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# General $L_p$-mixed-brightness integrals

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**Abstract**

The notion of mixed-brightness integrals was introduced by Li and Zhu. In this paper, motivated by the notion of general $L_p$-projection bodies, introduced by Haberl and Schuster, we define general $L_p$-mixed-brightness integrals and determine their extremal values, as well as several other inequalities for them.

**MSC:** 52A20; 52A40

**Keywords:** mixed-brightness integrals; general $L_p$-mixed-brightness integrals; general $L_p$-projection body

## 1 Introduction

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in Euclidean space $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interiors and the set of origin-symmetric convex bodies in $\mathbb{R}^n$, we write $\mathcal{K}_o^n$ and $\mathcal{K}_{os}^n$, respectively. Let $\mathcal{S}_o^n$ denote the set of star bodies (about the origin) in $\mathbb{R}^n$ and let $S^{n-1}$ denote the unit sphere in $\mathbb{R}^n$. By $V(K)$ we denote the $n$-dimensional volume of a body $K$ and for the standard unit ball $B$ in $\mathbb{R}^n$, we write $\omega_n$ for its volume.

If $K \in \mathcal{K}^n$, then its support function, $h_K = h(K,\cdot) : \mathbb{R}^n \to (-\infty,\infty)$, is defined by [1, 2]

$$h(K,x) = \max\{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n,$$

where $x \cdot y$ denotes the standard inner product of $x$ and $y$.

Projection bodies of convex bodies were introduced at the turn of the previous century by Minkowski [1]. For $K \in \mathcal{K}^n$, the projection body, $\Pi K$, of $K$ is the origin-symmetric convex body, defined by

$$h(\Pi K,u) = \frac{1}{2}\int_{S^{n-1}} |u \cdot v| \, dS(K,v)$$

for all $u \in S^{n-1}$. Here $S(K,\cdot)$ denotes the surface area measure of $K$.

Using the classical notion of projection bodies, Li and Zhu [3] recently introduced the mixed-brightness integral: For $K_1,\dots,K_n \in \mathcal{K}^n$, the mixed-brightness integral, $D(K_1,\dots,K_n)$, is defined by

$$D(K_1,\dots,K_n) = \frac{1}{n}\int_{S^{n-1}} \delta(K_1,u)\cdots\delta(K_n,u)\, dS(u), \tag{1.1}$$

where $\delta(K,u) = \frac{1}{2}h(\Pi K,u)$ is the half brightness of $K \in \mathcal{K}^n$ in the direction $u$. Convex bodies $K_1,\ldots,K_n$ are said to have similar brightness if there exist constants $\lambda_1,\ldots,\lambda_n > 0$ such that $\lambda_1\delta(K_1,u) = \lambda_2\delta(K_2,u) = \cdots = \lambda_n\delta(K_n,u)$ for all $u \in S^{n-1}$.

Further, Li and Zhu [3] established the following Fenchel-Aleksandrov type inequality for mixed-brightness integrals.

**Theorem 1.A** *If $K_1,\ldots,K_n \in \mathcal{K}^n$ and $1 < m \leq n$, then*

$$D(K_1,\ldots,K_n)^m \leq \prod_{i=0}^{m-1} D(K_1,\ldots,K_{n-m},K_{n-i},\ldots,K_{n-i}), \tag{1.2}$$

*with equality if and only if $K_{n-m+1},K_{n-m+2},\ldots,K_n$ are all of similar brightness.*

More recently, Zhou *et al.* [4] obtained Brunn-Minkowski type inequalities for mixed-brightness integrals.

The notion of $L_p$-projection bodies was introduced by Lutwak *et al.* [5]. For each $K \in \mathcal{K}_o^n$ and $p \geq 1$, the $L_p$-projection body, $\Pi_pK$, is the origin-symmetric convex body whose support function is defined by

$$h^p_{\Pi_pK}(u) = \alpha_{n,p} \int_{S^{n-1}} |u,v|^p \, dS_p(K,\upsilon), \tag{1.3}$$

for all $u \in S^{n-1}$, where $\alpha_{n,p} = 1/n\omega_n c_{n-2,p}$ with $c_{n,p} = \omega_{n+p}/\omega_2\omega_n\omega_{p-1}$, and $S_p(K,\cdot)$ is the $L_p$-surface measure of $K$. The normalization in definition (1.3) is chosen such that $\Pi_pB = B$.

