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Impulsive inequalities with nonlocal jumps and their applications to impulsive fractional integral conditions

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Abstract
In this paper we establish some impulsive differential and integral inequalities with nonlocal jumps. Two applications to impulsive differential and integral inequalities with Riemann-Liouville fractional integral jump conditions are given.

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Keywords: impulsive differential inequality; impulsive integral inequality; nonlocal jump condition; Riemann-Liouville fractional integral

1 Introduction

Impulsive differential and integral inequalities play a fundamental role in the global existence, uniqueness, oscillation, stability, and other properties of the solutions of various nonlinear impulsive differential and integral equations; see [1–27] and the references given therein.

Let $0 \leq t_0 < t_1 < t_2 < \cdots$, $\lim_{k\to\infty} t_k = \infty$, $\mathbb{R}_+ = [0, +\infty)$, and $I \subset \mathbb{R}$. We introduce the following function spaces: $PC(\mathbb{R}_+, I) = \{u : \mathbb{R}_+ \to I; u(t)$ is continuous for $t \neq t_k$, and $u(0^+)$, $u(t_k^-)$ and $u(t_k^+)$ exist, and $u(t_k^-) = u(t_k), k = 1, 2, \dots\}$ and $PC^1(\mathbb{R}_+, I) = \{u \in PC(\mathbb{R}_+, I) : u'(t)$ is continuous everywhere for $t \neq t_k$, and $u'(0^+), u'(t_k^+)$ and $u'(t_k^-)$ exist, and $u'(t_k^-) = u'(t_k)$, $k = 1, 2, \dots\}$.

In [1], Lakshmikantham *et al.* developed a famous impulsive differential inequality given in the next theorem.

Theorem 1.1 *Assume that*:

(H_0) *the sequence* $\{t_k\}$ *satisfies* $0 \leq t_0 < t_1 < t_2 < \cdots$, $\lim_{k\to\infty} t_k = \infty$;
(H_1) $m \in PC^1[\mathbb{R}_+, \mathbb{R}]$ *and* $m(t)$ *is left-continuous at* t_k, $k = 1, 2, \dots$;
(H_2) *for* $k = 1, 2, \dots$, $t \geq t_0$,

$$m'(t) \leq p(t)m(t) + q(t), \quad t \neq t_k, \tag{1.1}$$

$$m(t_k^+) \leq d_k m(t_k) + b_k, \tag{1.2}$$

where $q, p \in C[\mathbb{R}_+, \mathbb{R}]$, $d_k \geq 0$ *and* b_k, $k = 1, 2, \dots$, *are constants*.

Then

$$m(t) \leq m(t_0) \prod_{t_0<t_k<t} d_k e^{\int_{t_0}^{t} p(s)\,ds} + \sum_{t_0<t_k<t} \left(\prod_{t_k<t_j<t} d_j e^{\int_{t_k}^{t} p(s)\,ds} \right) b_k$$
$$+ \int_{t_0}^{t} \prod_{s<t_k<t} d_k e^{\int_s^t p(\sigma)\,d\sigma} q(s)\,ds, \quad t \geq t_0. \tag{1.3}$$

There are many results on the impulsive differential and integral inequalities (see for example [28–36]). However, most of these papers deal with jump conditions at impulse point t_k depending on the left hand limit $m(t_k)$ or a time-delay value, $m(t_k - \tau)$, $\tau > 0$.

Recently, in [37], Theorem 1.1 was generalized to obtain differential inequalities for integral jump conditions by replacing the inequality in (1.2) by the following inequality:

$$m\bigl(t_k^+\bigr) \leq d_k m(t_k) + c_k \int_{t_k-\tau_k}^{t_k-\sigma_k} m(s)\,ds + b_k, \quad k = 1, 2, \ldots, \tag{1.4}$$

where $0 \leq \sigma_k \leq \tau_k \leq t_k - t_{k-1}$.

In the present paper we generalize further Theorem 1.1 by replacing the inequality in (1.2) by the inequality

$$m\bigl(t_k^+\bigr) \leq \frac{c_k}{\Gamma(\beta_k)} \int_{t_{k-1}}^{t_k} (t_k - s)^{\beta_k - 1} m(s)\,ds + d_k m(t_k) + b_k, \tag{1.5}$$

where $c_k, d_k \geq 0$, $\beta_k > 0$ and b_k, $k = 1, 2, \ldots$, are constants. Some new impulsive differential and integral inequalities are obtained. Two applications to impulsive differential and integral inequalities with Riemann-Liouville fractional integral jump conditions are given. In the first one we study the maximum principle of an impulsive differential inequality and in the second one we show the boundedness of solution of impulsive differential equation with Riemann-Liouville fractional integral jump conditions.

Nonlocality and memory effects can be represented by the concepts of fractional calculus which contains definitions of fractional derivatives and fractional integrals in the form of weighted integrals. It is learnt through experimentation that the integral operators of fractional order take care of some of the hereditary properties of many phenomena and processes. Impulsive equations and inequalities with nonlocal fractional jump conditions provide a tool to describe systems which have a sudden change of the state values via memorizing previous events. For details of nonlocal theory and memory effects, we refer to [38].

2 Impulsive inequalities with nonlocal jumps

In this section, we state and prove some new impulsive differential and integral inequalities with nonlocal jumps. Throughout of this paper we denote $t_l = \max\{t_k : t \geq t_k, k = 1, 2, \ldots\}$.

Theorem 2.1 *Let* (H_0) *and* (H_1) *hold. Suppose that* $p, q \in C[\mathbb{R}_+, \mathbb{R}]$ *and, for* $k = 1, 2, \ldots$, $t \geq t_0$,

$$m'(t) \leq p(t)m(t) + q(t), \quad t \neq t_k, \tag{2.1}$$

$$m(t_k^+) \le \frac{c_k}{\Gamma(\beta_k)} \int_{t_{k-1}}^{t_k} (t_k - s)^{\beta_k - 1} m(s)\,ds + d_k m(t_k) + b_k, \tag{2.2}$$

where $c_k, d_k \ge 0$, $\beta_k > 0$ and b_k, $k = 1, 2, \ldots$ are constants.

