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# Some new nonlinear integral inequalities with weakly singular kernel and their applications to FDEs

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**Abstract**
In this paper, we investigate some nonlinear integral inequalities with weakly singular kernel which can be used as tools in deriving boundedness of the solutions of certain fractional differential equations and integral equations. Our results generalize and improve some results in the literature. Besides, we give some applications for some fractional differential equations involving the Riemann-Liouville derivative and the Caputo derivative, respectively.

**Keywords:** integral inequalities; singular kernel; fractional differential equations; boundedness

## 1 Introduction

In the study of the qualitative and quantitative properties of solutions of some fractional differential equations, many inequalities with singular kernels have been developed (for example, see [1–18] and the references therein). In these inequalities, Medveď [3] discussed the following useful integral inequality:

$$u(t) \leq a(t) + \int_0^t (t-s)^{\beta-1} f(s) w\bigl(u(s)\bigr)\,\mathrm{d}s, \quad t > 0,$$

which came from the study of a global existence and an exponential decay result for a parabolic Cauchy problem by Henry [19]. Ma and Pečarić [10] used the modification of Medveď's method [3] to study some new weakly singular integral inequality of Henry's type:

$$u^p(t) \leq a(t) + b(t) \int_0^t \bigl(t^\alpha - s^\alpha\bigr)^{\beta-1} s^{\gamma-1} f(s) u^q(s)\,\mathrm{d}s, \quad t > 0,$$

and used it to study the boundedness of certain fractional differential equations with the Caputo fractional derivatives and integral equations involving the Erdélyi-Kober fractional integrals. Recently, Ye and Gao [17] studied a Henry-Gronwall type retarded integral inequalities:

$$\begin{cases} u(t) \leq a(t) + \int_{t_0}^t (t-s)^{\beta-1} b(s) u(s-r)\,\mathrm{d}s, & t \in [t_0, T), \\ u(t) \leq \phi(t), & t \in [t_0 - r, t_0), \end{cases}$$

and used it to obtain boundedness of a class of fractional differential equations. Very recently, Lin [18] established some new weakly singular integral inequality of Gronwall-Bellman type:

$$u(t) \le a(t) + \sum_{i=1}^{n} b_i(t) \int_{t_0}^{t} (t-s)^{\beta_i - 1} c_i(s) u^{\gamma_i}(s)\,\mathrm{d}s,$$

with the conditions that $b_i(t)$ $(i = 1, 2, \ldots, n)$ are bounded and monotonically increasing and used it to deal with the uniqueness of solutions for fractional differential equations.

In this paper, we discuss the more general integral inequality with weakly singular kernel:

$$\begin{aligned} u(t) \le{}& a(t) + \sum_{i=1}^{l} b_i(t) \int_{t_0}^{t} (t-s)^{\alpha_i - 1} f(s) u(s)\,\mathrm{d}s \\ &+ \sum_{j=1}^{n} c_j(t) \int_{t_0}^{t} (t-s)^{\beta_j - 1} g_j(s) u^{\gamma_j}(s)\,\mathrm{d}s, \quad t \in [t_0, T), \end{aligned}$$

where the real constants $\alpha_i > 0$ $(i = 1, 2, \ldots, l)$, $\beta_j > 0$, $0 \le \gamma_j < 1$ $(j = 1, 2, \ldots, n)$, and we have the following retarded integral inequality with weakly singular kernel:

$$\begin{cases} u(t) \le a(t) + \sum_{i=1}^{l} b_i(t) \int_{t_0}^{t} (t-s)^{\alpha_i - 1} f(s) u(s-r)\,\mathrm{d}s \\ \qquad + \sum_{j=1}^{n} c_j(t) \int_{t_0}^{t} (t-s)^{\beta_j - 1} g_j(s) u^{\gamma_j}(s-r)\,\mathrm{d}s, \quad t \in [t_0, T), \\ u(t) \le \phi(t), \quad t \in [t_0 - r, t_0). \end{cases}$$

Our results not only generalize some integral inequalities that have been studied in [17] and improve the results of [18] by removing the conditions that $c_j(t)$ $(j = 1, 2, \ldots, n)$ are bounded and monotonically increasing but also provide a handy tool to derive the boundedness of the solutions of certain fractional differential equations and integral equations.

Throughout the present paper, $\mathbb{N}$ denotes the set of the natural numbers; $\mathbb{R}$ denotes the set of the real numbers; $\mathbb{R}_+ = [0, +\infty)$ is the subset of $\mathbb{R}$; and $C(D, E)$ denotes the class of all continuous functions defined on the set $D$ with range in the set $E$.

## 2 Preliminaries and main results

The following lemmas are useful in our main results.

**Lemma 2.1** (Jensen's inequality) *Let $n \in \mathbb{N}$, and $a_1, a_2, \ldots, a_n \in \mathbb{R}_+$. Then, for $r > 1$,*

$$\left( \sum_{i=1}^{n} a_i \right)^r \le n^{r-1} \sum_{i=1}^{n} a_i^r.$$

**Lemma 2.2** [20] *Let $c \ge 0$, $x \ge 0$, and $0 \le \lambda < 1$. Then, for any $k > 0$,*

$$cx^{\lambda} \le kx + \theta(c, k, \lambda)$$

*holds, where $\theta(c, k, \lambda) = (1 - \lambda)\lambda^{\lambda/(1-\lambda)} c^{1/(1-\lambda)} k^{\lambda/(\lambda-1)}$.*

**Lemma 2.3** *Let* $[t_0, T) \subset \mathbb{R}$ $(T \leq \infty)$, $a(t), b(t), f(t), c_i(t), g_i(t) \in C([t_0, T), \mathbb{R}_+)$ $(i = 1, 2, \ldots, n)$. *If* $u(t) \in C([t_0, T), \mathbb{R}_+)$ *and*

$$u(t) \leq a(t) + b(t) \int_{t_0}^{t} f(s)u(s)\,\mathrm{d}s + \sum_{i=1}^{n} c_i(t) \int_{t_0}^{t} g_i(s)u^{\gamma_i}(s)\,\mathrm{d}s, \quad t \in [t_0, T), \tag{1}$$

*where the real constants* $0 \leq \gamma_i < 1$ $(i = 1, 2, \ldots, n)$, *then, for any positive functions* $k_i(t) \in C([t_0, T), (0, +\infty))$ $(i = 1, 2, \ldots, n)$,

$$u(t) \leq A(t) \exp\left\{ B(t) \int_{t_0}^{t} f(\tau)\,\mathrm{d}\tau + \sum_{i=1}^{n} C_i(t) \int_{t_0}^{t} k_i(\tau)\,\mathrm{d}\tau \right\}, \quad t \in [t_0, T), \tag{2}$$

*where*

$$A(t) = \max_{t_0 \leq s \leq t} a(s) + \sum_{i=1}^{n} C_i(t) \int_{t_0}^{t} \theta\bigl(g_i(s), k_i(s), \gamma_i\bigr)\,\mathrm{d}s,$$

$$B(t) = \max_{t_0 \leq s \leq t} b(s), \qquad C_i(t) = \max_{t_0 \leq s \leq t} c_i(s), \quad i = 1, 2, \ldots, n.$$

*Proof* From Lemma 2.2 and (1), we get, for $t \in [t_0, T)$,

$$\begin{aligned} u(t) &\leq a(t) + b(t) \int_{t_0}^{t} f(s)u(s)\,\mathrm{d}s + \sum_{i=1}^{n} c_i(t) \int_{t_0}^{t} g_i(s)u^{\gamma_i}(s)\,\mathrm{d}s \\ &\leq a(t) + b(t) \int_{t_0}^{t} f(s)u(s)\,\mathrm{d}s + \sum_{i=1}^{n} c_i(t) \int_{t_0}^{t} \bigl[k_i(s)u(s) + \theta\bigl(g_i(s), k_i(s), \gamma_i\bigr)\bigr]\,\mathrm{d}s \\ &= a(t) + \sum_{i=1}^{n} c_i(t) \int_{t_0}^{t} \theta\bigl(g_i(s), k_i(s), \gamma_i\bigr)\,\mathrm{d}s + b(t) \int_{t_0}^{t} f(s)u(s)\,\mathrm{d}s \\ &\quad + \sum_{i=1}^{n} c_i(t) \int_{t_0}^{t} k_i(s)u(s)\,\mathrm{d}s, \end{aligned} \tag{3}$$

where $k_i(t) \in C([t_0, T), (0, +\infty))$ $(i = 1, 2, \ldots, n)$ are any positive functions. Given any $T_0 \in (t_0, T)$, for $t \in [t_0, T_0]$, from (3), we have

$$u(t) \leq A(T_0) + B(T_0) \int_{t_0}^{t} f(s)u(s)\,\mathrm{d}s + \sum_{i=1}^{n} C_i(T_0) \int_{t_0}^{t} k_i(s)u(s)\,\mathrm{d}s, \tag{4}$$

where $A(t) = \max_{t_0 \leq s \leq t} a(s) + \sum_{i=1}^{n} C_i(t) \int_{t_0}^{t} \theta(g_i(s), k_i(s), \gamma_i)\,\mathrm{d}s$, $B(t) = \max_{t_0 \leq s \leq t} b(s)$, $C_i(t) = \max_{t_0 \leq s \leq t} c_i(s)$, $i = 1, 2, \ldots, n$.