As part of the tremendous progress in the theory of Minkowski valuations (see [6–14]), Ludwig [15] discovered more general $L_p$-projection bodies $\Pi_p^\tau K \in \mathcal{K}_o^n$, which can be defined using the function $\varphi_\tau : \mathbb{R} \to [0,\infty)$ given by

$$\varphi_\tau(t) = |t| + \tau t,$$

where $\tau \in [-1,1]$. Now for $K \in \mathcal{K}_o^n$ and $p \geq 1$, let $\Pi_p^\tau K \in \mathcal{K}_o^n$ with support function

$$h^p_{\Pi_p^\tau K}(u) = \alpha_{n,p}(\tau) \int_{S^{n-1}} \varphi_\tau(u \cdot v)^p \, dS_p(K,\upsilon), \tag{1.4}$$

where

$$\alpha_{n,p}(\tau) = \frac{\alpha_{n,p}}{(1+\tau)^p + (1-\tau)^p}.$$

The normalization is again chosen such that $\Pi_p^\tau B = B$ for every $\tau \in [-1,1]$. Obviously, if $\tau = 0$, then $\Pi_p^\tau K = \Pi_pK$.

For general $L_p$-projection bodies, Haberl and Schuster [16] proved the general $L_p$-Petty projection inequality and determined the extremal values of volume for polars of general $L_p$-projection bodies. Wang and Wan [17] investigated Shephard type problems for general $L_p$-projection bodies. Wang and Feng [18] established general $L_p$-Petty affine projection inequality. These investigations were the starting point of a new and rapidly evolving asymmetric $L_p$-Brunn-Minkowski theory (see [13–32]).

In this article, using the notion of general $L_p$-projection bodies, we define general $L_p$-mixed-brightness integrals as follows: For $K_1,\ldots,K_n \in \mathcal{K}_o^n$, $p \geq 1$ and $\tau \in [-1,1]$, the general

$L_p$-mixed-brightness integral, $D_p^{(\tau)}(K_1,\dots,K_n)$, of $K_1,\dots,K_n$ is defined by

$$D_p^{(\tau)}(K_1,\dots,K_n)=\frac{1}{n}\int_{S^{n-1}}\delta_p^{(\tau)}(K_1,u)\cdots\delta_p^{(\tau)}(K_n,u)\,dS(u), \tag{1.5}$$

where $\delta_p^{(\tau)}(K,u)=\frac{1}{2}h(\Pi_p^\tau K,u)$ denotes the half general $L_p$-brightness of $K\in\mathcal{K}_o^n$ in the direction $u$. Convex bodies $K_1,\dots,K_n$ are said to have similar general $L_p$-brightness if there exist constants $\lambda_1,\dots,\lambda_n>0$ such that, for all $u\in S^{n-1}$,

$$\lambda_1\delta_p^{(\tau)}(K_1,u)=\lambda_2\delta_p^{(\tau)}(K_2,u)=\cdots=\lambda_n\delta_p^{(\tau)}(K_n,u).$$

**Remark 1.1** For $\tau=0$ in (1.5), we write $D_p^{(\tau)}(K_1,\dots,K_n)=D_p(K_1,\dots,K_n)$ and $\delta_p^{(\tau)}(K,u)=\delta_p(K,u)$ for all $u\in S^{n-1}$. Then

$$D_p(K_1,\dots,K_n)=\frac{1}{n}\int_{S^{n-1}}\delta_p(K_1,u)\cdots\delta_p(K_n,u)\,dS(u), \tag{1.6}$$

where $\delta_p(K,u)=\frac{1}{2}h(\Pi_pK,u)$. Here $D_p(K_1,\dots,K_n)$ is called the $L_p$-mixed-brightness integral of $K_1,\dots,K_n\in\mathcal{K}_o^n$. Obviously, for $p=1$, (1.6) is just the mixed-brightness integral from (1.1).