Then, for $t \ge t_0$,

$$\begin{aligned} m(t) \le {} & \Biggl\{ m(t_0) \prod_{t_0 < t_k < t} \left(\frac{c_k}{\Gamma(\beta_k)} \int_{t_{k-1}}^{t_k} (t_k - s)^{\beta_k - 1} e^{\int_{t_{k-1}}^{s} p(\xi)\,d\xi}\,ds + d_k e^{\int_{t_{k-1}}^{t_k} p(\xi)\,d\xi} \right) \\ & + \sum_{t_0 < t_k < t} \Biggl[\prod_{t_k < t_j < t} \left(\frac{c_j}{\Gamma(\beta_j)} \int_{t_{j-1}}^{t_j} (t_j - s)^{\beta_j - 1} e^{\int_{t_{j-1}}^{s} p(\xi)\,d\xi}\,ds + d_j e^{\int_{t_{j-1}}^{t_j} p(\xi)\,d\xi} \right) \\ & \times \Biggl(\frac{c_k}{\Gamma(\beta_k)} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} (t_k - s)^{\beta_k - 1} q(v) e^{\int_{v}^{s} p(\xi)\,d\xi}\,dv\,ds \\ & + d_k \int_{t_{k-1}}^{t_k} q(s) e^{\int_{s}^{t_k} p(\xi)\,d\xi}\,ds + b_k \Biggr) \Biggr] \Biggr\} e^{\int_{t_l}^{t} p(\xi)\,d\xi} + \int_{t_l}^{t} q(s) e^{\int_{s}^{t} p(\xi)\,d\xi}\,ds. \end{aligned} \tag{2.3}$$

Proof For $t \in [t_0, t_1]$, inequality (2.1) can be written as

$$\frac{d}{dt}\bigl[m(t) e^{-\int_{t_0}^{t} p(\xi)\,d\xi}\bigr] \le q(t) e^{-\int_{t_0}^{t} p(\xi)\,d\xi}. \tag{2.4}$$

Integrating (2.4) from t_0 to t for $t \in [t_0, t_1]$, we have

$$m(t) \le m(t_0) e^{\int_{t_0}^{t} p(\xi)\,d\xi} + \int_{t_0}^{t} q(s) e^{\int_{s}^{t} p(\xi)\,d\xi}\,ds. \tag{2.5}$$

Hence (2.3) is valid on $[t_0, t_1]$. Assume that (2.3) holds for $t \in [t_0, t_n]$ for some integer $n > 1$. Then, for $t \in [t_n, t_{n+1}]$, it follows from (2.1) and (2.5) that

$$m(t) \le m(t_n^+) e^{\int_{t_n}^{t} p(\xi)\,d\xi} + \int_{t_n}^{t} q(s) e^{\int_{s}^{t} p(\xi)\,d\xi}\,ds. \tag{2.6}$$

Applying (2.2) with (2.6), one has

$$\begin{aligned} m(t) \le {} & \left(\frac{c_n}{\Gamma(\beta_n)} \int_{t_{n-1}}^{t_n} (t_n - s)^{\beta_n - 1} m(s)\,ds + d_n m(t_n) + b_n \right) e^{\int_{t_n}^{t} p(\xi)\,d\xi} \\ & + \int_{t_n}^{t} q(s) e^{\int_{s}^{t} p(\xi)\,d\xi}\,ds. \end{aligned} \tag{2.7}$$

By the principle of mathematical induction, (2.7) can be expressed as

$$\begin{aligned} m(t) \le {} & \Biggl\{ \frac{c_n}{\Gamma(\beta_n)} \int_{t_{n-1}}^{t_n} (t_n - s)^{\beta_n - 1} \\ & \times \Biggl\{ \Biggl\{ m(t_0) \prod_{t_0 < t_k < s} \left(\frac{c_k}{\Gamma(\beta_k)} \int_{t_{k-1}}^{t_k} (t_k - v)^{\beta_k - 1} e^{\int_{t_{k-1}}^{v} p(\xi)\,d\xi}\,dv + d_k e^{\int_{t_{k-1}}^{t_k} p(\xi)\,d\xi} \right) \\ & + \sum_{t_0 < t_k < s} \Biggl[\prod_{t_k < t_j < s} \left(\frac{c_j}{\Gamma(\beta_j)} \int_{t_{j-1}}^{t_j} (t_j - v)^{\beta_j - 1} e^{\int_{t_{j-1}}^{v} p(\xi)\,d\xi}\,dv + d_j e^{\int_{t_{j-1}}^{t_j} p(\xi)\,d\xi} \right) \\ & \times \Biggl(\frac{c_k}{\Gamma(\beta_k)} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{v} (t_k - v)^{\beta_k - 1} q(r) e^{\int_{r}^{v} p(\xi)\,d\xi}\,dr\,dv \end{aligned}$$

$$
\begin{aligned}
&+ d_k \int_{t_{k-1}}^{t_k} q(\nu) e^{\int_\nu^{t_k} p(\xi)\,d\xi}\,d\nu + b_k \biggr) \biggr] \biggr\} e^{\int_{t_{n-1}}^{s} p(\xi)\,d\xi} + \int_{t_{n-1}}^{s} q(\nu) e^{\int_\nu^{s} p(\xi)\,d\xi}\,d\nu \biggr\}\,ds \\
&+ d_n \biggl(\biggl\{ m(t_0) \prod_{t_0<t_k<t_n} \biggl(\frac{c_k}{\Gamma(\beta_k)} \int_{t_{k-1}}^{t_k} (t_k - s)^{\beta_k - 1} e^{\int_{t_{k-1}}^{s} p(\xi)\,d\xi}\,ds + d_k e^{\int_{t_{k-1}}^{t_k} p(\xi)\,d\xi} \biggr) \\
&+ \sum_{t_0<t_k<t_n} \biggl[\prod_{t_k<t_j<t_n} \biggl(\frac{c_j}{\Gamma(\beta_j)} \int_{t_{j-1}}^{t_j} (t_j - s)^{\beta_j - 1} e^{\int_{t_{j-1}}^{s} p(\xi)\,d\xi}\,ds + d_j e^{\int_{t_{j-1}}^{t_j} p(\xi)\,d\xi} \biggr) \\
&\times \biggl(\frac{c_k}{\Gamma(\beta_k)} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} (t_k - s)^{\beta_k - 1} q(\nu) e^{\int_\nu^{s} p(\xi)\,d\xi}\,d\nu\,ds \\
&+ d_k \int_{t_{k-1}}^{t_k} q(s) e^{\int_s^{t_k} p(\xi)\,d\xi}\,ds + b_k \biggr) \biggr] \biggr\} e^{\int_{t_{n-1}}^{t_n} p(\xi)\,d\xi} \\
&+ \int_{t_{n-1}}^{t_n} q(s) e^{\int_s^{t_n} p(\xi)\,d\xi}\,ds \biggr) + b_n \biggr\} e^{\int_{t_n}^{t} p(\xi)\,d\xi} + \int_{t_n}^{t} q(s) e^{\int_s^{t} p(\xi)\,d\xi}\,ds. \qquad (2.8)
\end{aligned}
$$

