Xiang et al. *Journal of Inequalities and Applications* (2015) 2015:197
DOI 10.1186/s13660-015-0715-3

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Connecting orbits for Newtonian-like N-body problems

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Abstract
Using variational minimizing methods, we prove the existence of a connecting orbit between the center of mass and infinity of Newtonian-like N-body problems with Newtonian-type weak force potentials.

MSC: 34A34; 34C25; 34C37; 35A15

Keywords: N-body problems; variational minimizers; connecting orbits

1 Introduction

In the 1989 paper of Rabinowitz [1], we find the first substantial use of variational methods to study heteroclinic orbits for Hamiltonian systems. The perspective of that work appears influential for a number of papers by several authors which followed [2–15]. Especially, we would like to draw attention to Souissi [13], Maderna and Venturelli [14] and Zhang [15] for a study of the parabolic orbits for restricted 3-body problems and complete N-body problems. From those studies, we draw motivation for the present work: namely, we extend the results and methods of Souissi [13] and Zhang [15] to Newtonian-like N-body problems.

Given masses $m_1,\dots,m_N>0$ of N bodies, we study the following system of equations with Newtonian-type weak force potentials:

$$m_i\ddot{q}_i(t)+\frac{\partial U(q)}{\partial q_i}=0, \tag{1.1}$$

where $q_i\in R^k$, $q=(q_1,\dots,q_N)$, $0<\alpha<2$, and

$$U(q)=\sum_{1\le i<j\le N}\frac{m_im_j}{|q_i-q_j|^\alpha}. \tag{1.2}$$

We apply the variational minimizing method to prove the following.

Theorem 1.1 *For* (1.1), *there exists one connecting orbit* $\tilde{q}(t)=(\tilde{q}_1(t),\dots,\tilde{q}_N(t))$ *between the center of mass and infinity such that*:

(i) *For any* $1\le i\ne j\le N$,

$$\max_{0\le t\le+\infty}\left|\tilde{q}_i(t)-\tilde{q}_j(t)\right|=+\infty. \tag{1.3}$$

(ii)

$$\min_{0\le t\le +\infty}\sum_{1}^{N} m_i\left|\dot{\tilde{q}}_i(t)\right|^2 = 2E \ge 0. \tag{1.4}$$

2 Variational minimizing critical points

In order to find a connecting orbit of (1.1), we shall first find a solution of the system (1.1) on the open interval $(0,\tau)$ and then consider the limit orbit as $\tau \to +\infty$. To find a solution on $(0,\tau)$, we define the functional

$$f(q) = \int_0^\tau \left(\frac{1}{2}\sum_{i=1}^{N} m_i\left|\dot{q}_i(t)\right|^2 + U(q)\right) dt, \tag{2.1}$$

where

$$q_i \in H_\tau = \left\{x, \dot{x} \in L^2[0,\tau] | x_i(0) = 0, x_i(\tau) = a_i\right\}, \tag{2.2}$$

where $(a_1,\dots,a_i,\dots,a_N)$ is a central configuration for the N-body problems which satisfies $a_j \neq a_i$, $1 \le j \neq i \le N$, and there is $\lambda \in R$ such that

$$\sum_{j\neq i}\frac{m_j m_i(a_j - a_i)}{|a_j - a_i|^{\alpha+2}} = \lambda m_i a_i. \tag{2.3}$$

Since $\forall q_i \in H_\tau$, $q_i(0) = 0$, for $q = (q_1,\dots,q_N) \in H_\tau \times \cdots \times H_\tau$ we have the equivalent norm

$$\|q\|_\tau = \left(\sum_{i=1}^{N} m_i \int_0^\tau \left|\dot{q}_i(t)\right|^2 dt\right)^{1/2}. \tag{2.4}$$

Lemma 2.1 (Tonelli [16]) *Let X be a reflexive Banach space and $f : X \to R \cup \{+\infty\}$. If f does not always take $+\infty$ and is weakly lower semi-continuous and coercive* ($f(x) \to +\infty$, *as* $\|x\| \to +\infty$), *then f attains its infimum on X.*

Lemma 2.2 *The functional $f(q)$ defined in* (2.1) *is weakly lower semi-continuous* (*w.l.s.c.*) *on $H_\tau \times \cdots \times H_\tau$.*

Proof (1) It is well known that the norm and its square are w.l.s.c.