Let $\underbrace{K_1=\cdots=K_{n-i}}_{n-i}=K$ and $\underbrace{K_{n-i+1}=\cdots=K_n}_{i}=L$ $(i=0,1,\dots,n)$ in (1.5), we denote $D_{p,i}^\tau(K,L)=D_p^{(\tau)}(\underbrace{K,\dots,K}_{n-i},\underbrace{L,\dots,L}_{i})$. More general, if $i$ is any real, we define for $K,L\in\mathcal{K}_o^n$, $p\geq 1$, and $\tau\in[-1,1]$, the general $L_p$-mixed-brightness integral, $D_{p,i}^\tau(K,L)$, of $K$ and $L$ by

$$D_{p,i}^{(\tau)}(K,L)=\frac{1}{n}\int_{S^{n-1}}\delta_p^{(\tau)}(K,u)^{n-i}\delta_p^{(\tau)}(L,u)^i\,dS(u). \tag{1.7}$$

For $L=B$ in (1.7), we write $D_{p,i}^{(\tau)}(K,B)=\frac{1}{2^i}D_{p,i}^{(\tau)}(K)$ and notice that $\delta_p^{(\tau)}(B,u)=\frac{1}{2}h(\Pi_p^\tau B,u)=\frac{1}{2}$ for all $u\in S^{n-1}$, which together with (1.7) yields

$$D_{p,i}^{(\tau)}(K)=\frac{1}{2^i\cdot n}\int_{S^{n-1}}\delta_p^{(\tau)}(K,u)^{n-i}\,dS(u), \tag{1.8}$$

where $D_{p,i}^{(\tau)}(K)$ is called the $i$th general $L_p$-mixed-brightness integral of $K$. If $\tau=0$, then $D_{p,i}^{(\tau)}(K)=D_{p,i}(K)$. For $\tau=\pm 1$, we write $D_{p,i}^{(\tau)}(K)=D_{p,i}^{\pm}(K)$.

For $L=K$ in (1.7), write $D_{p,i}^{(\tau)}(K,K)=D_p^{(\tau)}(K)$, which is called the general $L_p$-brightness integral of $K$. Clearly,

$$D_p^{(\tau)}(K)=\frac{1}{n}\int_{S^{n-1}}\delta_p^{(\tau)}(K,u)^n\,dS(u). \tag{1.9}$$

Obviously, by (1.5), (1.7), (1.8), and (1.9), we have

$$D_p^{(\tau)}(K,\dots,K)=D_p^{(\tau)}(K); \tag{1.10}$$

$$D_{p,0}^{(\tau)}(K)=D_p^{(\tau)}(K);$$

$$D_{p,0}^{(\tau)}(K,L)=D_p^{(\tau)}(K),\qquad D_{p,n}^{(\tau)}(K,L)=D_p^{(\tau)}(L). \tag{1.11}$$

In this paper, we establish several inequalities for general $L_p$-mixed-brightness integrals. First, we determine the extremal values of general $L_p$-mixed-brightness integrals.

**Theorem 1.1** *If $K \in \mathcal{K}_o^n$, $p \geq 1$, and $\tau \in [-1,1]$, then*

$$D_{p,2n}(K) \leq D_{p,2n}^{(\tau)}(K) \leq D_{p,2n}^{\pm}(K). \tag{1.12}$$

*If $K$ is not origin-symmetric and $p$ is not an odd integer, there is equality in the left inequality if and only if $\tau = 0$ and equality in the right inequality if and only if $\tau = \pm 1$.*

Next, we obtain a Brunn-Minkowski type inequality for general $L_p$-mixed-brightness integrals.

**Theorem 1.2** *If $K, L \in \mathcal{K}_{os}^n$, $p \geq 1$, $\tau \in [-1,1]$, and $i \in \mathbb{R}$, and such that $i \neq n$, then for $i < n - p$,*

$$D_{p,i}^{(\tau)}(\lambda \circ K \oplus_p \mu \circ L)^{\frac{p}{n-i}} \leq \lambda D_{p,i}^{(\tau)}(K)^{\frac{p}{n-i}} + \mu D_{p,i}^{(\tau)}(L)^{\frac{p}{n-i}}. \tag{1.13}$$

*For $n - p < i < n$ or $i > n$, we have*

$$D_{p,i}^{(\tau)}(\lambda \circ K \oplus_p \mu \circ L)^{\frac{p}{n-i}} \geq \lambda D_{p,i}^{(\tau)}(K)^{\frac{p}{n-i}} + \mu D_{p,i}^{(\tau)}(L)^{\frac{p}{n-i}}. \tag{1.14}$$

*In each case, equality holds if and only if $K$ and $L$ have similar general $L_p$-brightness. For $i = n - p$, equality always holds in* (1.13) *or* (1.14).