Define $z(t)$ by the right side of (4), then $z(t_0) = A(T_0)$, $u(t) \leq z(t)$, $z(t)$ is nonnegative and nondecreasing, and

$$\begin{aligned} z'(t) &= B(T_0)f(t)u(t) + \sum_{i=1}^{n} C_i(T_0)k_i(t)u(t) \\ &= \left[ B(T_0)f(t) + \sum_{i=1}^{n} C_i(T_0)k_i(t) \right] u(t) \end{aligned}$$

$$\begin{aligned} &\leq \left[B(T_0)f(t) + \sum_{i=1}^{n} C_i(T_0)k_i(t)\right]z(t) \\ &= F(t)z(t), \end{aligned} \tag{5}$$

where $F(t) = B(T_0)f(t) + \sum_{i=1}^{n} C_i(T_0)k_i(t)$. Based on a straightforward computation, we have, for $t \in [t_0, T_0]$,

$$z(t) \leq A(T_0)\exp\left\{\int_{t_0}^{t} F(\tau)\,\mathrm{d}\tau\right\}, \tag{6}$$

which implies that

$$z(T_0) \leq A(T_0)\exp\left\{\int_{t_0}^{T_0} F(\tau)\,\mathrm{d}\tau\right\}, \tag{7}$$

and then we get

$$\begin{aligned} u(T_0) &\leq A(T_0)\exp\left\{\int_{t_0}^{T} F(\tau)\,\mathrm{d}\tau\right\} \\ &= A(T_0)\exp\left\{\int_{t_0}^{T_0}\left[B(T_0)f(\tau) + \sum_{i=1}^{n} C_i(T_0)k_i(\tau)\right]\mathrm{d}\tau\right\} \\ &= A(T_0)\exp\left\{B(T_0)\int_{t_0}^{T_0} f(\tau)\,\mathrm{d}\tau + \sum_{i=1}^{n} C_i(T_0)\int_{t_0}^{T_0} k_i(\tau)\,\mathrm{d}\tau\right\}. \end{aligned}$$

By the arbitrariness of $T_0 \in (t_0, T)$, we obtain the inequality (2). The proof is complete. □

**Lemma 2.4** *Let $[t_0, T) \subset \mathbb{R}$ $(T \leq \infty)$, $a(t), b(t), f(t), c_i(t), g_i(t) \in C([t_0, T), \mathbb{R}_+)$ $(i = 1, 2, \ldots, n)$, $\phi(t) \in C([t_0 - r, t_0], \mathbb{R}_+)$, and $a(t_0) = \phi(t_0)$. If $u(t) \in C([t_0 - r, T), \mathbb{R}_+)$, and*

$$\begin{cases} u(t) \leq a(t) + b(t)\int_{t_0}^{t} f(s)u(s-r)\,\mathrm{d}s \\ \qquad + \sum_{i=1}^{n} c_i(t)\int_{t_0}^{t} g_i(s)u^{\gamma_i}(s-r)\,\mathrm{d}s, \quad t \in [t_0, T), \\ u(t) \leq \phi(t), \quad t \in [t_0 - r, t_0), \end{cases} \tag{8}$$

*where the real constants $0 \leq \gamma_i < 1$ $(i = 1, 2, \ldots, n)$. Then, for any positive functions $k_i(t) \in C([t_0, T), (0, +\infty))$ $(i = 1, 2, \ldots, n)$,*

$$\begin{cases} u(t) \leq A(t) + [B(t)\int_{t_0}^{t_0+r} f(s)\phi(s-r)\,\mathrm{d}s + \sum_{i=1}^{n} C_i(t)\int_{t_0}^{t_0+r} k_i(s)\phi(s-r)\,\mathrm{d}s] \\ \qquad \cdot \exp\{B(t)\int_{t_0+r}^{t} f(\tau)\,\mathrm{d}\tau + \sum_{i=1}^{n} C_i(t)\int_{t_0+r}^{t} k_i(\tau)\,\mathrm{d}\tau\} \\ \qquad + B(t)\int_{t_0+r}^{t} A(s-r)f(s)\exp\{B(t)\int_{s}^{t} f(\tau)\,\mathrm{d}\tau + \sum_{i=1}^{n} C_i(t)\int_{s}^{t} k_i(\tau)\,\mathrm{d}\tau\}\,\mathrm{d}s \\ \qquad + \sum_{i=1}^{n} C_i(t)\int_{t_0+r}^{t} A(s-r)k_i(s) \\ \qquad \cdot \exp\{B(t)\int_{s}^{t} f(\tau)\,\mathrm{d}\tau + \sum_{i=1}^{n} C_i(t)\int_{s}^{t} k_i(\tau)\,\mathrm{d}\tau\}\,\mathrm{d}s, \quad t \in [t_0 + r, T), \\ u(t) \leq a(t) + b(t)\int_{t_0}^{t} f(s)\phi(s-r)\,\mathrm{d}s \\ \qquad + \sum_{i=1}^{n} c_i(t)\int_{t_0}^{t} g_i(s)\phi^{\gamma_i}(s-r)\,\mathrm{d}s, \quad t \in [t_0, t_0 + r), \end{cases} \tag{9}$$

*where*

$$A(t) = a(t) + \sum_{i=1}^{n} c_i(t) \int_{t_0}^{t} \theta\big(g_i(s), k_i(s), \gamma_i\big)\,\mathrm{d}s,$$

$$B(t) = \max_{t_0 \le s \le t} b(s), \qquad C_i(t) = \max_{t_0 \le s \le t} c_i(s), \quad i = 1, 2, \ldots, n.$$

*Proof* From (8), for $t \in [t_0, t_0 + r)$, we can easily get

$$u(t) \le a(t) + b(t) \int_{t_0}^{t} f(s)\phi(s-r)\,\mathrm{d}s + \sum_{i=1}^{n} c_i(t) \int_{t_0}^{t} g_i(s)\phi^{\gamma_i}(s-r)\,\mathrm{d}s. \tag{10}$$

From Lemma 2.2 and (8), for $t \in [t_0 + r, T)$ and any positive functions $k_i(t) \in C([t_0, T), (0, +\infty))$ $(i = 1, 2, \ldots, n)$, we obtain the following inequality:

$$\begin{aligned} u(t) &\le a(t) + b(t) \int_{t_0}^{t} f(s)u(s-r)\,\mathrm{d}s + \sum_{i=1}^{n} c_i(t) \int_{t_0}^{t} g_i(s)u^{\gamma_i}(s-r)\,\mathrm{d}s \\ &\le a(t) + b(t) \int_{t_0}^{t} f(s)u(s-r)\,\mathrm{d}s + \sum_{i=1}^{n} c_i(t) \int_{t_0}^{t} \big[k_i(s)u(s-r) + \theta\big(g_i(s), k_i(s), \gamma_i\big)\big]\,\mathrm{d}s \\ &= a(t) + \sum_{i=1}^{n} c_i(t) \int_{t_0}^{t} \theta\big(g_i(s), k_i(s), \gamma_i\big)\,\mathrm{d}s + b(t) \int_{t_0}^{t} f(s)u(s-r)\,\mathrm{d}s \\ &\quad + \sum_{i=1}^{n} c_i(t) \int_{t_0}^{t} k_i(s)u(s-r)\,\mathrm{d}s \\ &= A(t) + b(t) \int_{t_0}^{t} f(s)u(s-r)\,\mathrm{d}s + \sum_{i=1}^{n} c_i(t) \int_{t_0}^{t} k_i(s)u(s-r)\,\mathrm{d}s, \end{aligned} \tag{11}$$

where $A(t) = a(t) + \sum_{i=1}^{n} c_i(t) \int_{t_0}^{t} \theta(g_i(s), k_i(s), \gamma_i)\,\mathrm{d}s$. Given any $T_0 \in (t_0 + r, T)$, for $t \in [t_0 + r, T_0]$, from (11), we obtain

$$u(t) \le A(t) + B(T_0) \int_{t_0}^{t} f(s)u(s-r)\,\mathrm{d}s + \sum_{i=1}^{n} C_i(T_0) \int_{t_0}^{t} k_i(s)u(s-r)\,\mathrm{d}s, \tag{12}$$

where $B(t) = \max_{t_0 \le s \le t} b(s)$, $C_i(t) = \max_{t_0 \le s \le t} c_i(s)$, $i = 1, 2, \ldots, n$.

