractive mapping, from (3.1), we have

$$\begin{aligned} H_p(Tx_0, Tx_1) &\leq \zeta\left(\max\left\{p(x_0,x_1), \frac{p(x_0,Tx_0)+p(x_1,Tx_1)}{2}, \frac{p(x_0,Tx_1)+p(x_1,Tx_0)}{2}\right\}\right) \\ &< \max\left\{p(x_0,x_1), \frac{p(x_0,Tx_0)+p(x_1,Tx_1)}{2}, \frac{p(x_0,Tx_1)+p(x_1,Tx_0)}{2}\right\}. \end{aligned} \tag{3.3}$$

Then there exists $a_1 \in (0,l]$, where $l = \sup_{t>0} \frac{\zeta(t)}{t}$, and obviously a_1 depends on x_0 and x_1, such that

$$\begin{aligned} &H_p(Tx_0, Tx_1) \\ &\quad\leq a_1 \max\left\{p(x_0,x_1), \frac{p(x_0,Tx_0)+p(x_1,Tx_1)}{2}, \frac{p(x_0,Tx_1)+p(x_1,Tx_0)}{2}\right\}. \end{aligned} \tag{3.4}$$

Since $a_1 < 1$, then $1/\sqrt{a_1} > 1$. Thus, by using Lemma 2.14, we have $x_2 \in Tx_1$ such that

$$p(x_1,x_2) \leq \frac{1}{\sqrt{a_1}} H_p(Tx_0, Tx_1). \tag{3.5}$$

From (3.4) and (3.5), we get

$$\begin{aligned} p(x_1,x_2) &\leq \sqrt{a_1} \max\left\{p(x_0,x_1), \frac{p(x_0,x_1)+p(x_1,x_2)}{2}, \frac{p(x_0,x_2)+p(x_1,x_1)}{2}\right\} \\ &\leq \sqrt{a_1} \max\bigl\{p(x_0,x_1), p(x_1,x_2)\bigr\}. \end{aligned} \tag{3.6}$$

If we assume that $\max\{p(x_0,x_1), p(x_1,x_2)\} = p(x_1,x_2)$, then we get a contradiction to (3.6). Thus, $\max\{p(x_0,x_1), p(x_1,x_2)\} = p(x_0,x_1)$. From (3.6), we have

$$p(x_1,x_2) \leq \sqrt{a_1} p(x_0,x_1) < p(x_0,x_1). \tag{3.7}$$

From (3.2) and (3.7), we have $(x_1,x_2) \in E$. If $x_1 = x_2$, then x_1 is a fixed point. Suppose that $x_1 \neq x_2$. Again, from (3.1), we have

$$\begin{aligned} H_p(Tx_1, Tx_2) &\leq \zeta\left(\max\left\{p(x_1,x_2), \frac{p(x_1,Tx_1)+p(x_2,Tx_2)}{2}, \frac{p(x_1,Tx_2)+p(x_2,Tx_1)}{2}\right\}\right) \\ &< \max\left\{p(x_1,x_2), \frac{p(x_1,Tx_1)+p(x_2,Tx_2)}{2}, \frac{p(x_1,Tx_2)+p(x_2,Tx_1)}{2}\right\}. \end{aligned}$$

Then there exists $a_2 \in (0,l]$, and obviously a_2 depends on x_1 and x_2, such that

$$\begin{aligned} &H_p(Tx_1, Tx_2) \\ &\quad \leq a_2 \max\left\{p(x_1,x_2), \frac{p(x_1,Tx_1)+p(x_2,Tx_2)}{2}, \frac{p(x_1,Tx_2)+p(x_2,Tx_1)}{2}\right\}. \end{aligned} \tag{3.8}$$

Since $a_2 < 1$, $1/\sqrt{a_2} > 1$. Again by using Lemma 2.14, we have $x_3 \in Tx_2$ such that

$$p(x_2,x_3) \leq \frac{1}{\sqrt{a_2}} H_p(Tx_1, Tx_2). \tag{3.9}$$

From (3.8) and (3.9), we get

$$\begin{aligned} p(x_2,x_3) &\leq \sqrt{a_1} \max\left\{p(x_1,x_2), \frac{p(x_1,x_2)+p(x_2,x_3)}{2}, \frac{p(x_1,x_3)+p(x_2,x_2)}{2}\right\} \\ &\leq \sqrt{a_1} \max\bigl\{p(x_1,x_2), p(x_2,x_3)\bigr\}. \end{aligned} \tag{3.10}$$