(2) $\forall\{q_i^n\} \subset H_\tau$, if $q_i^n \rightharpoonup q_i$ weakly, then by the compact embedding theorem, we have the following uniform convergence:

$$\max_{0\le t\le\tau}\left|q_i^n(t) - q_i(t)\right| \to 0, \quad n \to +\infty. \tag{2.5}$$

Let $S = \{\tilde{t} \in [0,\tau] : \exists 1 \le i_0 \neq j_0 \le N \text{ s.t. } q_{i_0}(t_0) = q_{j_0}(t_0)\}$ and let $m(S)$ denote the Lebesgue measure of S.

(i) If $m(S) = 0$, then $U(q^n(t)) \overset{\text{a.e.}}{\to} U(q(t))$. From Fatou's lemma we have

$$\int_0^\tau U(q)\,dt \le \varliminf_{n\to\infty}\int_0^\tau U\big(q^n(t)\big)\,dt. \tag{2.6}$$

(ii) If $m(S) > 0$, then $\int_0^\tau U(q)\,dt = +\infty$ and $f(q) = +\infty$.

Since $q^n(t) \to q(t)$ uniformly we have $\int_0^\tau U(q^n(t))\,dt \to +\infty$, and so

$$\varliminf_{n\to\infty} f(q^n) \geq f(q). \tag{2.7}$$

□

The proof of the next lemma is straightforward.

Lemma 2.3 *f is coercive on $H_\tau \times \cdots \times H_\tau$.*

Lemma 2.4

(1) *$f(q)$ attains its infimum on $H_\tau \times \cdots \times H_\tau$, and the minimizer $\tilde{q}^\tau(t) = (\tilde{q}_1^\tau(t), \ldots, \tilde{q}_N^\tau(t))$ is a generalized solution* [16].

(2) *Furthermore, when $\tau \to +\infty$ and $\tilde{q}_i^\tau(t) \to \tilde{q}_i(t)$, $\tilde{q}_i(t)$ has the following properties*:

(i) *for any* $1 \leq i \neq j \leq N$,

$$\max_{0\leq t\leq +\infty} \left|\tilde{q}_i(t) - \tilde{q}_j(t)\right| = +\infty, \tag{2.8}$$

(ii)

$$\min_{0\leq t\leq +\infty} \sum_1^N m_i \left|\dot{\tilde{q}}_i(t)\right|^2 = 2E. \tag{2.9}$$

Definition 2.5 Concerning the velocities of the solution of (1.1),

(1°) if, for all i,

$$\left|\dot{\tilde{q}}_i(t)\right| \to 0, \quad t \to +\infty \tag{2.10}$$

we say $\tilde{q}(t)$ is a parabolic solution;

(2°) if, for all i,

$$\left|\dot{\tilde{q}}_i(t)\right| \to v_i > 0, \quad t \to +\infty \tag{2.11}$$

we say $\tilde{q}(t)$ is a hyperbolic solution;

otherwise, we call it a mixed type solution.

The proof of (1) in Lemma 2.4 is obvious using Lemmas 2.1-2.3.

In the following, we will give the proofs of (2.8) and (2.9) of Lemma 2.4.