Let

$$z(t) = B(T_0) \int_{t_0}^{t} f(s)u(s-r)\,\mathrm{d}s + \sum_{i=1}^{n} C_i(T_0) \int_{t_0}^{t} k_i(s)u(s-r)\,\mathrm{d}s, \tag{13}$$

then $z(t_0) = 0$, $u(t) \le A(t) + z(t)$, $z(t)$ is nonnegative and nondecreasing and

$$\begin{aligned} z'(t) &= B(T_0)f(t)u(t-r) + \sum_{i=1}^{n} C_i(T_0)k_i(t)u(t-r) \\ &\le B(T_0)f(t)\big[A(t-r) + z(t-r)\big] + \sum_{i=1}^{n} C_i(T_0)k_i(t)\big[A(t-r) + z(t-r)\big] \end{aligned}$$

$$
\begin{aligned}
&= \left[ B(T_0)f(t) + \sum_{i=1}^{n} C_i(T_0)k_i(t) \right] A(t-r) + \left[ B(T_0)f(t) + \sum_{i=1}^{n} C_i(T_0)k_i(t) \right] z(t-r) \\
&\leq \left[ B(T_0)f(t) + \sum_{i=1}^{n} C_i(T_0)k_i(t) \right] A(t-r) + \left[ B(T_0)f(t) + \sum_{i=1}^{n} C_i(T_0)k_i(t) \right] z(t) \\
&= P(t)A(t-r) + P(t)z(t), \qquad (14)
\end{aligned}
$$

where $P(t) = B(T_0)f(t) + \sum_{i=1}^{n} C_i(T_0)k_i(t)$. Based on a straightforward computation, from (14), we have

$$
\begin{aligned}
z(t) \leq{}& \int_{t_0}^{t_0+r} P(s)\phi(s-r)\,\mathrm{d}s \cdot \exp\left\{ \int_{t_0+r}^{t} P(\tau)\,\mathrm{d}\tau \right\} \\
&+ \int_{t_0+r}^{t} A(s-r)P(s)\exp\left\{ \int_{s}^{t} P(\tau)\,\mathrm{d}\tau \right\} \mathrm{d}s \\
={}& \left[ B(T_0)\int_{t_0}^{t_0+r} f(s)\phi(s-r)\,\mathrm{d}s + \sum_{i=1}^{n} C_i(T_0)\int_{t_0}^{t_0+r} k_i(s)\phi(s-r)\,\mathrm{d}s \right] \\
&\cdot \exp\left\{ B(T_0)\int_{t_0+r}^{t} f(\tau)\,\mathrm{d}\tau + \sum_{i=1}^{n} C_i(T_0)\int_{t_0+r}^{t} k_i(\tau)\,\mathrm{d}\tau \right\} \\
&+ B(T_0)\int_{t_0+r}^{t} A(s-r)f(s) \\
&\cdot \exp\left\{ B(T_0)\int_{s}^{t} f(\tau)\,\mathrm{d}\tau + \sum_{i=1}^{n} C_i(T_0)\int_{s}^{t} k_i(\tau)\,\mathrm{d}\tau \right\} \mathrm{d}s \\
&+ \sum_{i=1}^{n} C_i(T_0)\int_{t_0+r}^{t} A(s-r)k_i(s) \\
&\cdot \exp\left\{ B(T_0)\int_{s}^{t} f(\tau)\,\mathrm{d}\tau + \sum_{i=1}^{n} C_i(T_0)\int_{s}^{t} k_i(\tau)\,\mathrm{d}\tau \right\} \mathrm{d}s, \qquad (15)
\end{aligned}
$$

then

$$
\begin{aligned}
z(T_0) \leq{}& \left[ B(T_0)\int_{t_0}^{t_0+r} f(s)\phi(s-r)\,\mathrm{d}s + \sum_{i=1}^{n} C_i(T_0)\int_{t_0}^{t_0+r} k_i(s)\phi(s-r)\,\mathrm{d}s \right] \\
&\cdot \exp\left\{ B(T_0)\int_{t_0+r}^{T_0} f(\tau)\,\mathrm{d}\tau + \sum_{i=1}^{n} C_i(T_0)\int_{t_0+r}^{T_0} k_i(\tau)\,\mathrm{d}\tau \right\} \\
&+ B(T_0)\int_{t_0+r}^{T_0} A(s-r)f(s) \\
&\cdot \exp\left\{ B(T_0)\int_{s}^{T_0} f(\tau)\,\mathrm{d}\tau + \sum_{i=1}^{n} C_i(T_0)\int_{s}^{T_0} k_i(\tau)\,\mathrm{d}\tau \right\} \mathrm{d}s \\
&+ \sum_{i=1}^{n} C_i(T_0)\int_{t_0+r}^{T_0} A(s-r)k_i(s) \\
&\cdot \exp\left\{ B(T_0)\int_{s}^{T_0} f(\tau)\,\mathrm{d}\tau + \sum_{i=1}^{n} C_i(T_0)\int_{s}^{T_0} k_i(\tau)\,\mathrm{d}\tau \right\} \mathrm{d}s. \qquad (16)
\end{aligned}
$$

By the arbitrariness of $T_0 \in (t_0 + r, T)$, we obtain the inequality (9). The proof is complete. □

**Remark 2.1** Assume that for Lemma 2.4, if $b(t) = 1$, $c_i(t) = 0$, $i = 1, 2, \dots, n$, then we can obtain Lemma 2.2 in [17].

**Theorem 2.1** *Let* $[t_0, T) \subset \mathbb{R}$ $(T \leq \infty)$, $a(t), f(t), b_i(t), c_j(t), g_j(t) \in C([t_0, T), \mathbb{R}_+)$ $(i = 1, 2, \dots, l; j = 1, 2, \dots, n)$. *If* $u(t) \in C([t_0, T), \mathbb{R}_+)$ *and*

$$\begin{aligned} u(t) \leq{}& a(t) + \sum_{i=1}^{l} b_i(t) \int_{t_0}^{t} (t-s)^{\alpha_i - 1} f(s) u(s)\,\mathrm{d}s \\ &+ \sum_{j=1}^{n} c_j(t) \int_{t_0}^{t} (t-s)^{\beta_j - 1} g_j(s) u^{\gamma_j}(s)\,\mathrm{d}s, \quad t \in [t_0, T), \end{aligned} \tag{17}$$

*where the real constants* $\alpha_i > 0$ $(i = 1, 2, \dots, l)$ *and* $\beta_j > 0$, $0 \leq \gamma_j < 1$ $(j = 1, 2, \dots, n)$. *Then, for any positive functions* $k_j(t) \in C([t_0, T), (0, +\infty))$ $(j = 1, 2, \dots, n)$, *the following assertions hold*:

(i) *Suppose that* $\alpha_i > \frac{1}{2}$ $(i = 1, 2, \dots, l)$, $\beta_j > \frac{1}{2}$ $(j = 1, 2, \dots, n)$, *then*

$$u(t) \leq \sqrt{A(t)} \exp\left\{ t + \frac{B(t)}{2} \int_{t_0}^{t} F(\tau)\,\mathrm{d}\tau + \sum_{j=1}^{n} \frac{B_j(t)}{2} \int_{t_0}^{t} k_j(\tau)\,\mathrm{d}\tau \right\}, \quad t \in [t_0, T), \tag{18}$$

*where*

$$\begin{aligned} &A(t) = \max_{t_0 \leq s \leq t} H(s) + \sum_{j=1}^{n} B_j(t) \int_{t_0}^{t} \theta\big(G_j(s), k_j(s), \gamma_j\big)\,\mathrm{d}s, \qquad H(t) = 2e^{-2t} a^2(t), \\ &B_j(t) = \max_{t_0 \leq s \leq t} K_j(s), \qquad K_j(t) = \frac{2n c_j^2(t) \Gamma(2\beta_j - 1)}{4^{\beta_j - 1}}, \\ &G_j(t) = g_j^2(t) M_j(t), \qquad M_j(t) = e^{2(\gamma_j - 1)t}, \quad j = 1, 2, \dots, n, \\ &B(t) = \max_{t_0 \leq s \leq t} K(s), \qquad K(t) = l \sum_{i=1}^{l} \frac{2 b_i^2(t) e^{2t} \Gamma(2\alpha_i - 1)}{4^{\alpha_i - 1}}, \qquad F(t) = f^2(t). \end{aligned} \tag{19}$$

(ii) *Suppose that* $0 < \alpha_i \leq \frac{1}{2}$ $(i = 1, 2, \dots, l)$ *or* $0 < \beta_j \leq \frac{1}{2}$ $(j = 1, 2, \dots, n)$, *then*

$$u(t) \leq \widetilde{A}^{1/q}(t) \exp\left\{ t + \frac{\widetilde{B}(t)}{q} \int_{t_0}^{t} \widetilde{F}(\tau)\,\mathrm{d}\tau + \sum_{j=1}^{n} \frac{\widetilde{B}_j(t)}{q} \int_{t_0}^{t} k_i(\tau)\,\mathrm{d}\tau \right\}, \quad t \in [t_0, T), \tag{20}$$