If we assume that $\max\{p(x_1,x_2), p(x_2,x_3)\} = p(x_2,x_3)$, then we get a contradiction to (3.10). Thus, $\max\{p(x_1,x_2), p(x_2,x_3)\} = p(x_1,x_2)$. From (3.10), we have

$$p(x_2,x_3) \leq \sqrt{a_2} p(x_1,x_2) < p(x_1,x_2). \tag{3.11}$$

Also, we have

$$p(x_2,x_3) \leq \sqrt{a_2} p(x_1,x_2) \leq \sqrt{a_2}\sqrt{a_1} p(x_0,x_1).$$

Continuing the same way we get sequences $\{a_n\} \subset (0,l]$ and $\{x_n\} \subset X$ such that $x_{n-1} \in Tx_n$, $x_{n-1} \neq x_n$, and $(x_{n-1},x_n) \in E$, with

$$p(x_n,x_{n+1}) \leq \sqrt{a_n}\sqrt{a_{n-1}}\cdots\sqrt{a_1}p(x_0,x_1) \quad \text{for each } n \in \mathbb{N}.$$

Let $n,m \in \mathbb{N}$, by using the triangular inequality, we have

$$\begin{aligned} p(x_n,x_{n+m}) &\leq p(x_n,x_{n+1}) + p(x_{n+1},x_{n+2}) + \cdots + p(x_{n+m-1},x_{n+m}) - \sum_{i=n+1}^{n+m-1} p(x_i,x_i) \\ &\leq p(x_n,x_{n+1}) + p(x_{n+1},x_{n+2}) + \cdots + p(x_{n+m-1},x_{n+m}) \\ &\leq \sqrt{a_n}\sqrt{a_{n-1}}\cdots\sqrt{a_1}p(x_0,x_1) + \sqrt{a_{n+1}}\sqrt{a_n}\cdots\sqrt{a_1}p(x_0,x_1) \\ &\quad + \cdots + \sqrt{a_{n+m-1}}\sqrt{a_{n+m-2}}\cdots\sqrt{a_1}p(x_0,x_1). \end{aligned} \tag{3.12}$$

Let $b = \sup\{\sqrt{a_i} : i \in \mathbb{N}\}$, clearly, $b < 1$. Then from (3.12), we get

$$\begin{aligned} p(x_n,x_{n+m}) &\leq \sqrt{a_n}\sqrt{a_{n-1}}\cdots\sqrt{a_1}p(x_0,x_1) + \sqrt{a_{n+1}}\sqrt{a_n}\cdots\sqrt{a_1}p(x_0,x_1) \\ &\quad + \cdots + \sqrt{a_{n+m-1}}\sqrt{a_{n+m-2}}\cdots\sqrt{a_1}p(x_0,x_1) \\ &\leq \left[b^n + b^{n+1} + \cdots + b^{n+m-1}\right]p(x_0,x_1) \\ &< \frac{b^n}{1-b}p(x_0,x_1). \end{aligned} \tag{3.13}$$

Consequently, we have

$$d_p(x_n,x_{n+m}) \leq 2p(x_n,x_{n+m}) < \frac{2b^n}{1-b}p(x_0,x_1).$$

Thus, we conclude that $\{x_n\}$ is a Cauchy sequence in (X,d_p). Since (X,p) is a complete partial metric space, by Lemma 2.9(b), (X,d_p) is a complete metric space. Then there exists $x^* \in X$ such that $x_n \to x^* \in X$ with respect to d_p, as $n \to \infty$. By Lemma 2.8, we have

$$p\left(x^*,x^*\right) = \lim_{n\to\infty} p\left(x_n,x^*\right) = \lim_{n\to\infty} p(x_n,x_{n+m}) = 0. \tag{3.14}$$