Here, $\lambda \circ K \oplus_p \mu \circ L$ denotes the $L_p$-Blaschke combination of $K$ and $L$.

Next, we extend inequality (1.2) to general $L_p$-mixed-brightness integrals.

**Theorem 1.3** *If $K_1, \ldots, K_n \in \mathcal{K}_o^n$, $p \geq 1$, $\tau \in [-1,1]$, and $1 < m \leq n$, then*

$$D_p^{(\tau)}(K_1, \ldots, K_n)^m \leq \prod_{i=1}^{m} D_p^{(\tau)}(K_1, \ldots, K_{n-m}, \underbrace{K_{n-i+1}, \ldots, K_{n-i+1}}_{m}), \tag{1.15}$$

*with equality if and only if $K_{n-m+1}, K_{n-m+2}, \ldots, K_n$ are all of similar general $L_p$-brightness.*

Taking $m = n$ in Theorem 1.3 and using (1.10), we obtain the following corollary.

**Corollary 1.1** *If $K_1, \ldots, K_n \in \mathcal{K}_o^n$, $p \geq 1$, and $\tau \in [-1,1]$, then*

$$D_p^{(\tau)}(K_1, \ldots, K_n)^n \leq D_p^{(\tau)}(K_1) \cdots D_p^{(\tau)}(K_n),$$

*with equality if and only if $K_1, K_2, \ldots, K_n$ are all of similar general $L_p$-brightness.*

Moreover, we also establish the following cyclic inequality for general $L_p$-mixed-brightness integrals.

**Theorem 1.4** *If $K, L \in \mathcal{K}_o^n$, $p \geq 1$, $\tau \in [-1,1]$, and $i,j,k \in \mathbb{R}$ such that $i < j < k$, then*

$$D_{p,j}^{(\tau)}(K,L)^{k-i} \leq D_{p,i}^{(\tau)}(K,L)^{k-j} D_{p,k}^{(\tau)}(K,L)^{j-i}, \tag{1.16}$$

*with equality if and only if $K$ and $L$ have similar general $L_p$-brightness.*

Taking $i = 0$, $k = n$ in Theorem 1.4 and using (1.11), we obtain the following result.

**Corollary 1.2** *If $K, L \in \mathcal{K}_o^n$, $p \geq 1$, and $\tau \in [-1,1]$, then for $0 < j < n$,*

$$D_{p,j}^{(\tau)}(K,L)^{n} \leq D_{p}^{(\tau)}(K)^{n-j} D_{p}^{(\tau)}(L)^{j}, \tag{1.17}$$

*with equality if and only if $K$ and $L$ have similar general $L_p$-brightness. For $j = 0$ or $j = n$, equality always holds in* (1.17).

Let $L = B$ in Theorem 1.4, we also have the following result.

**Corollary 1.3** *If $K \in \mathcal{K}_o^n$, $p \geq 1$, $\tau \in [-1,1]$, and $i,j,k \in \mathbb{R}$ such that $i < j < k$, then*

$$D_{p,j}^{(\tau)}(K)^{k-i} \leq D_{p,i}^{(\tau)}(K)^{k-j} D_{p,k}^{(\tau)}(K)^{j-i},$$

*with equality if and only if $K$ and $L$ have similar general $L_p$-brightness, i.e., $K$ has constant general $L_p$-brightness.*

## 2 Notation and background material

### 2.1 Radial function and polars of convex bodies

If $K$ is a compact star-shaped set (about the origin) in $\mathbb{R}^n$, then its radial function, $\rho_K = \rho(K,\cdot) : \mathbb{R}^n \to [0,\infty)$, is defined by (see [1])

$$\rho(K,x) = \max\{\lambda \geq 0 : \lambda x \in K\}, \quad x \in \mathbb{R}^n. \tag{2.1}$$

If $\rho_K$ is positive and continuous, $K$ will be called a star body (with respect to the origin). Two star bodies $K$ and $L$ are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

If $E$ is a nonempty set in $\mathbb{R}^n$, then the polar set of $E$, $E^*$, is defined by (see [1])

$$E^* = \{x : x \cdot y \leq 1, y \in E\}, \quad x \in \mathbb{R}^n.$$

From this, we see that (see [1]) if $K \in \mathcal{K}_o^n$, then $(K^*)^* = K$ and

$$h_{K^*} = \frac{1}{\rho_K}, \qquad \rho_{K^*} = \frac{1}{h_K}. \tag{2.2}$$

Lutwak in [33] defined dual quermassintegrals as follows. For $K \in S_o^n$ and any real $i$, the dual quermassintegral, $\widetilde{W}_i(K)$, of $K$ is defined by

$$\widetilde{W}_i(K) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n-i}\,du. \tag{2.3}$$