*where*

$$\begin{aligned} &\widetilde{A}(t) = \max_{t_0 \leq s \leq t} \widetilde{H}(s) + \sum_{j=1}^{n} \widetilde{B}_j(t) \int_{t_0}^{t} \theta\big(\widetilde{G}_j(s), k_j(s), \gamma_j\big)\,\mathrm{d}s, \qquad \widetilde{H}(t) = 2^{q-1} a^q(t) e^{-qt}, \\ &\widetilde{B}_j(t) = \max_{t_0 \leq s \leq t} \widetilde{K}_j(s), \qquad \widetilde{K}_j(t) = (4n)^{(q-1)} c_j^q(t) \left( \frac{\Gamma(1 - p(1 - \beta_i))}{p^{1 - p(1 - \beta_j)}} \right)^{\frac{q}{p}}, \\ &\widetilde{G}_j(t) = g_j^q(t) \widetilde{M}_j(t), \qquad \widetilde{M}_j(t) = e^{q(\gamma_j - 1)t}, \quad j = 1, 2, \dots, n, \end{aligned} \tag{21}$$

$$\widetilde{B}(t) = \max_{t_0 \le s \le t} \widetilde{K}(s), \qquad \widetilde{K}(t) = (4l)^{(q-1)} \sum_{i=1}^{l} b_i^q(t) \left( \frac{e^{pt}\Gamma(1-p(1-\alpha_i))}{p^{1-p(1-\alpha_i)}} \right)^{\frac{q}{p}}, \qquad \widetilde{F}(t) = f^q(t),$$

$$p = 1+\theta, \qquad q = 1+\frac{1}{\theta}, \qquad \theta = \min\{\alpha_i, \beta_j, i = 1,2,\dots,l; j = 1,2,\dots,n\}.$$

*Proof* (i) For $t \in [t_0, T)$, from (17), using the Cauchy-Schwarz inequality and a simple computation,

$$\begin{aligned} u(t) &\le a(t) + \sum_{i=1}^{l} b_i(t) \left( \int_{t_0}^{t} (t-s)^{2\alpha_i - 2} e^{2s}\, ds \right)^{\frac{1}{2}} \left( \int_{t_0}^{t} f^2(s) e^{-2s} u^2(s)\, ds \right)^{\frac{1}{2}} \\ &\quad + \sum_{j=1}^{n} c_j(t) \left( \int_{t_0}^{t} (t-s)^{2\beta_j - 2} e^{2s}\, ds \right)^{\frac{1}{2}} \left( \int_{t_0}^{t} g_j^2(s) e^{-2s} u^{2\gamma_j}(s)\, ds \right)^{\frac{1}{2}} \\ &< a(t) + \sum_{i=1}^{l} \left( \frac{2b_i^2(t) e^{2t} \Gamma(2\alpha_i - 1)}{4^{\alpha_i}} \right)^{\frac{1}{2}} \left( \int_{t_0}^{t} f^2(s) e^{-2s} u^2(s)\, ds \right)^{\frac{1}{2}} \\ &\quad + \sum_{j=1}^{n} \left( \frac{2c_j^2(t) e^{2t} \Gamma(2\beta_j - 1)}{4^{\beta_j}} \right)^{\frac{1}{2}} \left( \int_{t_0}^{t} g_i^2(s) e^{-2s} u^{2\gamma_j}(s)\, ds \right)^{\frac{1}{2}}. \end{aligned} \tag{22}$$

Using Lemma 2.2, we obtain

$$\begin{aligned} u^2(t) &\le 2a^2(t) + l \sum_{i=1}^{l} \frac{2b_i^2(t) e^{2t} \Gamma(2\alpha_i - 1)}{4^{\alpha_i - 1}} \int_{t_0}^{t} f^2(s) e^{-2s} u^2(s)\, ds \\ &\quad + n \sum_{j=1}^{n} \frac{2c_j^2(t) e^{2t} \Gamma(2\beta_j - 1)}{4^{\beta_j - 1}} \int_{t_0}^{t} g_j^2(s) e^{-2s} u^{2\gamma_j}(s)\, ds. \end{aligned} \tag{23}$$

Let $w(t) = [e^{-t}u(t)]^2$, we obtain

$$w(t) \le H(t) + K(t) \int_{t_0}^{t} F(s) w(s)\, ds + \sum_{j=1}^{n} K_j(t) \int_{t_0}^{t} G_j(s) w^{\gamma_j}(s)\, ds,$$

where $H(t)$, $F(t)$, $K_j(t)$, and $G_j(t)$ ($j = 1,2,\dots,n$) are defined in (19).

Using Lemma 2.3, we get, for any positive functions $k_j(t) \in C([t_0, T), (0, +\infty))$ ($j = 1,2,\dots,n$),

$$w(t) \le A(t) \exp\left\{ B(t) \int_{t_0}^{t} F(\tau)\, d\tau + \sum_{j=1}^{n} B_j(t) \int_{t_0}^{t} k_j(\tau)\, d\tau \right\}, \tag{24}$$

where $A(t)$, $B(t)$, and $B_j(t)$ ($j = 1,2,\dots,n$) are defined in (19). From the definition of $w(t)$, we get (18).

(ii) For $t \in [t_0, T)$, by the hypothesis, we get $\frac{1}{p} + \frac{1}{q} = 1$. Using the Hölder inequality and a simple computation, we obtain

$$\begin{aligned} u(t) &\le a(t) + \sum_{i=1}^{l} b_i(t) \left( \int_{t_0}^{t} (t-s)^{p\alpha_i - p} e^{ps}\, ds \right)^{\frac{1}{p}} \left( \int_{t_0}^{t} f^q(s) e^{-qs} u^q(s)\, ds \right)^{\frac{1}{q}} \\ &\quad + \sum_{j=1}^{n} c_j(t) \left( \int_{t_0}^{t} (t-s)^{p\beta_j - p} e^{ps}\, ds \right)^{\frac{1}{p}} \left( \int_{t_0}^{t} g_j^q(s) e^{-qs} u^{q\gamma_j}(s)\, ds \right)^{\frac{1}{q}} \end{aligned}$$

$$\leq a(t)+\sum_{i=1}^{l} b_i(t)\left(\frac{e^{pt}\Gamma(1-p(1-\alpha_i))}{p^{1-p(1-\alpha_i)}}\right)^{\frac{1}{p}}\left(\int_{t_0}^{t} f^q(s)e^{-qs}u^q(s)\,\mathrm{d}s\right)^{\frac{1}{q}}$$
$$+\sum_{j=1}^{n} c_j(t)\left(\frac{e^{pt}\Gamma(1-p(1-\beta_j))}{p^{1-p(1-\beta_j)}}\right)^{\frac{1}{p}}\left(\int_{t_0}^{t} g_j^q(s)e^{-qs}u^{q\gamma_j}(s)\,\mathrm{d}s\right)^{\frac{1}{q}}.$$

Obviously, $1-p(1-\alpha_i)=1-(1+\theta)(1-\alpha_i)\geq 1-(1+\alpha_i)(1-\alpha_i)=\alpha_i^2>0$, $i=1,2,\ldots,l$, $1-p(1-\beta_j)=1-(1+\theta)(1-\beta_j)\geq 1-(1+\beta_j)(1-\beta_j)=\beta_j^2>0$, $j=1,2,\ldots,n$.

Using Lemma 2.2, we obtain

$$u^q(t)\leq 2^{q-1}a^q(t)+(4l)^{(q-1)}\sum_{i=1}^{l} b_i^q(t)\left(\frac{e^{pt}\Gamma(1-p(1-\alpha_i))}{p^{1-p(1-\alpha_i)}}\right)^{\frac{q}{p}}\int_{t_0}^{t} f^q(s)e^{-qs}u^q(s)\,\mathrm{d}s$$
$$+(4n)^{(q-1)}\sum_{j=1}^{n} c_j^q(t)\left(\frac{e^{pt}\Gamma(1-p(1-\beta_j))}{p^{1-p(1-\beta_j)}}\right)^{\frac{q}{p}}\int_{t_0}^{t} g_j^q(s)e^{-qs}u^{q\gamma_j}(s)\,\mathrm{d}s. \tag{25}$$