Set

$$
E_k = \frac{c_k}{\Gamma(\beta_k)} \int_{t_{k-1}}^{t_k} (t_k - s)^{\beta_k - 1} e^{\int_{t_{k-1}}^{s} p(\xi)\,d\xi}\,ds + d_k e^{\int_{t_{k-1}}^{t_k} p(\xi)\,d\xi}, \qquad (2.9)
$$

$$
\begin{aligned}
G_k = {}& \frac{c_k}{\Gamma(\beta_k)} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} (t_k - s)^{\beta_k - 1} q(\nu) e^{\int_\nu^{s} p(\xi)\,d\xi}\,d\nu\,ds \\
&+ d_k \int_{t_{k-1}}^{t_k} q(s) e^{\int_s^{t_k} p(\xi)\,d\xi}\,ds + b_k. \qquad (2.10)
\end{aligned}
$$

Substituting (2.9), (2.10) into (2.8), we get for $t \in [t_n, t_{n+1}]$

$$
\begin{aligned}
m(t) \le {}& \biggl\{ \frac{c_n}{\Gamma(\beta_n)} \int_{t_{n-1}}^{t_n} (t_n - s)^{\beta_n - 1} \biggl\{ \biggl\{ m(t_0) \prod_{t_0<t_k<s} E_k + \sum_{t_0<t_k<s} \biggl[\prod_{t_k<t_j<s} E_j G_k \biggr] \biggr\} e^{\int_{t_{n-1}}^{s} p(\xi)\,d\xi} \\
&+ \int_{t_{n-1}}^{s} q(\nu) e^{\int_\nu^{s} p(\xi)\,d\xi}\,d\nu \biggr\}\,ds \\
&+ d_n \biggl(\biggl\{ m(t_0) \prod_{t_0<t_k<t_n} E_k + \sum_{t_0<t_k<t_n} \biggl[\prod_{t_k<t_j<t_n} E_j G_k \biggr] \biggr\} e^{\int_{t_{n-1}}^{t_n} p(\xi)\,d\xi} \\
&+ \int_{t_{n-1}}^{t_n} q(s) e^{\int_s^{t_n} p(\xi)\,d\xi}\,ds \biggr) + b_n \biggr\} e^{\int_{t_n}^{t} p(\xi)\,d\xi} + \int_{t_n}^{t} q(s) e^{\int_s^{t} p(\xi)\,d\xi}\,ds \\
= {}& \biggl\{ \biggl(m(t_0) \prod_{t_0<t_k<t_n} E_k + \sum_{t_0<t_k<t_n} \biggl[\prod_{t_k<t_j<t_n} E_j G_k \biggr] \biggr) \frac{c_n}{\Gamma(\beta_n)} \int_{t_{n-1}}^{t_n} (t_n - s)^{\beta_n - 1} e^{\int_{t_{n-1}}^{s} p(\xi)\,d\xi}\,ds \\
&+ \frac{c_n}{\Gamma(\beta_n)} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{s} (t_n - s)^{\beta_n - 1} q(\nu) e^{\int_\nu^{s} p(\xi)\,d\xi}\,d\nu\,ds \\
&+ \biggl(m(t_0) \prod_{t_0<t_k<t_n} E_k + \sum_{t_0<t_k<t_n} \biggl[\prod_{t_k<t_j<t_n} E_j G_k \biggr] \biggr) d_n e^{\int_{t_{n-1}}^{t_n} p(\xi)\,d\xi} \\
&+ d_n \int_{t_{n-1}}^{t_n} q(s) e^{\int_s^{t_n} p(\xi)\,d\xi}\,ds + b_n \biggr\} e^{\int_{t_n}^{t} p(\xi)\,d\xi} + \int_{t_n}^{t} q(s) e^{\int_s^{t} p(\xi)\,d\xi}\,ds \\
= {}& \biggl\{ \biggl(m(t_0) \prod_{t_0<t_k<t_n} E_k + \sum_{t_0<t_k<t_n} \biggl[\prod_{t_k<t_j<t_n} E_j G_k \biggr] \biggr) E_n + G_n \biggr\} e^{\int_{t_n}^{t} p(\xi)\,d\xi}\,ds
\end{aligned}
$$

$$+\int_{t_n}^{t} q(s)e^{\int_s^t p(\xi)\,d\xi}$$

$$=\left\{m(t_0)\prod_{t_0<t_k<t} E_k+\sum_{t_0<t_k<t}\left[\prod_{t_k<t_j<t} E_jG_k\right]\right\}e^{\int_{t_n}^t p(\xi)\,d\xi}\,ds+\int_{t_n}^{t} q(s)e^{\int_s^t p(\xi)\,d\xi}.$$