Lemma 2.6 *There exist constants $c > 0$ and $0 < \theta < 1$ independent of τ such that*

$$f\left(\tilde{q}^\tau\right) \leq c\tau^\theta. \tag{2.12}$$

Proof We choose a special orbit defined by

$$q_i(t) = a_i t^\beta, \quad t \in [0, \tau], a_i \in R^k, \tag{2.13}$$

where $(a_1, a_2, \ldots, a_N)$ can be a given central configuration, $\frac{1}{2} < \beta < \min\{1, \frac{1}{\alpha}\}$, then

$$\begin{aligned} f\bigl(q(t)\bigr) &= \frac{1}{2}\sum_{i=1}^{N} m_i|a_i|^2 \int_0^\tau \beta^2 t^{2(\beta-1)}\,dt + \int_0^\tau \sum_{1\le i<j\le N} \frac{m_i m_j}{|a_i - a_j|^\alpha} t^{-\alpha\beta}\,dt \\ &\le \frac{1}{2}\left(\sum_{i=1}^{N} m_i|a_i|^2\right)\frac{\beta^2}{2\beta-1}\tau^{2\beta-1} \\ &\quad + \left(\sum_{1\le i<j\le N} \frac{m_i m_j}{|a_i - a_j|^\alpha}\right)\frac{1}{1-\alpha\beta}\tau^{1-\alpha\beta} \\ &\le c\tau^\theta, \end{aligned} \tag{2.14}$$

where

$$\theta = \max(2\beta - 1, 1 - \alpha\beta) \tag{2.15}$$

and

$$c = \frac{1}{2}\sum_{1}^{N} m_i|a_i|^2 \frac{\beta^2}{2\beta-1} + \sum_{1\le i<j\le N} \frac{m_i m_j}{|a_i - a_j|^\alpha}\frac{1}{1-\alpha\beta} > 0. \tag{2.16}$$

When $0 < \alpha < 2$, we have $\frac{1}{\alpha} > \frac{1}{2}$. We can choose $\frac{1}{2} < \beta < \frac{1}{\alpha}$, then $2\beta - 1 > 0$, $1 - \alpha\beta > 0$, and hence $\theta > 0$. When $\beta < 1$, $2\beta - 1 < 1$, then $0 < \theta < 1$. □

Lemma 2.7 *Let $\tilde{q}^n(t) = (\tilde{q}_1^n(t), \ldots, \tilde{q}_N^n(t))$ be critical points corresponding to the minimizing critical values $\min_{H_n} f(q)$, where H_n was defined in* (2.2) *when $\tau = n$. Then the maximum distance between $\tilde{q}_i^n$ and $\tilde{q}_j^n$ on R^+ satisfies*

$$\bigl\|\tilde{q}_i^n(t) - \tilde{q}_j^n(t)\bigr\|_\infty \to +\infty, \quad \textit{when } n \to +\infty. \tag{2.17}$$

Proof By the definition of $f(\tilde{q}^n)$ and Lemma 2.6, we have the inequalities

$$cn^\theta \ge f\bigl(\tilde{q}^n\bigr) \ge \int_0^n \sum_{1\le i<j\le N} \frac{m_i m_j}{|\tilde{q}_i^n(t) - \tilde{q}_j^n(t)|^\alpha}\,dt. \tag{2.18}$$

Hence

$$\sum_{1\le i<j\le N} \frac{m_i m_j}{\|\tilde{q}_i^n(t) - \tilde{q}_j^n(t)\|_\infty^\alpha} \le cn^{\theta-1} \to 0, \tag{2.19}$$

from which it follows that $\forall 1 \le i < j \le N$, $\|\tilde{q}_i^n(t) - \tilde{q}_j^n(t)\|_\infty \to +\infty$, $n \to +\infty$. □

Lemma 2.8 $\{\tilde{q}^n(t)\}$ *is equi-continuous and uniformly bounded on any compact interval.*

Proof By the proof of Lemma 2.6, we can see $\forall T > 0$,

$$\sum_{i=1}^{N} m_i \int_0^T \bigl|\dot{\tilde{q}}_i^n(t)\bigr|^2\,dt \le cT^\theta. \tag{2.20}$$