By Property (A), we have $(x_n,x^*) \in E$ for each $n \in \mathbb{N}$. Now, we claim that $p(x^*,Tx^*) = 0$. On the contrary suppose that $p(x^*,Tx^*) > 0$. By using the triangular inequality and (3.1), we have

$$\begin{aligned} p\left(x^*,Tx^*\right) &\leq p\left(x^*,x_{n+1}\right) + p\left(x_{n+1},Tx^*\right) - p(x_{n+1},x_{n+1}) \\ &\leq p\left(x^*,x_{n+1}\right) + H_p\left(Tx_n,Tx^*\right) \\ &\leq p\left(x^*,x_{n+1}\right) + \zeta\left(\max\left\{p\left(x_n,x^*\right), \frac{p(x_n,Tx_n) + p(x^*,Tx^*)}{2},\right.\right. \\ &\quad \left.\left.\frac{p(x_n,Tx^*) + p(x^*,Tx_n)}{2}\right\}\right) \\ &< p\left(x^*,x_{n+1}\right) + \max\left\{p\left(x_n,x^*\right), \frac{p(x_n,Tx_n) + p(x^*,Tx^*)}{2},\right. \end{aligned}$$

$$\left.\frac{p(x_n, Tx^*) + p(x^*, Tx_n)}{2}\right\}$$

$$\leq p\bigl(x^*, x_{n+1}\bigr) + \max\left\{p\bigl(x_n, x^*\bigr), \frac{p(x_n, x_{n+1}) + p(x^*, Tx^*)}{2},\right.$$

$$\left.\frac{p(x_n, x^*) + p(x^*, Tx^*) - p(x^*, x^*) + p(x^*, x_{n+1})}{2}\right\}.$$

Letting $n \to \infty$ in the above inequality,

$$p\bigl(x^*, Tx^*\bigr) \leq \frac{p(x^*, Tx^*)}{2},$$

but this is impossible for $p(x^*, Tx^*) > 0$. Thus, $p(x^*, Tx^*) = 0$. Therefore, we have

$$p\bigl(x^*, Tx^*\bigr) = 0 = p\bigl(x^*, x^*\bigr).$$

This implies that $x^* \in Tx^*$. □

Example 3.5 Let $X = \mathbb{N} \cup \{0\}$ endowed with the partial metric $p(x, y) = \max\{x, y\}$ and a graph $G = (V, E)$ be defined as $V = X$ and $E = \{(x, y) : x, y \in \{0, 2, 4, 6, 8, 10, 12, 14, 16\}\} \cup \{(x, x) : x \in \mathbb{N}\}$. Let $T : X \to \mathrm{CB}_p(X)$ be defined by

$$Tx = \begin{cases} \{0\} & \text{if } x \in \{0, 2, 6, 10, 12, 14\}, \\ \{0, 2\} & \text{if } x = 4, \\ \{0, 4\} & \text{if } x = 8, \\ \{0, 8\} & \text{if } x = 16, \\ \{x + 1, x + 2\} & \text{otherwise}, \end{cases}$$

and $\zeta : [0, \infty) \to [0, \infty)$ by

$$\zeta(t) = \begin{cases} \frac{t}{2} & \text{if } 0 \leq t \leq 4, \\ \frac{2t}{3} & \text{if } 4 < t \leq 8, \\ \frac{t}{2} & \text{if } t > 8. \end{cases}$$

To see that (3.1) holds it is sufficient to consider the following cases:

(i) If $(x, y) \in E$ with $x \neq y \in \{0, 2, 6, 10, 12, 14\}$, then (3.1) trivially holds.