Hence,

$$\begin{aligned}m(t)\le{}&\left\{m(t_0)\prod_{t_0<t_k<t}\left(\frac{c_k}{\Gamma(\beta_k)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\beta_k-1}e^{\int_{t_{k-1}}^s p(\xi)\,d\xi}\,ds+d_ke^{\int_{t_{k-1}}^{t_k} p(\xi)\,d\xi}\right)\right.\\&+\sum_{t_0<t_k<t}\left[\prod_{t_k<t_j<t}\left(\frac{c_j}{\Gamma(\beta_j)}\int_{t_{j-1}}^{t_j}(t_j-s)^{\beta_j-1}e^{\int_{t_{j-1}}^s p(\xi)\,d\xi}\,ds+d_je^{\int_{t_{j-1}}^{t_j} p(\xi)\,d\xi}\right)\right.\\&\times\left(\frac{c_k}{\Gamma(\beta_k)}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}(t_k-s)^{\beta_k-1}q(v)e^{\int_v^s p(\xi)\,d\xi}\,dv\,ds\right.\\&\left.\left.\left.+d_k\int_{t_{k-1}}^{t_k}q(s)e^{\int_s^{t_k} p(\xi)\,d\xi}\,ds+b_k\right)\right]\right\}e^{\int_{t_n}^t p(\xi)\,d\xi}+\int_{t_n}^t q(s)e^{\int_s^t p(\xi)\,d\xi}\,ds,\end{aligned}$$

for $t_n \le t \le t_{n+1}$. Therefore, the estimate (2.3) holds for $t_0 \le t \le t_{n+1}$. This completes the proof. □

Theorem 2.2 *Assume that the hypotheses of Theorem* 2.1 *are fullfilled. Then, for* $t \ge t_0$, *we have*:

(i) $0<\beta_k\le\frac{1}{2}$ *for* $k=1,2,\dots,$

$$\begin{aligned}m(t)\le{}&\left\{m(t_0)\prod_{t_0<t_k<t}\left[\frac{c_k}{\Gamma(\beta_k)}\left(\frac{e^{\mu_kt_k}\Gamma(\beta_k^2)}{\mu_k^{\beta_k^2}}\right)^{\frac{1}{\mu_k}}\left(\int_{t_{k-1}}^{t_k}e^{\nu_k(\int_{t_{k-1}}^s p(\xi)\,d\xi-s)}\,ds\right)^{\frac{1}{\nu_k}}\right.\right.\\&\left.+d_ke^{\int_{t_{k-1}}^{t_k} p(\xi)\,d\xi}\right]\\&+\sum_{t_0<t_k<t}\left[\prod_{t_k<t_j<t}\left(\frac{c_j}{\Gamma(\beta_j)}\left(\frac{e^{\mu_jt_j}\Gamma(\beta_j^2)}{\mu_j^{\beta_j^2}}\right)^{\frac{1}{\mu_j}}\left(\int_{t_{j-1}}^{t_j}e^{\nu_j(\int_{t_{j-1}}^s p(\xi)\,d\xi-s)}\,ds\right)^{\frac{1}{\nu_j}}\right.\right.\\&\left.+d_je^{\int_{t_{j-1}}^{t_j} p(\xi)\,d\xi}\right)\\&\times\left(\frac{c_k}{\Gamma(\beta_k)}\left(\frac{e^{\mu_kt_k}\Gamma(\beta_k^2)}{\mu_k^{\beta_k^2}}\right)^{\frac{1}{\mu_k}}\left(\int_{t_{k-1}}^{t_k}e^{-\nu_ks}\left\{\int_{t_{k-1}}^s q(v)e^{\int_v^s p(\xi)\,d\xi}\,dv\right\}^{\nu_k}ds\right)^{\frac{1}{\nu_k}}\right.\\&\left.\left.\left.+d_k\int_{t_{k-1}}^{t_k}q(s)e^{\int_s^{t_k} p(\xi)\,d\xi}\,ds+b_k\right)\right]\right\}e^{\int_{t_l}^t p(\xi)\,d\xi}+\int_{t_l}^t q(s)e^{\int_s^t p(\xi)\,d\xi}\,ds,\end{aligned}\tag{2.11}$$

where $\mu_k=\beta_k+1$ *and* $\nu_k=1+\frac{1}{\beta_k}$,

(ii) $\beta_k>\frac{1}{2}$ *for* $k=1,2,\dots,$

$$\begin{aligned}m(t)\le{}&\left\{m(t_0)\prod_{t_0<t_k<t}\left[\frac{c_k}{\Gamma(\beta_k)}\left(\frac{e^{2t_k}\Gamma(2\beta_k-1)}{2^{2\beta_k-1}}\right)^{\frac{1}{2}}\left(\int_{t_{k-1}}^{t_k}e^{2(\int_{t_{k-1}}^s p(\xi)\,d\xi-s)}\,ds\right)^{\frac{1}{2}}\right.\right.\\&\left.+d_ke^{\int_{t_{k-1}}^{t_k} p(\xi)\,d\xi}\right]\end{aligned}$$

$$
\begin{aligned}
&+\sum_{t_0<t_k<t}\Biggl[\prod_{t_k<t_j<t}\biggl(\frac{c_j}{\Gamma(\beta_j)}\biggl(\frac{e^{2t_j}\Gamma(2\beta_j-1)}{2^{2\beta_j-1}}\biggr)^{\frac{1}{2}}\biggl(\int_{t_{j-1}}^{t_j}e^{2(\int_{t_{j-1}}^{s}p(\xi)\,d\xi-s)}\,ds\biggr)^{\frac{1}{2}}\\
&+d_je^{\int_{t_{j-1}}^{t_j}p(\xi)\,d\xi}\biggr)\\
&\times\biggl(\frac{c_k}{\Gamma(\beta_k)}\biggl(\frac{e^{2t_k}\Gamma(2\beta_k-1)}{2^{2\beta_k-1}}\biggr)^{\frac{1}{2}}\biggl(\int_{t_{k-1}}^{t_k}e^{-2s}\biggl\{\int_{t_{k-1}}^{s}q(v)e^{\int_v^s p(\xi)\,d\xi}\,dv\biggr\}^2\,ds\biggr)^{\frac{1}{2}}\\
&+d_k\int_{t_{k-1}}^{t_k}q(s)e^{\int_s^{t_k}p(\xi)\,d\xi}\,ds+b_k\biggr)\Biggr]\Biggr\}e^{\int_{t_l}^{t}p(\xi)\,d\xi}+\int_{t_l}^{t}q(s)e^{\int_s^t p(\xi)\,d\xi}\,ds. \qquad (2.12)
\end{aligned}
$$