Then, for any $0 \leq s, r \leq T$, we have

$$\begin{aligned} \left|\tilde{q}_i^n(s) - \tilde{q}_i^n(r)\right| &\leq \int_r^s \left|\dot{\tilde{q}}_i^n(t)\right| dt \\ &\leq |s-r|^{1/2} \left(\int_r^s \left|\dot{\tilde{q}}_i^n(t)\right|^2 dt \right)^{1/2} \\ &\leq \left(\frac{cT^\theta}{m_i} \right)^{1/2} |s-r|^{1/2}. \end{aligned} \tag{2.21}$$

By $q^n(0) = 0$ and the above inequality, for $0 < s < T$, we have

$$\left|\tilde{q}_i^n(s)\right| \leq \left(\frac{cT^\theta}{m_i} \right)^{1/2} |s|^{1/2} \leq \left(\frac{cT^\theta}{m_i} \right)^{1/2} T^{1/2}. \tag{2.22}$$

□

Now we can prove Theorem 1.1.

Proof of Theorem 1.1 For any compact interval $[a, b]$ of R^+, Marchal's theorem [17] implies that $\tilde{q}^n(t)$ has no collision on (a, b), so, by the Ascoli-Arzelà theorem, we know $\{\tilde{q}^n\}$ has a sub-sequence converging uniformly to a limit $\tilde{q}(t)$ on any compact set $[c, d] \subset (a, b)$, and $\tilde{q}(t) \in C^2(R^+, R^k)$ is a solution of (1.1). By the energy conservation law and (2.17), we have

$$E = \sum_{i=1}^N \frac{1}{2} m_i |\dot{\tilde{q}}_i|^2 - \sum_{1 \leq i < j \leq N} \frac{m_i m_j}{|\tilde{q}_i - \tilde{q}_j|^\alpha} \geq 0, \tag{2.23}$$

rewritten as

$$\sum_{i=1}^N \frac{1}{2} m_i |\dot{\tilde{q}}_i|^2 = \sum_{1 \leq i < j \leq N} \frac{m_i m_j}{|\tilde{q}_i - \tilde{q}_j|^\alpha} + E. \tag{2.24}$$

Now we claim:
(i) for any $1 \leq i \neq j \leq N$,

$$\max_{t \in R^+} \left|\tilde{q}_i(t) - \tilde{q}_j(t)\right| = +\infty \tag{2.25}$$

suppose there exist $1 \leq i_0 < j_0 \leq N$ and $d > 0$ such that

$$\left|\tilde{q}_{i_0}(t) - \tilde{q}_{j_0}(t)\right| < d, \quad \forall t \in R^+. \tag{2.26}$$

By (2.24), there exist $1 \leq k_0 \leq N$ and $e > 0$ such that

$$|\dot{\tilde{q}}_{k_0}| > e, \quad \forall t \in R^+, \tag{2.27}$$

then we have

$$ct^\theta \geq \frac{1}{2} \int_0^t \sum_{i=1}^N m_i |\dot{\tilde{q}}_i|^2 \, dt \geq \frac{1}{2} \int_0^t m_{k_0} |\dot{\tilde{q}}_{k_0}|^2 \, dt \geq \frac{1}{2} m_{k_0} e^2 t. \tag{2.28}$$

This is a contradiction, since $0 < \theta < 1$ and $t \in R^+$.

Now by (2.24), we have:

$$\text{(ii)} \quad \min_{t\in R^+} \sum_{i=1}^{N} m_i \left|\dot{\tilde{q}}_i(t)\right|^2 = 2E \geq 0. \tag{2.29}$$

□

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The research and writing of this manuscript was a collaborative effort from all the authors. All authors read and approved the final manuscript.

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Acknowledgements
The authors sincerely thank the referees for their many valuable comments which help us improving the paper. This paper was partially supported by NSF of China (No. 11071175 and No. 11426181) and Fundamental Research Funds for the Central Universities (JBK 130401 and JBK 150931).

Received: 16 November 2014 Accepted: 26 May 2015 Published online: 16 June 2015

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