(ii) If $(x, y) \in E$ with $x \in \{0, 2, 6, 10, 12, 14\}$ and $y = 4$, then

$$H_p(Tx, Ty) = 2 \leq \zeta\bigl(M_p(x, y)\bigr).$$

(iii) If $(x, y) \in E$ with $x \in \{0, 2, 6, 10, 12, 14\}$ and $y = 8$, then

$$H_p(Tx, Ty) = 4 \leq \zeta\bigl(M_p(x, y)\bigr).$$

(iv) If $(x, y) \in E$ with $x \in \{0, 2, 6, 10, 12, 14\}$ and $y = 16$, then

$$H_p(Tx, Ty) = 8 \leq \zeta\bigl(M_p(x, y)\bigr).$$

(v) If $(x,y) \in E$ with $x = 4$ and $y = 8$, then

$$H_p(Tx,Ty) = 4 \leq \zeta\big(M_p(x,y)\big).$$

(vi) If $(x,y) \in E$ with $x = 4$ and $y = 16$, then

$$H_p(Tx,Ty) = 8 \leq \zeta\big(M_p(x,y)\big).$$

(vii) If $(x,y) \in E$ with $x = 8$ and $y = 16$, then

$$H_p(Tx,Ty) = 8 \leq \zeta\big(M_p(x,y)\big),$$

where $M_p(x,y) = \max\{p(x,y), \frac{p(x,Tx)+p(y,Ty)}{2}, \frac{p(x,Ty)+p(x,Ty)}{2}\}$. Thus, (3.1) holds. Further it can be observed that for $(x,y) \in E$ with $x \neq y$, if $s \in Tx$ and $t \in Ty$ are such that $p(s,t) < p(x,y)$, then we have $(s,t) \in E$. For $x_0 = 8$, we have $x_1 = 4 \in Tx_0$ such that $(8,4) \in E$. Moreover, Property (A) holds. Therefore, Theorem 3.4 guarantees the existence of a fixed point of T.

Example 3.6 Let $X = [0,\infty) \times [0,\infty)$ be endowed with the partial metric $p(x,y) = p((x_1,x_2),(y_1,y_2)) = \max\{x_1,y_1\} + \max\{x_2,y_2\}$ and a graph $G = (V,E)$ be defined as $V = X$ and $E = \{(r,s) : r = (r_1,0), s = (s_1,0) \text{ with } r_1,s_1 \geq 0\} \cup \{(x,x) : x \in X\}$. Let $T : X \to \mathrm{CB}_p(X)$ be defined by

$$T(b,a) = \big\{(0,0),(b/2,a)\big\} \quad \text{for each } (b,a) \in X$$

and $\zeta : [0,\infty) \to [0,\infty)$ be defined as

$$\zeta(t) = \begin{cases} \frac{2t}{3} & \text{if } 0 \leq t \leq 10, \\ \frac{t}{2} & \text{otherwise.} \end{cases}$$

To see that (3.1) holds, we consider the following cases:

(i) If $((u,0),(v,0)) \in E$ with $0 \leq u < v$, then

$$H_p\big(T(u,0),T(v,0)\big) = \frac{v}{2} \leq \zeta\big(M_p(x,y)\big).$$

(ii) If $((u,0),(v,0)) \in E$ with $0 \leq v < u$, then

$$H_p\big(T(u,0),T(v,0)\big) = \frac{u}{2} \leq \zeta\big(M_p(x,y)\big),$$

where $M_p(x,y) = \max\{p(x,y), \frac{p(x,Tx)+p(y,Ty)}{2}, \frac{p(x,Ty)+p(x,Ty)}{2}\}$. Thus, (3.1) holds. Further it can be observed that for $(x,y) \in E$ with $x \neq y$, if $s \in Tx$ and $t \in Ty$ are such that $p(s,t) < p(x,y)$, then we have $(s,t) \in E$. For $x_0 = (1,0)$, we have $x_1 = (0.5,0) \in Tx_0$ such that $((1,0),(0.5,0)) \in E$. Moreover, Property (A) holds. Therefore, Theorem 3.4 guarantees the existence of a fixed point of T.

Note that the following results are direct consequences of our result.