Proof To prove (i) we apply the Hölder inequality. We have for $k \in \mathbb{N}$

$$
\begin{aligned}
&\int_{t_{k-1}}^{t_k}(t_k-s)^{\beta_k-1}e^{\int_{t_{k-1}}^{s}p(\xi)\,d\xi}\,ds\\
&\quad\le\biggl(\int_{t_{k-1}}^{t_k}(t_k-s)^{\mu_k(\beta_k-1)}e^{\mu_k s}\,ds\biggr)^{\frac{1}{\mu_k}}\biggl(\int_{t_{k-1}}^{t_k}e^{-\nu_k s}e^{\nu_k\int_{t_{k-1}}^{s}p(\xi)\,d\xi}\,ds\biggr)^{\frac{1}{\nu_k}}\\
&\quad<\biggl(\frac{e^{\mu_k t_k}\Gamma(1-\mu_k(1-\beta_k))}{\mu_k^{1-\mu_k(1-\beta_k)}}\biggr)^{\frac{1}{\mu_k}}\biggl(\int_{t_{k-1}}^{t_k}e^{\nu_k(\int_{t_{k-1}}^{s}p(\xi)\,d\xi-s)}\,ds\biggr)^{\frac{1}{\nu_k}}\\
&\quad=\biggl(\frac{e^{\mu_k t_k}\Gamma(\beta_k^2)}{\mu_k^{\beta_k^2}}\biggr)^{\frac{1}{\mu_k}}\biggl(\int_{t_{k-1}}^{t_k}e^{\nu_k(\int_{t_{k-1}}^{s}p(\xi)\,d\xi-s)}\,ds\biggr)^{\frac{1}{\nu_k}},
\end{aligned}
$$

using

$$
\begin{aligned}
\int_{t_{k-1}}^{t_k}(t_k-s)^{\mu_k(\beta_k-1)}e^{\mu_k s}\,ds&=e^{\mu_k t_k}\int_0^{t_k-t_{k-1}}\tau^{\mu_k(\beta_k-1)}e^{-\mu_k\tau}\,d\tau\\
&=\frac{e^{\mu_k t_k}}{\mu_k^{1-\mu_k(1-\beta_k)}}\int_0^{\mu_k(t_k-t_{k-1})}\sigma^{\mu_k(\beta_k-1)}e^{-\sigma}\,d\sigma\\
&<\frac{e^{\mu_k t_k}}{\mu_k^{1-\mu_k(1-\beta_k)}}\Gamma\bigl(1-\mu_k(1-\beta_k)\bigr)\\
&=\frac{e^{\mu_k t_k}}{\mu_k^{\beta_k^2}}\Gamma\bigl(\beta_k^2\bigr),
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}(t_k-s)^{\beta_k-1}q(v)e^{\int_v^s p(\xi)\,d\xi}\,dv\,ds\\
&\quad\le\biggl(\int_{t_{k-1}}^{t_k}(t_k-s)^{\mu_k(\beta_k-1)}e^{\mu_k s}\,ds\biggr)^{\frac{1}{\mu_k}}\biggl(\int_{t_{k-1}}^{t_k}e^{-\nu_k s}\biggl\{\int_{t_{k-1}}^{s}q(v)e^{\int_v^s p(\xi)\,d\xi}\,dv\biggr\}^{\nu_k}\,ds\biggr)^{\frac{1}{\nu_k}}\\
&\quad<\biggl(\frac{e^{\mu_k t_k}\Gamma(\beta_k^2)}{\mu_k^{\beta_k^2}}\biggr)^{\frac{1}{\mu_k}}\biggl(\int_{t_{k-1}}^{t_k}e^{-\nu_k s}\biggl\{\int_{t_{k-1}}^{s}q(v)e^{\int_v^s p(\xi)\,d\xi}\,dv\biggr\}^{\nu_k}\,ds\biggr)^{\frac{1}{\nu_k}}.
\end{aligned}
$$

Substituting the above inequalities in (2.3), we obtain the desired inequality in (2.11).

To prove (ii), applying the Cauchy-Schwarz inequality, we get for $k \in \mathbb{N}$

$$\begin{aligned}
&\int_{t_{k-1}}^{t_k} (t_k - s)^{\beta_k - 1} e^{\int_{t_{k-1}}^{s} p(\xi)\,d\xi}\,ds \\
&\quad \le \left(\int_{t_{k-1}}^{t_k} (t_k - s)^{2(\beta_k - 1)} e^{2s}\,ds\right)^{\frac{1}{2}} \left(\int_{t_{k-1}}^{t_k} e^{2(\int_{t_{k-1}}^{s} p(\xi)\,d\xi - s)}\,ds\right)^{\frac{1}{2}} \\
&\quad < \left(\frac{e^{2t_k}\Gamma(2\beta_k - 1)}{2^{2\beta_k - 1}}\right)^{\frac{1}{2}} \left(\int_{t_{k-1}}^{t_k} e^{2(\int_{t_{k-1}}^{s} p(\xi)\,d\xi - s)}\,ds\right)^{\frac{1}{2}},
\end{aligned}$$

and

$$\begin{aligned}
&\int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} (t_k - s)^{\beta_k - 1} q(\nu) e^{\int_{\nu}^{s} p(\xi)\,d\xi}\,d\nu\,ds \\
&\quad \le \left(\int_{t_{k-1}}^{t_k} (t_k - s)^{2(\beta_k - 1)} e^{2s}\,ds\right)^{\frac{1}{2}} \left(\int_{t_{k-1}}^{t_k} e^{-2s}\left\{\int_{t_{k-1}}^{s} q(\nu) e^{\int_{\nu}^{s} p(\xi)\,d\xi}\,d\nu\right\}^2 ds\right)^{\frac{1}{2}} \\
&\quad < \left(\frac{e^{2t_k}\Gamma(2\beta_k - 1)}{2^{2\beta_k - 1}}\right)^{\frac{1}{2}} \left(\int_{t_{k-1}}^{t_k} e^{-2s}\left\{\int_{t_{k-1}}^{s} q(\nu) e^{\int_{\nu}^{s} p(\xi)\,d\xi}\,d\nu\right\}^2 ds\right)^{\frac{1}{2}}.
\end{aligned}$$

Substituting these two inequalities in (2.3), we get the required inequality in (2.12). The proof is completed. □

