Xue and Hu *Journal of Inequalities and Applications* (2015) 2015:194
DOI 10.1186/s13660-015-0721-5

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

On Fischer-type determinantal inequalities for accretive-dissipative matrices

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Abstract
This paper aims to give some refinements of recent results on Fischer-type determinantal inequalities for accretive-dissipative matrices.

MSC: 15A45

Keywords: accretive-dissipative matrix; Fischer determinantal inequality; Buckley matrix

1 Introduction

Let $M_n(C)$ be the set of $n \times n$ complex matrices. For any $A \in M_n(C)$, the conjugate transpose of A is denoted by A^*. $A \in M_n(C)$ is accretive-dissipative if it has the Hermitian decomposition

$$A = B + iC, \qquad B = B^*, \qquad C = C^*, \tag{1.1}$$

where both matrices B and C are positive definite. Conformally partition A, B, C as

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{pmatrix} + i \begin{pmatrix} C_{11} & C_{12} \\ C_{12}^* & C_{22} \end{pmatrix}, \tag{1.2}$$

such that all diagonal blocks are square. Say k and l ($k, l > 0$ and $k + l = n$) the order of A_{11} and A_{22}, respectively, and let $m = \min\{k, l\}$. In this article, we always partition A as in (1.2).

If $B = I_n$ in (1.1), then an accretive-dissipative matrix $A \in M_n(C)$ is called a Buckley matrix.

Let $A = \left(\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right) \in M_n(C)$. If A_{11} is invertible, then the Schur complement of A_{11} in A is denoted by $A/A_{11} := A_{22} - A_{21}A_{11}^{-1}A_{12}$. For a nonsingular matrix A, its condition number is denoted by $k(A) := \sqrt{\frac{\lambda_{\max}(A^*A)}{\lambda_{\min}(A^*A)}}$, which is the ratio of the largest and the smallest singular value of A. For Hermitian matrices $B, C \in M_n(C)$, we write $B > (\geq)\, C$ to mean that $B - C$ is Hermitian positive (semi)definite.

If $A \in M_n(C)$ is positive definite, then the famous Fischer-type determinantal inequality ([1], p.478) states that

$$\det A \leq \det A_{11} \cdot \det A_{22}. \tag{1.3}$$

If $A \in M_n(C)$ is accretive-dissipative, Ikramov [2] first proved the determinantal inequality

$$|\det A| \le 3^m |\det A_{11}| \cdot |\det A_{22}|. \tag{1.4}$$

If $A \in M_n(C)$ is accretive-dissipative, Lin [3] proved the determinantal inequality

$$|\det A| \le 2^{\frac{3m}{2}} |\det A_{11}| \cdot |\det A_{22}|. \tag{1.5}$$

Recently, Fu and He ([4], Theorem 1) got a stronger result than (1.5) as follows.
Let $A \in M_n(C)$ be accretive-dissipative and partitioned as in (1.2). Then

$$|\det A| \le 2^{\frac{m}{2}} \left[1 + \left(\frac{1-k}{1+k}\right)^2\right]^m |\det A_{11}| \cdot |\det A_{22}|, \tag{1.6}$$

where $k = \max(k(B), k(C))$.
For Buckley matrices, Ikramov [2] obtained the stronger bound

$$|\det A| \le \left(\frac{1+\sqrt{17}}{4}\right)^m |\det A_{11}| \cdot |\det A_{22}|. \tag{1.7}$$

In this paper, we will give refinements of (1.6) and (1.7) in Section 2. Other related studies of the Fischer-type determinantal inequalities for accretive-dissipative matrices can be found in [5–7].

2 Main results

We begin this section with the following lemmas.

Lemma 1 ([8], Property 6) *Let $A \in M_n(C)$ be accretive-dissipative and partitioned as in* (1.2). *Then A/A_{11} is also accretive-dissipative.*

Lemma 2 ([2], Lemma 1) *Let $A \in M_n(C)$ be accretive-dissipative as in* (1.1). *Then*

$$A^{-1} = E - iF, \qquad E = \left(B + CB^{-1}C\right)^{-1}, \qquad F = \left(C + BC^{-1}B\right)^{-1}.$$