Corollary 3.7 *Let (X,p) be a complete partial metric space endowed with a graph G and Property* (A). *Let $T : X \to \mathrm{CB}_p(X)$ be a mapping such that*

(i) *for each $(x,y) \in E$ with $x \neq y$, we have*

$$H_p(Tx,Ty) \leq \phi\bigl(M_p(x,y)\bigr)M_p(x,y),$$

where $M_p(x,y) = \max\{p(x,y), \frac{p(x,Tx)+p(y,Ty)}{2}, \frac{p(x,Ty)+p(x,Ty)}{2}\}$, and $\phi : [0,\infty) \to [0,1)$ is such that $\limsup_{t\to r^+} \phi(t) < 1$ for each $r \in [0,\infty)$;

(ii) *if $s \in Tx$ and $t \in Ty$ are such that $p(s,t) < p(x,y)$, then we have $(s,t) \in E$, whenever $(x,y) \in E$ with $x \neq y$.*

Further, assume that there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0,x_1) \in E$. Then T has a fixed point.

This result can be obtained from Theorem 3.4, by considering $\zeta(t) = \phi(t)t$ for each $t \geq 0$.

Corollary 3.8 *Let (X,d) be a complete metric space endowed with a graph G and Property* (A). *Let $T : X \to \mathrm{CB}(X)$ be a mapping such that*

(i) *for each $(x,y) \in E$ with $x \neq y$, we have*

$$H(Tx,Ty) \leq \zeta\left(\max\left\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(x,Ty)}{2}\right\}\right),$$

where ζ is a K-comparison function with $\sup_{t>0} \frac{\zeta(t)}{t} < 1$;

(ii) *if $s \in Tx$ and $t \in Ty$ are such that $d(s,t) < d(x,y)$, then we have $(s,t) \in E$, whenever $(x,y) \in E$ with $x \neq y$.*

Further, assume that there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0,x_1) \in E$. Then T has a fixed point.

Corollary 3.9 *Let (X,p) be a complete partial metric space endowed with a graph G and Property* (A). *Let $T : X \to \mathrm{CB}_p(X)$ be a mapping such that*

(i) *for each $(x,y) \in E$ with $x \neq y$, we have*

$$H_p(Tx,Ty) \leq \zeta\left(\max\left\{p(x,y), \frac{p(x,Tx)+p(y,Ty)}{2}, \frac{p(x,Ty)+p(x,Ty)}{2}\right\}\right),$$

where ζ is comparison function with $\sup_{t>0} \frac{\zeta(t)}{t} < 1$;

(ii) *if $s \in Tx$ and $t \in Ty$ are such that $p(s,t) < p(x,y)$, then we have $(s,t) \in E$, whenever $(x,y) \in E$ with $x \neq y$.*

Further, assume that there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0,x_1) \in E$. Then T has a fixed point.

Corollary 3.10 *Let (X,p) be a complete partial metric space endowed with a graph G and Property* (A). *Let $T : X \to \mathrm{CB}_p(X)$ be a mapping such that*

(i) *for each $(x,y) \in E$ with $x \neq y$, we have*

$$H_p(Tx,Ty) \leq \zeta\left(\max\left\{p(x,y), \frac{p(x,Tx)+p(y,Ty)}{2}, \frac{p(x,Ty)+p(x,Ty)}{2}\right\}\right),$$

where ζ is a (c)-comparison function with $\sup_{t>0} \frac{\zeta(t)}{t} < 1$;

(ii) *if $s \in Tx$ and $t \in Ty$ are such that $p(s,t) < p(x,y)$, then we have $(s,t) \in E$, whenever $(x,y) \in E$ with $x \neq y$.*

Further, assume that there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E$. Then T has a fixed point.

Corollary 3.11 *Let (X,p) be a complete partial metric space endowed with a graph G and Property* (A). *Let $T : X \to \mathrm{CB}_p(X)$ be a mapping such that*

(i) *for each $(x,y) \in E$ with $x \neq y$, we have*

$$H_p(Tx, Ty) \leq a \max\left\{p(x,y), \frac{p(x,Tx)+p(y,Ty)}{2}, \frac{p(x,Ty)+p(x,Ty)}{2}\right\},$$

where $a \in [0,1)$;

(ii) *if $s \in Tx$ and $t \in Ty$ are such that $p(s,t) < p(x,y)$, then we have $(s,t) \in E$, whenever $(x,y) \in E$ with $x \neq y$.*

Further, assume that there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E$. Then T has a fixed point.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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