Obviously, (2.3) implies that

$$V(K) = \widetilde{W}_0(K) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^n \, du. \tag{2.4}$$

### 2.2 $L_p$-combinations of convex and star bodies

For $K, L \in \mathcal{K}_o^n$, $p \geq 1$, and $\lambda, \mu \geq 0$ (not both zero), the Firey $L_p$-combination, $\lambda \cdot K +_p \mu \cdot L \in \mathcal{K}_o^n$, of $K$ and $L$ is defined by (see [34, 35])

$$h(\lambda \cdot K +_p \mu \cdot L, \cdot)^p = \lambda h(K, \cdot)^p + \mu h(L, \cdot)^p, \tag{2.5}$$

where the symbol $\cdot$ in $\lambda \cdot K$ denotes the Firey scalar multiplication. Note that $\lambda \cdot K = \lambda^{1/p} K$.

For $K, L \in \mathcal{S}_o^n$, $p \geq 1$, and $\lambda, \mu \geq 0$ (not both zero), the $L_p$-harmonic radial combination, $\lambda \star K +_{-p} \mu \star L \in \mathcal{S}_o^n$, of $K$ and $L$ is defined by (see [36])

$$\rho(\lambda \star K +_{-p} \mu \star L, \cdot)^{-p} = \lambda \rho(K, \cdot)^{-p} + \mu \rho(L, \cdot)^{-p}, \tag{2.6}$$

where $\lambda \star K = \lambda^{-1/p} K$.

From (2.2), (2.5), and (2.6), we easily find that if $K, L \in \mathcal{K}_o^n$, $p \geq 1$, and $\lambda, \mu \geq 0$ (not both zero), then

$$(\lambda \cdot K +_p \mu \cdot L)^* = \lambda \star K^* +_{-p} \mu \star L^*. \tag{2.7}$$

In [37] Wang and Leng established the following Brunn-Minkowski type inequality for dual quermassintegrals with respect to an $L_p$-harmonic radial combination of star bodies.

**Theorem 2.A** *If $K, L \in \mathcal{S}_o^n$, $p \geq 1$, $i \in \mathbb{R}$ and such that $i \neq n$, and $\lambda, \mu \geq 0$ (not both zero), then for $i < n$ or $n < i < n + p$,*

$$\widetilde{W}_i(\lambda \star K +_{-p} \mu \star L)^{-\frac{p}{n-i}} \geq \lambda \widetilde{W}_i(K)^{-\frac{p}{n-i}} + \mu \widetilde{W}_i(L)^{-\frac{p}{n-i}}; \tag{2.8}$$

*for $i > n + p$,*

$$\widetilde{W}_i(\lambda \star K +_{-p} \mu \star L)^{-\frac{p}{n-i}} \leq \lambda \widetilde{W}_i(K)^{-\frac{p}{n-i}} + \mu \widetilde{W}_i(L)^{-\frac{p}{n-i}}. \tag{2.9}$$

*In each inequality, equality holds if and only if $K$ and $L$ are dilates. For $i = n + p$, equality always holds in* (2.8) *and* (2.9).

The $L_p$-Blaschke combination of origin-symmetric convex bodies was introduced by Lutwak [35]. For $K, L \in \mathcal{K}_{os}^n$, $p \geq 1$, and $\lambda, \mu \geq 0$ (not both zero), the $L_p$-Blaschke combination, $\lambda \circ K \oplus_p \mu \circ L \in \mathcal{K}_{os}^n$, of $K$ and $L$ is defined by

$$dS_p(\lambda \circ K \oplus_p \mu \circ L, \cdot) = \lambda \, dS_p(K, \cdot) + \mu \, dS_p(L, \cdot), \tag{2.10}$$

where $\lambda \circ K = \lambda^{1/(n-p)} K$. For more information on these and other binary operations between convex and star bodies, see [38–42].