Corollary 2.3 *Let* (H_0) *and* (H_1) *hold. Suppose that* $q \in C[\mathbb{R}_+, \mathbb{R}]$ *and, for* $k = 1, 2, \ldots$, $t \ge t_0$,

$$\begin{cases} m'(t) \le \lambda m(t) + q(t), & t \ne t_k, \\ m(t_k^+) \le \frac{c_k}{\Gamma(\beta_k)} \int_{t_{k-1}}^{t_k} (t_k - s)^{\beta_k - 1} m(s)\,ds + b_k, \end{cases} \tag{2.13}$$

where λ, $c_k \ge 0$, $\beta_k > 0$ *and* b_k, $k = 1, 2, \ldots$ *are constants. Then, for* $t \ge t_0$, *we have the following two cases.*

Case I: $\lambda \ne 1$,

(i) $0 < \beta_k \le \frac{1}{2}$ *for* $k = 1, 2, \ldots$,

$$\begin{aligned}
m(t) \le{}& m(t_0)\left(\prod_{t_0 < t_k < t} A_k\right) e^{\lambda(t - t_0)} + \sum_{t_0 < t_k < t} \left(\prod_{t_k < t_j < t} A_j B_k e^{\lambda(t - t_k)}\right) \\
&+ \int_{t_l}^{t} q(s) e^{\lambda(t - s)}\,ds,
\end{aligned} \tag{2.14}$$

where

$$\begin{aligned}
A_k &= \frac{c_k}{\Gamma(\beta_k)} \left(\frac{\Gamma(\beta_k^2)}{\mu_k^{\beta_k^2}}\right)^{\frac{1}{\mu_k}} \left(\frac{1 - e^{\nu_k(t_k - t_{k-1})(1 - \lambda)}}{\nu_k(\lambda - 1)}\right)^{\frac{1}{\nu_k}}, \\
B_k &= \frac{c_k}{\Gamma(\beta_k)} \left(\frac{e^{\mu_k t_k}\Gamma(\beta_k^2)}{\mu_k^{\beta_k^2}}\right)^{\frac{1}{\mu_k}} \left(\int_{t_{k-1}}^{t_k} e^{\nu_k(\lambda - 1)s}\left\{\int_{t_{k-1}}^{s} q(\nu) e^{-\lambda \nu}\,d\nu\right\}^{\nu_k} ds\right)^{\frac{1}{\nu_k}} + b_k,
\end{aligned}$$

(ii) $\beta_k > \frac{1}{2}$ *for* $k = 1, 2, \ldots,$

$$m(t) \leq m(t_0)\left(\prod_{t_0 < t_k < t} C_k\right)e^{\lambda(t-t_0)} + \sum_{t_0 < t_k < t}\left(\prod_{t_k < t_j < t} C_j D_k e^{\lambda(t-t_k)}\right) + \int_{t_l}^{t} q(s)e^{\lambda(t-s)}\,ds, \tag{2.15}$$

where

$$C_k = \frac{c_k}{2^{\beta_k}\Gamma(\beta_k)}\left(\frac{\Gamma(2\beta_k - 1)}{\lambda - 1}\left[1 - e^{2(t_k - t_{k-1})(1-\lambda)}\right]\right)^{\frac{1}{2}},$$

$$D_k = \frac{c_k}{\Gamma(\beta_k)}\left(\frac{e^{2t_k}\Gamma(2\beta_k - 1)}{2^{2\beta_k - 1}}\right)^{\frac{1}{2}}\left(\int_{t_{k-1}}^{t_k} e^{2(\lambda-1)s}\left\{\int_{t_{k-1}}^{s} q(\nu)e^{-\lambda\nu}\,d\nu\right\}^2 ds\right)^{\frac{1}{2}} + b_k.$$

Case II: $\lambda = 1$,

(i) $0 < \beta_k \leq \frac{1}{2}$ *for* $k = 1, 2, \ldots,$

$$m(t) \leq m(t_0)\left(\prod_{t_0 < t_k < t} P_k\right)e^{(t-t_0)} + \sum_{t_0 < t_k < t}\left(\prod_{t_k < t_j < t} P_j V_k e^{(t-t_k)}\right) + \int_{t_l}^{t} q(s)e^{(t-s)}\,ds, \tag{2.16}$$

where

$$P_k = \frac{c_k}{\Gamma(\beta_k)}\left(\frac{\Gamma(\beta_k^2)}{\mu_k^{\beta_k^2}}\right)^{\frac{1}{\mu_k}}(t_k - t_{k-1})^{\frac{1}{\nu_k}},$$

$$V_k = \frac{c_k}{\Gamma(\beta_k)}\left(\frac{e^{\mu_k t_k}\Gamma(\beta_k^2)}{\mu_k^{\beta_k^2}}\right)^{\frac{1}{\mu_k}}\left(\int_{t_{k-1}}^{t_k}\left\{\int_{t_{k-1}}^{s} q(\nu)e^{-\nu}\,d\nu\right\}^{\nu_k} ds\right)^{\frac{1}{\nu_k}} + b_k,$$

(ii) $\beta_k > \frac{1}{2}$ *for* $k = 1, 2, \ldots,$

$$m(t) \leq m(t_0)\left(\prod_{t_0 < t_k < t} S_k\right)e^{(t-t_0)} + \sum_{t_0 < t_k < t}\left(\prod_{t_k < t_j < t} S_j U_k e^{(t-t_k)}\right) + \int_{t_l}^{t} q(s)e^{(t-s)}\,ds, \tag{2.17}$$

where

$$S_k = \frac{c_k}{\Gamma(\beta_k)}\left(\frac{\Gamma(2\beta_k - 1)(t_k - t_{k-1})}{2^{2\beta_k - 1}}\right)^{\frac{1}{2}},$$

$$U_k = \frac{c_k}{\Gamma(\beta_k)}\left(\frac{e^{2t_k}\Gamma(2\beta_k - 1)}{2^{2\beta_k - 1}}\right)^{\frac{1}{2}}\left(\int_{t_{k-1}}^{t_k}\left\{\int_{t_{k-1}}^{s} q(\nu)e^{-\nu}\,d\nu\right\}^2 ds\right)^{\frac{1}{2}} + b_k.$$