Let $w(t)=[e^{-t}u(t)]^q$, we get

$$w(t)\leq \widetilde{H}(t)+\widetilde{K}(t)\int_{t_0}^{t}\widetilde{F}(s)w(s)\,\mathrm{d}s+\sum_{j=1}^{n}\widetilde{K}_j(t)\int_{t_0}^{t}\widetilde{G}_j(s)w^{\gamma_j}(s)\,\mathrm{d}s,$$

where $\widetilde{H}(t)$, $\widetilde{K}_j(t)$ $(j=1,2,\ldots,n)$, $\widetilde{F}(t)$, and $\widetilde{G}_j(t)$ $(j=1,2,\ldots,n)$ are defined in (21). Using Lemma 2.3, we have, for any positive functions $k_j(t)\in C([t_0,T),(0,+\infty))$ $(j=1,2,\ldots,n)$,

$$w(t)\leq \widetilde{A}(t)\exp\left\{\widetilde{B}(t)\int_{t_0}^{t}\widetilde{F}(\tau)\,\mathrm{d}\tau+\sum_{j=1}^{n}\widetilde{B}_j(t)\int_{t_0}^{t}k_j(\tau)\,\mathrm{d}\tau\right\}, \tag{26}$$

where $\widetilde{A}(t)$, $\widetilde{B}(t)$, and $\widetilde{B}_j(t)$ $(j=1,2,\ldots,n)$ are defined in (21). From the definition of $w(t)$, we get (20). The proof is complete. □

**Theorem 2.2** *Let* $[t_0,T)\subset\mathbb{R}$ $(T\leq\infty)$, $a(t),f(t),b_i(t),c_j(t),g_j(t)\in C([t_0,T),\mathbb{R}_+)$ $(i=1,2,\ldots,l;j=1,2,\ldots,n)$. $\phi(t)\in C([t_0-r,t_0],\mathbb{R}_+)$ *and* $a(t_0)=\phi(t_0)$. *If* $u(t)\in C([t_0-r,T),\mathbb{R}_+)$, *and*

$$\begin{cases} u(t)\leq a(t)+\sum_{i=1}^{l} b_i(t)\int_{t_0}^{t}(t-s)^{\alpha_i-1}f(s)u(s-r)\,\mathrm{d}s \\ \qquad +\sum_{j=1}^{n} c_j(t)\int_{t_0}^{t}(t-s)^{\beta_j-1}g_j(s)u^{\gamma_j}(s-r)\,\mathrm{d}s, \quad t\in[t_0,T), \\ u(t)\leq\phi(t), \quad t\in[t_0-r,t_0), \end{cases} \tag{27}$$

*where the real constants* $\alpha_i>0$ $(i=1,2,\ldots,l)$, $\beta_j>0$, $0\leq\gamma_j<1$ $(j=1,2,\ldots,n)$. *Then, for any positive functions* $k_i(t)\in C([t_0,T),(0,+\infty))$ $(i=1,2,\ldots,n)$, *the following assertions hold*:

(i) *Suppose that* $\alpha_i>\frac{1}{2}$ $(i=1,2,\ldots,l)$, $\beta_j>\frac{1}{2}$ $(j=1,2,\ldots,n)$, *then*

$$\begin{cases} u(t)\leq e^t\{A(t)+[B(t)\int_{t_0}^{t_0+r}F(s)\phi_1(s-r)\,\mathrm{d}s+\sum_{j=1}^{n}B_j(t)\int_{t_0}^{t_0+r}k_j(s)\phi_1(s-r)\,\mathrm{d}s] \\ \qquad\cdot\exp\{B(t)\int_{t_0+r}^{t}F(\tau)\,\mathrm{d}\tau+\sum_{j=1}^{n}B_j(t)\int_{t_0+r}^{t}k_j(\tau)\,\mathrm{d}\tau\} \\ \qquad +B(t)\int_{t_0+r}^{t}A(s-r)F(s)\exp\{B(t)\int_{s}^{t}F(\tau)\,\mathrm{d}\tau+\sum_{j=1}^{n}B_j(t)\int_{s}^{t}k_j(\tau)\,\mathrm{d}\tau\}\,\mathrm{d}s \\ \qquad +\sum_{j=1}^{n}B_j(t)\int_{t_0+r}^{t}A(s-r)k_j(s) \\ \qquad\cdot\exp\{B(t)\int_{s}^{t}F(\tau)\,\mathrm{d}\tau+\sum_{j=1}^{n}B_j(t)\int_{s}^{t}k_j(\tau)\,\mathrm{d}\tau\}\,\mathrm{d}s\}^{1/2}, \quad t\in[t_0+r,T), \\ u(t)\leq a(t)+\sum_{i=1}^{l}b_i(t)\int_{t_0}^{t}(t-s)^{\alpha_i-1}f(s)\phi(s-r)\,\mathrm{d}s \\ \qquad +\sum_{j=1}^{n}c_j(t)\int_{t_0}^{t}(t-s)^{\beta_j-1}g_j(s)\phi^{\gamma_j}(s-r)\,\mathrm{d}s, \quad t\in[t_0,t_0+r), \end{cases} \tag{28}$$

*where*

$$
\begin{aligned}
&A(t) = H(t) + \sum_{j=1}^{n} K_j(t) \int_{t_0}^{t} \theta\big(G_j(s), k_j(s), \gamma_j\big)\,\mathrm{d}s, \\
&H(t) = 2e^{-2t}a^2(t), \qquad K_j(t) = \frac{2nc_i^2(t)e^{-2\gamma_j r}\Gamma(2\beta_j - 1)}{4^{\beta_j-1}}, \\
&G_j(t) = g_j^2(t)M_j(t), \qquad M_j(t) = e^{2(\gamma_j-1)t}, \quad j = 1,2,\ldots,n, \\
&B(t) = \max_{t_0\le s\le t} K(s), \qquad K(t) = l\sum_{i=1}^{l} \frac{2b_i^2(t)e^{2t}\Gamma(2\alpha_i - 1)}{4^{\alpha_i-1}}, \\
&F(t) = f^2(t), \qquad \phi_1(t) = e^{-2t}\phi^2(t), \qquad B_j(t) = \max_{t_0\le s\le t} K_j(s), \quad j = 1,2,\ldots,n.
\end{aligned}
\tag{29}
$$

(ii) *Suppose that* $0 < \alpha_i \le \frac{1}{2}$ $(i = 1,2,\ldots,l)$ *or* $0 < \beta_j \le \frac{1}{2}$ $(j = 1,2,\ldots,n)$ *then*

$$
\begin{cases}
u(t) \le e^t\{\widetilde{A}(t) + [\widetilde{B}(t)\int_{t_0}^{t_0+r}\widetilde{F}(s)\phi_2(s-r)\,\mathrm{d}s + \sum_{j=1}^{n}\widetilde{B}_j(t)\int_{t_0}^{t_0+r}k_j(s)\phi_2(s-r)\,\mathrm{d}s] \\
\qquad \cdot \exp\{\widetilde{B}(t)\int_{t_0+r}^{t}\widetilde{F}(\tau)\,\mathrm{d}\tau + \sum_{j=1}^{n}\widetilde{B}_j(t)\int_{t_0+r}^{t}k_j(\tau)\,\mathrm{d}\tau\} \\
\qquad + \widetilde{B}(t)\int_{t_0+r}^{t}\widetilde{A}(s-r)\widetilde{F}(s)\exp\{\widetilde{B}(t)\int_s^t\widetilde{F}(\tau)\,\mathrm{d}\tau + \sum_{j=1}^{n}\widetilde{B}_j(t)\int_s^t k_j(\tau)\,\mathrm{d}\tau\}\,\mathrm{d}s \\
\qquad + \sum_{j=1}^{n}\widetilde{B}_j(t)\int_{t_0+r}^{t}\widetilde{A}(s-r)k_j(s) \\
\qquad \cdot \exp\{\widetilde{B}(t)\int_s^t\widetilde{F}(\tau)\,\mathrm{d}\tau + \sum_{j=1}^{n}\widetilde{B}_j(t)\int_s^t k_j(\tau)\,\mathrm{d}\tau\}\,\mathrm{d}s\}^{1/q}, \quad t \in [t_0 + r, T), \\
u(t) \le a(t) + \sum_{i=1}^{l} b_i(t)\int_{t_0}^{t}(t-s)^{\alpha_i-1}f(s)\phi(s-r)\,\mathrm{d}s \\
\qquad + \sum_{j=1}^{n} c_j(t)\int_{t_0}^{t}(t-s)^{\beta_j-1}g_j(s)\phi^{\gamma_j}(s-r)\,\mathrm{d}s, \quad t \in [t_0, t_0 + r),
\end{cases}
\tag{30}
$$