### 2.3 General $L_p$-projection bodies

For $p \geq 1$, Ludwig [15] discovered the asymmetric $L_p$-projection body, $\Pi_p^+ K$, of $K \in \mathcal{K}_o^n$, whose support function is defined by

$$h^p_{\Pi_p^+ K}(u) = \alpha_{n,p} \int_{S^{n-1}} (u \cdot v)_+^p \, dS_p(K, v),$$

where $(u \cdot v)_+ = \max\{u \cdot v, 0\}$. In [16], Haberl and Schuster also defined

$$\Pi_p^- K = \Pi_p^+ (-K).$$

Using definition (1.4) of general $L_p$-projection bodies, Haberl and Schuster [16] showed that, for $K \in \mathcal{K}_o^n$, $p \geq 1$, and $\tau \in [-1, 1]$,

$$\Pi_p^\tau K = f_1(\tau) \cdot \Pi_p^+ K +_p f_2(\tau) \cdot \Pi_p^- K,$$

where

$$f_1(\tau) = \frac{(1+\tau)^p}{(1+\tau)^p + (1-\tau)^p}, \qquad f_2(\tau) = \frac{(1-\tau)^p}{(1+\tau)^p + (1-\tau)^p}.$$

Moreover, they [16] determined the following extremal values of the volume for polars of general $L_p$-projection bodies.

**Theorem 2.B** *If $K \in \mathcal{K}_o^n$, $p \geq 1$, and $\tau \in [-1, 1]$, then*

$$V\left(\Pi_p^* K\right) \leq V\left(\Pi_p^{\tau,*} K\right) \leq V\left(\Pi_p^{\pm,*} K\right). \tag{2.11}$$

*If $K$ is not origin-symmetric and $p$ is not an odd integer, there is equality in the left inequality if and only if $\tau = 0$ and equality in the right inequality if and only if $\tau = \pm 1$.*

Here, $\Pi_p^{\tau,*} K$ denotes the polar of the general $L_p$-projection body $\Pi_p^\tau K$.

## 3 Proofs of the main theorems

In this section, we will prove Theorems 1.1-1.3.

To complete the proofs of Theorems 1.1-1.2, we require the following a lemma.

**Lemma 3.1** *If $K \in \mathcal{K}_o^n$, $p \geq 1$, $\tau \in [-1, 1]$, and $i$ is any real, then*

$$D_{p,i}^{(\tau)}(K) = \frac{1}{2^n} \widetilde{W}_{2n-i}\left(\Pi_p^{\tau,*} K\right). \tag{3.1}$$

*Proof* By (1.8), (2.2), and (2.3), we have

$$\begin{aligned} D_{p,i}^{(\tau)}(K) &= \frac{1}{2^i \cdot n} \int_{S^{n-1}} \delta_p^{(\tau)}(K, u)^{n-i} \, dS(u) \\ &= \frac{1}{2^n \cdot n} \int_{S^{n-1}} h\left(\Pi_p^\tau K, u\right)^{n-i} \, dS(u) \end{aligned}$$

$$= \frac{1}{2^n \cdot n} \int_{S^{n-1}} \rho\left(\Pi_p^{\tau,*} K, u\right)^{i-n} dS(u)$$
$$= \frac{1}{2^n} \widetilde{W}_{2n-i}\left(\Pi_p^{\tau,*} K\right). \qquad \square$$

*Proof of Theorem* 1.1 Taking $i = 2n$ in (3.1) and using (2.4), we obtain

$$D_{p,2n}^{(\tau)}(K) = \frac{1}{2^n} V\left(\Pi_p^{\tau,*} K\right). \tag{3.2}$$

Therefore, by inequality (2.11) together with (3.2), we immediately obtain

$$D_{p,2n}(K) \le D_{p,2n}^{(\tau)}(K) \le D_{p,2n}^{\pm}(K).$$

This is inequality (1.12).