Corollary 2.4 *Let* (H_0) *and* (H_1) *hold. Suppose that* $q \in C[\mathbb{R}_+, \mathbb{R}]$ *and, for* $k = 1, 2, \ldots$, $t \geq t_0$,

$$\begin{cases} m'(t) \leq q(t), & t \neq t_k, \\ \Delta m(t_k) \leq \frac{c_k}{\Gamma(\beta_k)} \int_{t_{k-1}}^{t_k} (t_k - s)^{\beta_k - 1} m(s)\,ds + b_k, \end{cases} \tag{2.18}$$

where $c_k \geq 0$, $\beta_k > 0$ *and* b_k, $k = 1, 2, \ldots$ *are constants*, $\Delta m(t_k) = m(t_k^+) - m(t_k)$. *Then, for* $t \geq t_0$, *the following assertions hold*:

(i) $0 < \beta_k \leq \frac{1}{2}$ *for* $k = 1, 2, \ldots$,

$$m(t) \leq m(t_0)\left(\prod_{t_0 < t_k < t} F_k\right) + \sum_{t_0 < t_k < t}\left(\prod_{t_k < t_j < t} F_j H_k\right) + \int_{t_l}^{t} q(s)\,ds, \tag{2.19}$$

where

$$F_k = 1 + \frac{c_k}{\Gamma(\beta_k)}\left(\frac{\Gamma(\beta_k^2)}{\mu_k^{\beta_k^2}}\right)^{\frac{1}{\mu_k}} \left(\frac{e^{\nu_k(t_k - t_{k-1})} - 1}{\nu_k}\right)^{\frac{1}{\nu_k}},$$

$$H_k = \int_{t_{k-1}}^{t_k} q(s)\,ds + \frac{c_k}{\Gamma(\beta_k)}\left(\frac{e^{\mu_k t_k}\Gamma(\beta_k^2)}{\mu_k^{\beta_k^2}}\right)^{\frac{1}{\mu_k}} \left(\int_{t_{k-1}}^{t_k} e^{-\nu_k s}\left\{\int_{t_{k-1}}^{s} q(v)\,dv\right\}^{\nu_k} ds\right)^{\frac{1}{\nu_k}} + b_k,$$

(ii) $\beta_k > \frac{1}{2}$ *for* $k = 1, 2, \ldots$,

$$m(t) \leq m(t_0)\left(\prod_{t_0 < t_k < t} M_k\right) + \sum_{t_0 < t_k < t}\left(\prod_{t_k < t_j < t} M_j N_k\right) + \int_{t_l}^{t} q(s)\,ds, \tag{2.20}$$

where

$$M_k = 1 + \frac{c_k}{\Gamma(\beta_k)}\left(\frac{\Gamma(2\beta_k - 1)}{2^{2\beta_k}}\left[e^{2(t_k - t_{k-1})} - 1\right]\right)^{\frac{1}{2}},$$

$$N_k = \int_{t_{k-1}}^{t_k} q(s)\,ds + \frac{c_k}{\Gamma(\beta_k)}\left(\frac{e^{2t_k}\Gamma(2\beta_k - 1)}{2^{2\beta_k - 1}}\right)^{\frac{1}{2}} \left(\int_{t_{k-1}}^{t_k} e^{-2s}\left\{\int_{t_{k-1}}^{s} q(v)\,dv\right\}^{2} ds\right)^{\frac{1}{2}} + b_k.$$

Now we state and prove impulsive integral inequalities with nonlocal jump conditions.

Theorem 2.5 *Assume that* (H_0) *and* (H_1) *hold. Suppose that* $p \in C[\mathbb{R}_+, \mathbb{R}_+]$ *and, for* $k = 1, 2, \ldots$, $t \geq t_0$,

$$\begin{aligned} m(t) \leq{}& C + \int_{t_0}^{t} p(s) m(s)\,ds + \sum_{t_0 < t_k < t} \gamma_k m(t_k) \\ &+ \sum_{t_0 < t_k < t} \frac{\alpha_k}{\Gamma(\beta_k)} \int_{t_{k-1}}^{t_k} (t_k - s)^{\beta_k - 1} m(s)\,ds, \end{aligned} \tag{2.21}$$

where $\alpha_k \geq 0$, $\gamma_k \geq -1$, $\beta_k > 0$, $k = 1, 2, \ldots$, *and* C *are constants. Then, for* $t \geq t_0$, *the following assertions hold*:

(i) $0 < \beta_k \le \frac{1}{2}$ *for* $k = 1, 2, \ldots,$

$$m(t) \le C \prod_{t_0<t_k<t} \biggl\{ (1+\gamma_k) e^{\int_{t_{k-1}}^{t_k} p(\xi)\,d\xi} + \frac{\alpha_k}{\Gamma(\beta_k)} \biggl(\frac{e^{\mu_k t_k}}{\mu_k^{\beta_k^2}} \Gamma\bigl(\beta_k^2\bigr) \biggr)^{\frac{1}{\mu_k}} \times \biggl(\int_{t_{k-1}}^{t_k} e^{\nu_k (\int_{t_{k-1}}^{s} p(\xi)\,d\xi - s)}\,ds \biggr)^{\frac{1}{\nu_k}} \biggr\} e^{\int_{t_l}^{t} p(\xi)\,d\xi}, \tag{2.22}$$

where $\mu_k = \beta_k + 1$ *and* $\nu_k = 1 + \frac{1}{\beta_k}$,

(ii) $\beta_k > \frac{1}{2}$ *for* $k = 1, 2, \ldots,$

$$m(t) \le C \prod_{t_0<t_k<t} \biggl\{ (1+\gamma_k) e^{\int_{t_{k-1}}^{t_k} p(\xi)\,d\xi} + \frac{\alpha_k}{\Gamma(\beta_k)} \biggl(\frac{e^{2t_k}}{2^{2\beta_k-1}} \Gamma(2\beta_k - 1) \biggr)^{\frac{1}{2}} \times \biggl(\int_{t_{k-1}}^{t_k} e^{2(\int_{t_{k-1}}^{s} p(\xi)\,d\xi - s)}\,ds \biggr)^{\frac{1}{2}} \biggr\} e^{\int_{t_l}^{t} p(\xi)\,d\xi}. \tag{2.23}$$