*where*

$$
\begin{aligned}
&\widetilde{A}(t) = \max_{t_0\le s\le t}\widetilde{H}(s) + \sum_{j=1}^{n}\widetilde{B}_j(t)\int_{t_0}^{t}\theta\big(\widetilde{G}_j(s), k_j(s), \gamma_j\big)\,\mathrm{d}s, \\
&\widetilde{H}(t) = 2^{q-1}e^{-qt}a^q(t), \qquad \widetilde{B}_j(t) = \max_{t_0\le s\le t}\widetilde{K}_j(s), \\
&\widetilde{K}_j(t) = (4n)^{(q-1)}c_j^q(t)e^{-q\gamma_j r}\left(\frac{\Gamma(1-p(1-\beta_j))}{p^{1-p(1-\beta_j)}}\right)^{\frac{q}{p}}, \quad j = 1,2,\ldots,n, \\
&\widetilde{G}_j(t) = g_j^q(t)\widetilde{M}_j(t), \qquad \widetilde{M}_j(t) = e^{q(\gamma_j-1)t}, \quad j = 1,2,\ldots,n, \\
&\widetilde{B}(t) = \max_{t_0\le s\le t}\widetilde{K}(s), \qquad \widetilde{K}(t) = (4l)^{(q-1)}\sum_{i=1}^{l} b_i^q(t)\left(\frac{e^{pt}\Gamma(1-p(1-\alpha_i))}{p^{1-p(1-\alpha_i)}}\right)^{\frac{q}{p}}, \\
&\phi_2(t) = e^{-qt}\phi^q(t), \qquad p = 1 + \theta, \qquad q = 1 + \frac{1}{\theta}, \\
&\theta = \min\{\alpha_i, \beta_j, i = 1,2,\ldots,l; j = 1,2,\ldots,n\}, \qquad \widetilde{F}(t) = f^q(t).
\end{aligned}
\tag{31}
$$

*Proof* From (27), for $t \in [t_0, t_0 + r)$, we can easily get

$$
\begin{aligned}
u(t) \le{}& a(t) + \sum_{i=1}^{l} b_i(t)\int_{t_0}^{t}(t-s)^{\alpha_i-1}f(s)\phi(s-r)\,\mathrm{d}s \\
&+ \sum_{j=1}^{n} c_j(t)\int_{t_0}^{t}(t-s)^{\beta_j-1}g_j(s)\phi^{\gamma_j}(s-r)\,\mathrm{d}s.
\end{aligned}
$$

(i) For $t \in [t_0, T)$, from a proof procedure similar to (i) of Theorem 2.1, we obtain

$$\begin{aligned} u^2(t) \le{}& 2a^2(t) + l\sum_{i=1}^{l} \frac{2b_i^2(t)e^{2t}\Gamma(2\alpha_i - 1)}{4^{\alpha_i-1}} \int_{t_0}^{t} f^2(s)e^{-2s}u^2(s-r)\,\mathrm{d}s \\ &+ n\sum_{j=1}^{n} \frac{2c_j^2(t)e^{2t}\Gamma(2\beta_j - 1)}{4^{\beta_j-1}} \int_{t_0}^{t} g_j^2(s)e^{-2s}u^{2\gamma_j}(s-r)\,\mathrm{d}s. \end{aligned} \tag{32}$$

Let $w(t) = [e^{-t}u(t)]^2$, we obtain

$$\begin{aligned} w(t) \le{}& H(t) + K(t)\int_{t_0}^{t} F(s)w(s-r)\,\mathrm{d}s \\ &+ \sum_{j=1}^{n} K_j(t)\int_{t_0}^{t} G_j(s)w^{\gamma_j}(s-r)\,\mathrm{d}s, \quad t \in [t_0, T) \end{aligned} \tag{33}$$

and

$$w(t) \le \phi_1(t), \quad t \in [t_0 - r, t_0),$$

where $H(t)$, $K(t)$, $F(t)$, $\phi_1(t)$, and $K_j(t)$, $G_j(t)$ ($j = 1, 2, \ldots, n$) are defined in (29). Using Lemma 2.4, we get, for any positive functions $k_j(t) \in C([t_0, T), (0, +\infty))$ ($j = 1, 2, \ldots, n$),

$$\begin{aligned} w(t) \le{}& A(t) + \left[B(t)\int_{t_0}^{t_0+r} F(s)\phi_1(s-r)\,\mathrm{d}s + \sum_{j=1}^{n} B_j(t)\int_{t_0}^{t_0+r} k_j(s)\phi_1(s-r)\,\mathrm{d}s\right] \\ &\cdot \exp\left\{B(t)\int_{t_0+r}^{t} F(\tau)\,\mathrm{d}\tau + \sum_{j=1}^{n} B_j(t)\int_{t_0+r}^{t} k_j(\tau)\,\mathrm{d}\tau\right\} \\ &+ B(t)\int_{t_0+r}^{t} A(s-r)F(s)\exp\left\{B(t)\int_{s}^{t} F(\tau)\,\mathrm{d}\tau + \sum_{j=1}^{n} B_j(t)\int_{s}^{t} k_j(\tau)\,\mathrm{d}\tau\right\}\mathrm{d}s \\ &+ \sum_{j=1}^{n} B_j(t)\int_{t_0+r}^{t} A(s-r)k_j(s)\exp\left\{B(t)\int_{s}^{t} F(\tau)\,\mathrm{d}\tau + \sum_{j=1}^{n} B_j(t)\int_{s}^{t} k_j(\tau)\,\mathrm{d}\tau\right\}\mathrm{d}s, \\ & t \in [t_0 + r, T), \end{aligned}$$

where $A(t)$, $B(t)$, and $B_j(t)$ ($i = 1, 2, \ldots, n$) are defined in (21). From the definition of $w(t)$, we get (28).

(ii) For $t \in [t_0, T)$, from the similar proof procedure of (ii) of Theorem 2.1, we have

$$\begin{aligned} u^q(t) \le{}& 2^{q-1}a^q(t) + (4l)^{(q-1)}\sum_{i=1}^{l} b_i^q(t)\left(\frac{e^{pt}\Gamma(1 - p(1-\alpha_i))}{p^{1-p(1-\alpha_i)}}\right)^{\frac{q}{p}} \\ &\cdot \int_{t_0}^{t} f^q(s)e^{-qs}u^q(s-r)\,\mathrm{d}s \\ &+ (4n)^{(q-1)}\sum_{j=1}^{n} c_j^q(t)\left(\frac{e^{pt}\Gamma(1 - p(1-\beta_j))}{p^{1-p(1-\beta_j)}}\right)^{\frac{q}{p}} \int_{t_0}^{t} g_j^q(s)e^{-qs}u^{q\gamma_j}(s-r)\,\mathrm{d}s. \end{aligned} \tag{34}$$

Let $w(t) = [e^{-t}u(t)]^q$, we get

$$w(t) \le \widetilde{H}(t) + \widetilde{K}(t)\int_{t_0}^{t}\widetilde{F}(s)w(s-r)\,\mathrm{d}s + \sum_{j=1}^{n}\widetilde{K}_j(t)\int_{t_0}^{t}\widetilde{G}_j(s)w^{\gamma_j}(s-r)\,\mathrm{d}s, \quad t \in [t_0, T)$$

and

$$w(t) \le \phi_2(t), \quad t \in [t_0 - r, t_0),$$

where $\widetilde{H}(t)$, $\widetilde{F}(t)$, $\phi_2(t)$, $\widetilde{K}_j(t)$ and $\widetilde{G}_j(t)$ $(j = 1, 2, \ldots, n)$ are defined in (31). Using Lemma 2.4, we get, for any positive functions $k_j(t) \in C([t_0, T), (0, +\infty))$ $(j = 1, 2, \ldots, n)$,

$$\begin{aligned} w(t) \le{}& \widetilde{A}(t) + \left[\widetilde{B}(t)\int_{t_0}^{t_0+r}\widetilde{F}(s)\phi_2(s-r)\,\mathrm{d}s + \sum_{j=1}^{n}\widetilde{B}_j(t)\int_{t_0}^{t_0+r}k_j(s)\phi_1(s-r)\,\mathrm{d}s\right] \\ &\cdot \exp\left\{\widetilde{B}(t)\int_{t_0+r}^{t}\widetilde{F}(\tau)\,\mathrm{d}\tau + \sum_{j=1}^{n}\widetilde{B}_j(t)\int_{t_0+r}^{t}k_j(\tau)\,\mathrm{d}\tau\right\} \\ &+ \widetilde{B}(t)\int_{t_0+r}^{t}\widetilde{A}(s-r)\widetilde{F}(s)\exp\left\{\widetilde{B}(t)\int_{s}^{t}\widetilde{F}(\tau)\,\mathrm{d}\tau + \sum_{j=1}^{n}\widetilde{B}_j(t)\int_{s}^{t}k_j(\tau)\,\mathrm{d}\tau\right\}\mathrm{d}s \\ &+ \sum_{j=1}^{n}\widetilde{B}_j(t)\int_{t_0+r}^{t}\widetilde{A}(s-r)k_j(s)\exp\left\{\widetilde{B}(t)\int_{s}^{t}\widetilde{F}(\tau)\,\mathrm{d}\tau + \sum_{j=1}^{n}\widetilde{B}_j(t)\int_{s}^{t}k_j(\tau)\,\mathrm{d}\tau\right\}\mathrm{d}s, \\ & t \in [t_0 + r, T), \end{aligned}$$

where $\widetilde{A}(t)$, $\widetilde{B}(t)$, and $\widetilde{B}_j(t)$ $(j = 1, 2, \ldots, n)$ are defined in (31). From the definition of $w(t)$, we get (30). The proof is complete. □

## 3 Applications

In this section, we present some applications in studying the boundedness of the solutions of certain fractional differential equations with a Riemann-Liouville fractional derivative and a Caputo fractional derivative, respectively.