According to the equality conditions of inequality (2.11), we know that if $K$ is not origin-symmetric and $p$ is not an odd integer, there is equality in the left inequality of (1.12) if and only if $\tau = 0$ and equality in the right inequality of (1.12) if and only if $\tau = \pm 1$. $\square$

*Proof of Theorem* 1.2 By (1.4) and (2.10), we have, for all $u \in S^{n-1}$,

$$h\left(\Pi_p^\tau(\lambda \circ K \oplus_p \mu \circ L), u\right)^p = \lambda h\left(\Pi_p^\tau K, u\right)^p + \mu h\left(\Pi_p^\tau L, u\right)^p,$$

*i.e.*,

$$\Pi_p^\tau(\lambda \circ K \oplus_p \mu \circ L) = \lambda \cdot \Pi_p^\tau K +_p \mu \cdot \Pi_p^\tau L.$$

This together with (2.7), yields

$$\Pi_p^{\tau,*}(\lambda \circ K \oplus_p \mu \circ L) = \left(\lambda \cdot \Pi_p^\tau K +_p \mu \cdot \Pi_p^\tau L\right)^* = \lambda \star \Pi_p^{\tau,*} K +_{-p} \mu \star \Pi_p^{\tau,*} L. \tag{3.3}$$

Hence, if $i < n - p$, then $2n - i > n + p$. From this, (3.1), (3.3), and inequality (2.9), we obtain

$$\begin{aligned}
&\left(2^n D_{p,i}^{(\tau)}(\lambda \circ K \oplus_p \mu \circ L)\right)^{\frac{p}{n-i}} \\
&\quad = \widetilde{W}_{2n-i}\left(\Pi_p^{\tau,*}(\lambda \circ K \oplus_p \mu \circ L)\right)^{-\frac{p}{n-(2n-i)}} \\
&\quad = \widetilde{W}_{2n-i}\left(\lambda \star \Pi_p^{\tau,*} K +_{-p} \mu \star \Pi_p^{\tau,*} L\right)^{-\frac{p}{n-(2n-i)}} \\
&\quad \le \lambda \widetilde{W}_{2n-i}\left(\Pi_p^{\tau,*} K\right)^{-\frac{p}{n-(2n-i)}} + \mu \widetilde{W}_{2n-i}\left(\Pi_p^{\tau,*} L\right)^{-\frac{p}{n-(2n-i)}} \\
&\quad = \lambda\left(2^n D_{p,i}^{(\tau)}(K)\right)^{\frac{p}{n-i}} + \mu\left(2^n D_{p,i}^{(\tau)}(L)\right)^{\frac{p}{n-i}}.
\end{aligned}$$

This yields inequality (1.13).

From the equality conditions of inequality (2.9), we see that equality holds in (1.13) if and only if $\Pi_p^{\tau,*} K$ and $\Pi_p^{\tau,*} L$ are dilates, *i.e.*, $\Pi_p^\tau K$ and $\Pi_p^\tau L$ are dilates. This means equality holds in (1.13) if and only if $K$ and $L$ have similar general $L_p$-brightness.

Similarly, if $n - p < i < n$ or $i > n$, then $2n - i < n$ or $n < 2n - i < n + p$. Thus, using (3.1), (3.3), and inequality (2.8), we obtain inequality (1.14).

If $i = n - p$, then $2n - i = n + p$. This combined with Theorem 2.A, shows that equality always holds in (1.13) or (1.14). □

The proof of Theorem 1.3 requires the following inequality [3].

**Lemma 3.2** *If $f_0, f_1, \dots, f_m$ are (strictly) positive continuous functions defined on $S^{n-1}$ and $\lambda_1, \dots, \lambda_m$ are positive constants the sum of whose reciprocals is unity, then*

$$\int_{S^{n-1}} f_0(u) \cdots f_m(u)\,dS(u) \le \prod_{i=1}^{m} \left( \int_{S^{n-1}} f_0(u) f_i^{\lambda_i}(u)\,dS(u) \right)^{\frac{1}{\lambda_i}}, \tag{3.4}$$

*with equality if and only if there exist positive constants $\alpha_1, \alpha_2, \dots, \alpha_m$ such that $\alpha_1 f_1^{\lambda_1}(u) = \cdots = \alpha_m f_m^{\lambda_m}(u)$ for all $u \in S^{n-1}$.*

*Proof of Theorem* 1.3 For $K_1, \dots, K_n \in \mathcal{K}_o^n$, take $\lambda_i = m$ in (3.4) ($1 \le i \le n$), and

$$f_0 = \delta_p^{(\tau)}(K_1, u) \cdots \delta_p^{(\tau)}(K_{n-m}, u) \quad (f_0 = 1 \text{ if } m = n),$$
$$f_i = \delta_p^{(\tau)}(K_{n-i+1}, u) \quad (1 \le i \le m).$$