Lemma 3 ([9], Lemma 3.2) *Let $B, C \in M_n(C)$ be Hermitian and assume B is positive definite. Then*

$$B + CB^{-1}C \ge 2C.$$

Lemma 4 ([10], (6)) *Let $B = \left(\begin{smallmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{smallmatrix}\right)$ be Hermitian positive definite. Then*

$$B_{12}^* B_{11}^{-1} B_{12} \le \left(\frac{1-k(B)}{1+k(B)}\right)^2 B_{22}.$$

Lemma 5 ([3], Lemma 6) *Let $B, C \in M_n(C)$ be positive semidefinite. Then*

$$\left|\det(B + iC)\right| \le \det(B + C).$$

Lemma 6 ([11], (1.2)) *Let $a, b > 0$. Then*

$$\left[1 + \frac{(\ln a - \ln b)^2}{8}\right]\sqrt{ab} \le \frac{a+b}{2}.$$

Lemma 7 *Let $B, C \in M_n(C)$ be positive definite. Then*

$$\det(B + C) \le r^n \left|\det(B + iC)\right|,$$

where $r = \max_{1\le j\le n}\{\sqrt{1 + \frac{2}{2+(\ln\lambda_j)^2}}\}$, λ_j are the eigenvalues of $B^{-1/2}CB^{-1/2}$, and $B^{1/2}$ means the unique positive definite square root of B.

Proof Letting $a = \lambda_j$, $b = \frac{1}{a}$ in Lemma 6 gives $1 + \lambda_j \le \sqrt{1 + \frac{2}{2+(\ln\lambda_j)^2}}|1 + i\lambda_j|$, $j = 1, \dots, n$. Then

$$\begin{aligned}
\det(B + C) &= \det B \cdot \det\bigl(I + B^{-1/2}CB^{-1/2}\bigr) \\
&= \det B \cdot \prod_{j=1}^{n} (1 + \lambda_j) \\
&\le \det B \cdot \prod_{j=1}^{n} \left(\sqrt{1 + \frac{2}{2 + (\ln\lambda_j)^2}}|1 + i\lambda_j|\right) \\
&\le \det B \cdot \prod_{j=1}^{n} \bigl(r|1 + i\lambda_j|\bigr) \\
&= r^n \det B \cdot \left|\det\bigl(I + iB^{-1/2}CB^{-1/2}\bigr)\right| \\
&= r^n \left|\det(B + iC)\right|.
\end{aligned}$$

This completes the proof. □

Theorem 1 *Let $A \in M_n(C)$ be accretive-dissipative and partitioned as in* (1.2). *Then*

$$|\det A| \le \left[1 + \left(\frac{1-k}{1+k}\right)^2\right]^m r^m |\det A_{11}| \cdot |\det A_{22}|, \tag{2.1}$$

where $r = \max_{1\le j\le n}\{\sqrt{1 + \frac{2}{2+(\ln\lambda_j)^2}}\}$, λ_j are the eigenvalues of $B^{-1/2}CB^{-1/2}$, $B^{1/2}$ means the unique positive definite square root of B, and $k = \max(k(B), k(C))$.

Proof By Lemma 2 and Lemma 3, we have

$$\begin{aligned}
A/A_{11} &= A_{22} - A_{21}A_{11}^{-1}A_{12} \\
&= B_{22} + iC_{22} - \bigl(B_{12}^* + iC_{12}^*\bigr)(B_{11} + iC_{11})^{-1}(B_{12} + iC_{12}) \\
&= B_{22} + iC_{22} - \bigl(B_{12}^* + iC_{12}^*\bigr)(E_k - iF_k)(B_{12} + iC_{12})
\end{aligned}$$

with

$$E_k = \bigl(B_{11} + C_{11}B_{11}^{-1}C_{11}\bigr)^{-1} \le \frac{1}{2}C_{11}^{-1}, \qquad F_k = \bigl(C_{11} + B_{11}C_{11}^{-1}B_{11}\bigr)^{-1} \le \frac{1}{2}B_{11}^{-1}. \tag{2.2}$$

Set $A/A_{11} = R + iS$ with $R = R^*$ and $S = S^*$. By Lemma 1, we obtain

$$R = B_{22} - B_{12}^* E_k B_{12} + C_{12}^* E_k C_{12} - B_{12}^* F_k C_{12} - C_{12}^* F_k B_{12},$$
$$S = C_{22} + B_{12}^* F_k B_{12} - C_{12}^* F_k C_{12} - C_{12}^* E_k B_{12} - B_{12}^* E_k C_{12}.$$