First, we consider some fractional differential equations with the Riemann-Liouville fractional derivative. The Riemann-Liouville fractional order derivative and fractional integral are defined as follows.

**Definition 1** [21] For any $0 < \beta < 1$, the $\beta$th Riemann-Liouville fractional order derivative of a function $f : [0, +\infty) \to \mathbb{R}$ is defined by

$$D_R^{\beta}f(t) = \frac{1}{\Gamma(1-\beta)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-s)^{-\beta}f(s)\,\mathrm{d}s,$$

where $\Gamma$ is the gamma function.

**Definition 2** [21] The $\beta$th Riemann-Liouville fractional order integral of a function $f :$ $[0, +\infty) \to \mathbb{R}$ is defined by

$$I_R^{\beta}f(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}f(s)\,\mathrm{d}s,$$

where $\beta > 0$.

Let us consider the initial value problems for fractional differential equations in the following form:

$$\sum_{i=1}^{n} D_R^{\beta_i} y(t) = f\bigl(t, y(t)\bigr), \tag{35}$$

$$\sum_{i=1}^{n} I_R^{1-\beta_i} y(t)|_{t=0} = \delta, \tag{36}$$

where $0 < \beta_1 < \beta_2 < \cdots < \beta_n < 1$, $t \in [0, T)$. The solution $y(t)$ of the initial value problem (IVP) (35)-(36) can be written as (see [18])

$$\begin{aligned} y(t) &= I_R^{\beta_n} f\bigl(t, y(t)\bigr) - \sum_{i=1}^{n-1} I_R^{\beta_n-\beta_i} y(t) + \delta \frac{t^{\beta_n-1}}{\Gamma(\beta_n)} \\ &= \frac{1}{\Gamma(\beta_n)} \int_0^t (t-s)^{\beta_n-1} f\bigl(s, y(s)\bigr)\,\mathrm{d}s \\ &\quad + \sum_{i=1}^{n-1} \frac{1}{\Gamma(\beta_n-\beta_i)} \int_0^t (t-s)^{\beta_n-\beta_i-1} y(s)\,\mathrm{d}s + \delta \frac{t^{\beta_n-1}}{\Gamma(\beta_n)}. \end{aligned} \tag{37}$$

The following theorem deals with the boundedness of the solutions of the initial value problem (35)-(36).

**Theorem 3.1** *Suppose that the function* $f \in C([0,T) \times \mathbb{R}, \mathbb{R})$ *satisfies*

$$\bigl|f(t,y)\bigr| \le h_1(t)|y|^{\gamma_1} + h_2(t)|y|^{\gamma_2} + h_3(t)|y|^{\gamma_3}, \quad (t,y) \in [0,T) \times \mathbb{R}, \tag{38}$$

*where* $0 < \gamma_1, \gamma_2, \gamma_3 < 1$ *are constants independent of* $t$, $y$ *in* $\mathbb{R}$, $h_1(t)$, $h_2(t)$, *and* $h_3(t)$ *are nonnegative continuous functions defined on* $[0,T)$. *If* $y(t)$ *is any solution of the initial value problem* (35)-(36), *then, for any positive functions* $k_j(t) \in C([t_0,T), (0,+\infty))$ $(j = 1,2,3)$, *the following assertions hold*:

(i) *Suppose that* $\beta_n - \beta_i > \frac{1}{2}$ $(i = 1,2,\ldots,n-1)$, $\beta_n > \frac{1}{2}$, *then*

$$u(t) \le \sqrt{A(t)} \exp\left\{ \frac{t(2+B(t))}{2} + \sum_{j=1}^{3} \frac{B_j(t)}{2} \int_0^t k_j(\tau)\,\mathrm{d}\tau \right\}, \quad t \in [0,T), \tag{39}$$

*where*

$$A(t) = \max_{0 \le s \le t} H(s) + \sum_{j=1}^{3} B_j(t) \int_0^t \theta\bigl(G_j(s), k_j(s), \gamma_j\bigr)\,\mathrm{d}s, \qquad H(t) = \frac{\delta^2 2e^{-2t} t^{2(\beta_n-1)}}{\Gamma^2(\beta_n)},$$

$$B_j(t) = \frac{6\Gamma(2\beta_n-1)}{4^{\beta_n-1}\Gamma^2(\beta_n)}, \qquad G_j(t) = h_j^2(t) e^{2(\gamma_j-1)t}, \quad j = 1,2,3,$$

$$B(t) = (n-1) \sum_{i=1}^{n-1} \frac{2e^{2t}\Gamma(2\beta_n-2\beta_i-1)}{4^{\beta_n-\beta_i-1}\Gamma^2(\beta_n-\beta_i)}.$$

(ii) *Suppose that* $\beta_n - \beta_i \le \frac{1}{2}$ $(i = 1, 2, \ldots, n-1)$, *or* $0 < \beta_n \le \frac{1}{2}$, *then*

$$u(t) \le \widetilde{A}^{1/q}(t) \exp\left\{ \frac{t(q + \widetilde{B}(t))}{q} + \sum_{j=1}^{3} \frac{\widetilde{B}_j(t)}{q} \int_0^t k_i(\tau)\,\mathrm{d}\tau \right\}, \quad t \in [0, T), \tag{40}$$

*where*

$$\widetilde{A}(t) = \max_{0\le s\le t} \widetilde{H}(s) + \sum_{j=1}^{3} \widetilde{B}_j(t) \int_0^t \theta\bigl(\widetilde{G}_j(s), k_j(s), \gamma_j\bigr)\,\mathrm{d}s, \qquad \widetilde{H}(t) = \frac{|\delta|^q 2^{q-1} e^{-qt} t^{q(\beta_n - 1)}}{\Gamma^q(\beta_n)},$$

$$\widetilde{B}_j(t) = (12)^{(q-1)} \frac{1}{\Gamma^q(\beta_n)} \left( \frac{\Gamma(1 - p(1-\beta_n))}{p^{1-p(1-\beta_n)}} \right)^{\frac{q}{p}},$$

$$\widetilde{G}_j(t) = h_j^q(t) e^{q(\gamma_j - 1)t}, \quad j = 1, 2, 3, \qquad \widetilde{B}(t) = \max_{0\le s\le t} \widetilde{K}(s),$$

$$\widetilde{K}(t) = \bigl(4(n-1)\bigr)^{(q-1)} \sum_{i=1}^{n-1} \frac{1}{\Gamma^q(\beta_n - \beta_i)} \left( \frac{e^{pt}\Gamma(1 - p(1-\alpha_i))}{p^{1-p(1-\alpha_i)}} \right)^{\frac{q}{p}},$$

$$p = 1 + \theta, \qquad q = 1 + \frac{1}{\theta}, \qquad \theta = \min\{\beta_n - \beta_i, \beta_n, i = 1, 2, \ldots, n-1\}.$$

*Proof* From (37) and (38) we have

$$\begin{aligned}
\bigl|y(t)\bigr| &\le \frac{1}{\Gamma(\beta_n)} \int_0^t (t-s)^{\beta_n - 1} \bigl|f\bigl(s, y(s)\bigr)\bigr|\,\mathrm{d}s \\
&\quad + \sum_{i=1}^{n-1} \frac{1}{\Gamma(\beta_n - \beta_i)} \int_0^t (t-s)^{\beta_n - \beta_i - 1} \bigl|y(s)\bigr|\,\mathrm{d}s + |\delta| \frac{t^{\beta_n - 1}}{\Gamma(\beta_n)} \\
&\le |\delta| \frac{t^{\beta_n - 1}}{\Gamma(\beta_n)} + \frac{1}{\Gamma(\beta_n)} \int_0^t (t-s)^{\beta_n - 1} \bigl(h_1(s)\bigl|y(s)\bigr|^{\gamma_1} + h_2(s)\bigl|y(s)\bigr|^{\gamma_2} + h_3(s)\bigl|y(s)\bigr|^{\gamma_3}\bigr)\,\mathrm{d}s \\
&\quad + \sum_{i=1}^{n-1} \frac{1}{\Gamma(\beta_n - \beta_i)} \int_0^t (t-s)^{\beta_n - \beta_i - 1} \bigl|y(s)\bigr|\,\mathrm{d}s.
\end{aligned}$$

As an application of Theorem 2.1, we obtain the inequalities (39) and (40). This process completes the proof of Theorem 3.1. □

Next, we consider some fractional differential equations with the Caputo fractional derivative. The Caputo fractional order derivative is defined as follows.