Then we have

$$\begin{aligned} &\int_{S^{n-1}} \delta_p^{(\tau)}(K_1, u) \cdots \delta_p^{(\tau)}(K_n, u)\,dS(u) \\ &\quad \le \prod_{i=1}^{m} \left( \int_{S^{n-1}} \delta_p^{(\tau)}(K_1, u) \cdots \delta_p^{(\tau)}(K_{n-m}, u) \delta_p^{(\tau)}(K_{n-i+1}, u)^m\,dS(u) \right)^{\frac{1}{m}}, \end{aligned} \tag{3.5}$$

*i.e.*

$$D_p^{(\tau)}(K_1, \dots, K_n)^m \le \prod_{i=1}^{m} D_p^{(\tau)}(K_1, \dots, K_{n-m}, \underbrace{K_{n-i+1}, \dots, K_{n-i+1}}).$$

According to the equality conditions of Lemma 3.2, we see that equality holds in (3.5) if and only if there exist positive constants $\lambda_1, \lambda_2, \dots, \lambda_m$ such that

$$\lambda_1 \delta_p^{(\tau)}(K_{n-m+1}, u)^m = \lambda_2 \delta_p^{(\tau)}(K_{n-m+2}, u)^m = \cdots = \lambda_m \delta_p^{(\tau)}(K_n, u)^m$$

for all $u \in S^{n-1}$. Thus equality holds in (1.15) if and only if $K_{n-m+1}, K_{n-m+2}, \dots, K_n$ are all of similar general $L_p$-brightness. □

*Proof of Theorem* 1.4 From (1.7) and the Hölder inequality, we obtain

$$\begin{aligned} &D_{p,i}^{(\tau)}(K, L)^{\frac{k-j}{k-i}} D_{p,k}^{(\tau)}(K, L)^{\frac{j-i}{k-i}} \\ &\quad = \left[ \frac{1}{n} \int_{S^{n-1}} \delta_p^{(\tau)}(K, u)^{n-i} \delta_p^{(\tau)}(L, u)^i\,dS(u) \right]^{\frac{k-j}{k-i}} \\ &\qquad \times \left[ \frac{1}{n} \int_{S^{n-1}} \delta_p^{(\tau)}(K, u)^{n-k} \delta_p^{(\tau)}(L, u)^k\,dS(u) \right]^{\frac{j-i}{k-i}} \end{aligned}$$

$$\begin{aligned}
&= \left[\frac{1}{n}\int_{S^{n-1}}\left[\delta_p^{(\tau)}(K,u)^{\frac{(n-i)(k-j)}{(k-i)}}\delta_p^{(\tau)}(L,u)^{\frac{i(k-j)}{k-i}}\right]^{\frac{k-i}{k-j}}dS(u)\right]^{\frac{k-j}{k-i}} \\
&\quad\times\left[\frac{1}{n}\int_{S^{n-1}}\left[\delta_p^{(\tau)}(K,u)^{\frac{(n-k)(j-i)}{k-i}}\delta_p^{(\tau)}(L,u)^{\frac{k(j-i)}{k-i}}\right]^{\frac{k-i}{j-i}}dS(u)\right]^{\frac{j-i}{k-i}} \\
&\geq \frac{1}{n}\int_{S^{n-1}}\delta_p^{(\tau)}(K,u)^{n-j}\delta_p^{(\tau)}(L,u)^{j}\,dS(u) \\
&= D_{p,j}^{(\tau)}(K,L).
\end{aligned}$$

This gives the desired inequality (1.16). According to the equality conditions of the Hölder inequality, we know that equality holds in (1.16) if and only if there exists a constant $\lambda > 0$ such that

$$\left[\delta_p^{(\tau)}(K,u)^{\frac{(n-i)(k-j)}{(k-i)}}\delta_p^{(\tau)}(L,u)^{\frac{i(k-j)}{k-i}}\right]^{\frac{k-i}{k-j}} = \lambda\left[\delta_p^{(\tau)}(K,u)^{\frac{(n-k)(j-i)}{k-i}}\delta_p^{(\tau)}(L,u)^{\frac{k(j-i)}{k-i}}\right]^{\frac{k-i}{j-i}},$$

*i.e.* $\delta_p^{(\tau)}(K,u) = \lambda\delta_p^{(\tau)}(L,u)$ for all $u \in S^{n-1}$. Thus equality holds in (1.16) if and only if $K$ and $L$ have similar general $L_p$-brightness. □

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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