It can be proved that

$$\pm\left(B_{12}^* F_k C_{12} + C_{12}^* F_k B_{12}\right) \le B_{12}^* F_k B_{12} + C_{12}^* F_k C_{12},$$
$$\pm\left(C_{12}^* E_k B_{12} + B_{12}^* E_k C_{12}\right) \le C_{12}^* E_k C_{12} + B_{12}^* E_k B_{12}.$$

Thus,

$$R + S \le B_{22} + 2B_{12}^* F_k B_{12} + C_{22} + 2C_{12}^* E_k C_{12}. \tag{2.3}$$

As B, C are positive definite, by Lemma 4, we have

$$B_{12}^* B_{11}^{-1} B_{12} \le \left(\frac{1-k(B)}{1+k(B)}\right)^2 B_{22}, \qquad C_{12}^* C_{11}^{-1} C_{12} \le \left(\frac{1-k(C)}{1+k(C)}\right)^2 C_{22}. \tag{2.4}$$

Without loss of generality, we assume $m = l$, then

$$\begin{aligned}
\left|\det(A/A_{11})\right| &= \left|\det(R+iS)\right| \\
&\le \det(R+S) \quad \text{(by Lemma 5)} \\
&\le \det\left(B_{22} + 2B_{12}^* F_k B_{12} + C_{22} + 2C_{12}^* E_k C_{12}\right) \quad \text{(by (2.3))} \\
&\le \det\left(B_{22} + B_{12}^* B_{11}^{-1} B_{12} + C_{22} + C_{12}^* C_{11}^{-1} C_{12}\right) \quad \text{(by (2.2))} \\
&\le \det\left\{\left[1 + \left(\frac{1-k(B)}{1+k(B)}\right)^2\right] B_{22} + \left[1 + \left(\frac{1-k(C)}{1+k(C)}\right)^2\right] C_{22}\right\} \quad \text{(by (2.4))} \\
&\le \left[1 + \left(\frac{1-k}{1+k}\right)^2\right]^m \det(B_{22} + C_{22}) \\
&\le \left[1 + \left(\frac{1-k}{1+k}\right)^2\right]^m r^m \left|\det(B_{22} + iC_{22})\right| \quad \text{(by Lemma 7)} \\
&= \left[1 + \left(\frac{1-k}{1+k}\right)^2\right]^m r^m |\det A_{22}|,
\end{aligned}$$

where $k = \max(k(B), k(C))$.

The proof is completed by noting $|\det A| = |\det A_{11}| \cdot |\det(A/A_{11})|$. □

Remark 1 Because of $r \le \sqrt{2}$, inequality (2.1) is a refinement of inequality (1.6).

Theorem 2 *Let* $A \in M_n(C)$ *be accretive-dissipative and partitioned as in* (1.2) *with* $B_{12} = 0$. *Then*

$$|\det A| \le \left(\frac{\sqrt{17}+1}{4}\right)^m |\det A_{11}| \cdot |\det A_{22}|. \tag{2.5}$$

Proof Compute

$$\begin{aligned}
|\det A| &= \left|\det(B + iC)\right| \\
&= \det B \cdot \left|\det\left(I + iB^{-1/2}CB^{-1/2}\right)\right| \\
&\le \left(\frac{\sqrt{17}+1}{4}\right)^m \det B \cdot \left|\det\left(I_k + iB_{11}^{-1/2}C_{11}B_{11}^{-1/2}\right)\right| \\
&\quad \cdot \left|\det\left(I_l + iB_{22}^{-1/2}C_{22}B_{22}^{-1/2}\right)\right| \quad \text{(by (1.7))} \\
&= \left(\frac{\sqrt{17}+1}{4}\right)^m \left|\det(B_{11} + iC_{11})\right| \cdot \left|\det(B_{22} + iC_{22})\right| \\
&= \left(\frac{\sqrt{17}+1}{4}\right)^m |\det A_{11}| \cdot |\det A_{22}|.
\end{aligned}$$

This completes the proof. □

Remark 2 It is clear that inequality (2.5) is an extension of inequality (1.7).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Acknowledgements
The authors wish to express their heartfelt thanks to the referees for their detailed and helpful suggestions for revising the manuscript. At the same time, we are grateful to Prof. Xiaorong Gan for her fruitful discussions.

Received: 23 March 2015 Accepted: 31 May 2015 Published online: 14 June 2015

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