**Definition 3** [21] The Caputo fractional derivative of order $\alpha$ ($n - 1 < \alpha < n$, $n$ is a positive integer) of a continuous function $f : \mathbb{R}_+ \to \mathbb{R}$ is given by

$$D_t^\alpha f(x) := \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^x (x-t)^{-\alpha+n-1} f^{(n)}(t)\,\mathrm{d}t.$$

Let us consider the initial value problems for fractional differential equations with delay in the following form:

$$D_t^\beta y(t) = f\bigl(t, y(t-r)\bigr), \quad t \in [t_0, T), \tag{41}$$

with the given initial condition

$$y(t) = \phi(t), \quad t \in [t_0 - r, t_0], \tag{42}$$

where $f \in C([t_0, T) \times \mathbb{R}, \mathbb{R})$ and $\phi$ is a given continuously differentiable function on $[t_0 - r, t_0]$ up to order $n$ ($n = -[-\beta]$), and we denote $\phi^k(t_0) = b_k$, $k = 0, 1, 2, \dots, n-1$.

**Theorem 3.2** *Suppose that*

$$\bigl|f(t,y)\bigr| \le h_0(t)|y| + h_1(t)|y|^{\gamma_1} + h_2(t)|y|^{\gamma_2}, \quad (t,y) \in [t_0, T) \times \mathbb{R}, \tag{43}$$

*where* $0 < \gamma_1, \gamma_2 < 1$ *are constants independent of* $t$, $y$ *in* $\mathbb{R}$, $h_0(t)$, $h_1(t)$, *and* $h_2(t)$ *are nonnegative continuous functions defined on* $[t_0, T)$. *If* $y(t)$ *is any solution of the initial value problem* (41)-(42), *then, for any positive functions* $k_j(t) \in C([t_0, T), (0, +\infty))$ ($j = 1, 2$), *the following assertions hold*:

(i) *Suppose that* $\beta > \frac{1}{2}$, *then*

$$\begin{aligned} u(t) \le e^t \Biggl\{ & A(t) + \Biggl[ B(t) \int_{t_0}^{t_0+r} F(s)\phi_1(s-r)\,\mathrm{d}s + \sum_{j=1}^{2} B_j(t) \int_{t_0}^{t_0+r} k_j(s)\phi_1(s-r)\,\mathrm{d}s \Biggr] \\ & \cdot \exp\Biggl\{ B(t) \int_{t_0+r}^{t} F(\tau)\,\mathrm{d}\tau + \sum_{j=1}^{2} B_j(t) \int_{t_0+r}^{t} k_j(\tau)\,\mathrm{d}\tau \Biggr\} \\ & + B(t) \int_{t_0+r}^{t} A(s-r)F(s) \\ & \cdot \exp\Biggl\{ B(t) \int_{s}^{t} F(\tau)\,\mathrm{d}\tau + \sum_{j=1}^{2} B_j(t) \int_{s}^{t} k_j(\tau)\,\mathrm{d}\tau \Biggr\}\,\mathrm{d}s \\ & + \sum_{j=1}^{2} B_j(t) \int_{t_0+r}^{t} A(s-r)k_j(s) \\ & \cdot \exp\Biggl\{ B(t) \int_{s}^{t} F(\tau)\,\mathrm{d}\tau + \sum_{j=1}^{2} B_j(t) \int_{s}^{t} k_j(\tau)\,\mathrm{d}\tau \Biggr\}\,\mathrm{d}s \Biggr\}^{1/2}, \quad t \in [t_0 + r, T), \end{aligned} \tag{44}$$

*where*

$$\begin{aligned} A(t) &= H(t) + \sum_{j=1}^{2} K_j(t) \int_{t_0}^{t} \theta\bigl(G_j(s), k_j(s), \gamma_j\bigr)\,\mathrm{d}s, \\ H(t) &= 2e^{-2t} \Biggl( \sum_{k=0}^{n-1} \frac{b_k}{k!}(t-t_0)^k \Biggr)^2, \\ B_j(t) &= K_j(t) = \frac{e^{-2\gamma_j r}\Gamma(2\beta-1)}{4^{\beta-2}\Gamma^2(\beta)}, \qquad G_j(t) = h_j^2(t)e^{2(\gamma_j-1)t}, \quad j = 1, 2, \\ B(t) &= \frac{2e^{2t}\Gamma(2\beta-1)}{4^{\beta-1}\Gamma^2(\beta)}, \qquad F(t) = h_0^2(t), \qquad \phi_1(t) = e^{-2t}\phi^2(t). \end{aligned}$$

(ii) *Suppose that* $0 < \beta \le \frac{1}{2}$, *then*

$$\begin{aligned}
u(t) \le e^t \Bigg\{ & \widetilde{A}(t) + \Bigg[\widetilde{B}(t)\int_{t_0}^{t_0+r} \widetilde{F}(s)\phi_2(s-r)\,\mathrm{d}s + \sum_{j=1}^{2}\widetilde{B}_j(t)\int_{t_0}^{t_0+r} k_j(s)\phi_2(s-r)\,\mathrm{d}s\Bigg] \\
& \cdot \exp\Bigg\{\widetilde{B}(t)\int_{t_0+r}^{t}\widetilde{F}(\tau)\,\mathrm{d}\tau + \sum_{j=1}^{2}\widetilde{B}_j(t)\int_{t_0+r}^{t} k_j(\tau)\,\mathrm{d}\tau\Bigg\} \\
& + \widetilde{B}(t)\int_{t_0+r}^{t}\widetilde{A}(s-r)\widetilde{F}(s) \\
& \cdot \exp\Bigg\{\widetilde{B}(t)\int_{s}^{t}\widetilde{F}(\tau)\,\mathrm{d}\tau + \sum_{j=1}^{2}\widetilde{B}_j(t)\int_{s}^{t} k_j(\tau)\,\mathrm{d}\tau\Bigg\}\,\mathrm{d}s \\
& + \sum_{j=1}^{2}\widetilde{B}_j(t)\int_{t_0+r}^{t}\widetilde{A}(s-r)k_j(s) \\
& \cdot \exp\Bigg\{\widetilde{B}(t)\int_{s}^{t}\widetilde{F}(\tau)\,\mathrm{d}\tau + \sum_{j=1}^{2}\widetilde{B}_j(t)\int_{s}^{t} k_j(\tau)\,\mathrm{d}\tau\Bigg\}\,\mathrm{d}s\Bigg\}^{1/q}, \quad t \in [t_0+r, T), \qquad (45)
\end{aligned}$$

*where*

$$\begin{aligned}
&\widetilde{A}(t) = \max_{t_0\le s\le t}\widetilde{H}(s) + \sum_{j=1}^{2}\widetilde{B}_j(t)\int_{t_0}^{t}\theta\big(\widetilde{G}_j(s), k_j(s), \gamma_j\big)\,\mathrm{d}s, \\
&\widetilde{H}(t) = 2^{q-1}e^{-qt}\left(\sum_{k=0}^{n-1}\frac{b_k}{k!}(t-t_0)^k\right)^q, \\
&\widetilde{B}_j(t) = \frac{8^{q-1}e^{-q\gamma_j r}}{\Gamma^q(\beta)}\left(\frac{\Gamma(1-p(1-\beta))}{p^{1-p(1-\beta)}}\right)^{\frac{q}{p}}, \qquad \widetilde{G}_j(t) = g_j^q(t)e^{q(\gamma_j-1)t}, \quad j = 1, 2, \\
&\widetilde{B}(t) = \frac{4^{q-1}}{\Gamma^q(\beta)}\left(\frac{e^{pt}\Gamma(1-p(1-\beta))}{p^{1-p(1-\beta)}}\right)^{\frac{q}{p}}, \qquad \phi_2(t) = e^{-qt}\phi^q(t), \\
&p = 1+\beta, \qquad q = 1+\frac{1}{\beta}, \qquad \widetilde{F}(t) = h_0^q(t).
\end{aligned}$$

*Proof* The solution $y(t)$ of the initial value problem (41)-(42) can be written as (see [21])

$$\begin{cases} y(t) = \sum_{k=0}^{n-1}\frac{b_k}{k!}(t-t_0)^k + \frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^\beta f(s, y(s))\,\mathrm{d}s, & t \in [t_0, T), \\ u(t) = \phi(t), \quad t \in [t_0-r, t_0]. \end{cases}$$

So

$$\begin{aligned}
\big|y(t)\big| \le {} & \sum_{k=0}^{n-1}\frac{|b_k|}{k!}(t-t_0)^k + \frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^\beta \\
& \cdot \big(h_0(s)\big|y(s)\big| + h_1(s)\big|y(s)\big|^{\gamma_1} + h_2(s)\big|y(s)\big|^{\gamma_2}\big)\,\mathrm{d}s, \quad t \in [t_0, T).
\end{aligned}$$

As an application of Theorem 2.2, we obtain the inequalities (44) and (45). This process completes the proof of Theorem 3.2. □

**Competing interests**
The authors declare that there is no conflict of interests regarding the publication of this paper.

**Authors' contributions**
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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