Proof Define a function $g(t)$ by the right-hand side of (2.21). Then we have

$$\begin{cases} g'(t) = p(t)m(t), \quad t \neq t_k, \qquad g(t_0) = C, \\ g(t_k^+) = g(t_k) + \gamma_k m(t_k) + \frac{\alpha_k}{\Gamma(\beta_k)} \int_{t_{k-1}}^{t_k} (t_k - s)^{\beta_k - 1} m(s)\,ds. \end{cases}$$

Since $m(t) \le g(t)$, we obtain

$$\begin{cases} g'(t) \le p(t)g(t), \quad t \neq t_k, \qquad g(t_0) = C, \\ g(t_k^+) = (1+\gamma_k) g(t_k) + \frac{\alpha_k}{\Gamma(\beta_k)} \int_{t_{k-1}}^{t_k} (t_k - s)^{\beta_k - 1} g(s)\,ds. \end{cases}$$

Applying Theorem 2.2, we deduce that:

(i) $0 < \beta_k \le \frac{1}{2}$, $k = 1, 2, \ldots,$ for $t \ge t_0$,

$$g(t) \le C \prod_{t_0<t_k<t} \biggl\{ (1+\gamma_k) e^{\int_{t_{k-1}}^{t_k} p(\xi)\,d\xi} + \frac{\alpha_k}{\Gamma(\beta_k)} \biggl(\frac{e^{\mu_k t_k}}{\mu_k^{\beta_k^2}} \Gamma\bigl(\beta_k^2\bigr) \biggr)^{\frac{1}{\mu_k}} \times \biggl(\int_{t_{k-1}}^{t_k} e^{\nu_k (\int_{t_{k-1}}^{s} p(\xi)\,d\xi - s)}\,ds \biggr)^{\frac{1}{\nu_k}} \biggr\} e^{\int_{t_l}^{t} p(\xi)\,d\xi},$$

(ii) $\beta_k > \frac{1}{2}$, $k = 1, 2, \ldots,$ for $t \ge t_0$,

$$g(t) \le C \prod_{t_0<t_k<t} \biggl\{ (1+\gamma_k) e^{\int_{t_{k-1}}^{t_k} p(\xi)\,d\xi} + \frac{\alpha_k}{\Gamma(\beta_k)} \biggl(\frac{e^{2t_k}}{2^{2\beta_k-1}} \Gamma(2\beta_k - 1) \biggr)^{\frac{1}{2}} \times \biggl(\int_{t_{k-1}}^{t_k} e^{2(\int_{t_{k-1}}^{s} p(\xi)\,d\xi - s)}\,ds \biggr)^{\frac{1}{2}} \biggr\} e^{\int_{t_l}^{t} p(\xi)\,d\xi},$$

which are the results in (2.22) and (2.23), respectively. □

In the case when in place of the constant C involved in Theorem 2.5 we have a function $h(t)$, we obtain the following result.

Theorem 2.6 *Assume that* (H_0) *and* (H_1) *hold. Suppose that* $p \in C[\mathbb{R}_+, \mathbb{R}_+]$, $h \in PC[\mathbb{R}_+, \mathbb{R}]$, *and, for* $k = 1, 2, \ldots$, $t \geq t_0$,

$$\begin{aligned} m(t) &\leq h(t) + \int_{t_0}^{t} p(s)m(s)\,ds + \sum_{t_0<t_k<t} \gamma_k m(t_k) \\ &\quad + \sum_{t_0<t_k<t} \frac{\alpha_k}{\Gamma(\beta_k)} \int_{t_{k-1}}^{t_k} (t_k - s)^{\beta_k - 1} m(s)\,ds, \end{aligned} \tag{2.24}$$

where $\alpha_k \geq 0$, $\gamma_k \geq -1$ *and* $\beta_k > 0$, $k = 1, 2, \ldots$, *are constants. Then, for* $t \geq t_0$, *the following assertions hold*:

(i) $0 < \beta_k \leq \frac{1}{2}$ *for* $k = 1, 2, \ldots$,

$$\begin{aligned} m(t) &\leq h(t) + \Biggl\{ \sum_{t_0<t_k<t} \Biggl[\prod_{t_k<t_j<t} \Biggl\{ (1+\gamma_j) e^{\int_{t_{j-1}}^{t_j} p(\xi)\,d\xi} + \frac{\alpha_j}{\Gamma(\beta_j)} \biggl(\frac{e^{\mu_j t_j}}{\mu_j^{\beta_j^2}} \Gamma\bigl(\beta_j^2\bigr) \biggr)^{\frac{1}{\mu_j}} \\ &\quad \times \biggl(\int_{t_{j-1}}^{t_j} e^{\nu_j (\int_{t_{j-1}}^{s} p(\xi)\,d\xi - s)}\,ds \biggr)^{\frac{1}{\nu_j}} \Biggr\} \Biggl\{ (1+\gamma_k) \int_{t_{k-1}}^{t_k} p(s)h(s) e^{\int_s^{t_k} p(\xi)\,d\xi}\,ds \\ &\quad + \frac{\alpha_k}{\Gamma(\beta_k)} \biggl(\frac{e^{\mu_k t_k}}{\mu_k^{\beta_k^2}} \Gamma\bigl(\beta_k^2\bigr) \biggr)^{\frac{1}{\mu_k}} \\ &\quad \times \biggl(\int_{t_{k-1}}^{t_k} e^{-\nu_k s} \biggl\{ \int_{t_{k-1}}^{s} p(\nu)h(\nu) e^{\int_\nu^s p(\xi)\,d\xi}\,d\nu \biggr\}^{\nu_k} ds \biggr)^{\frac{1}{\nu_k}} \\ &\quad + \gamma_k h(t_k) + \frac{\alpha_k}{\Gamma(\beta_k)} \biggl(\frac{e^{\mu_k t_k}}{\mu_k^{\beta_k^2}} \Gamma\bigl(\beta_k^2\bigr) \biggr)^{\frac{1}{\mu_k}} \biggl(\int_{t_{k-1}}^{t_k} e^{-\nu_k s} h^{\nu_k}(s)\,ds \biggr)^{\frac{1}{\nu_k}} \Biggr\} \Biggr] \Biggr\} e^{\int_{t_l}^{t} p(\xi)\,d\xi} \\ &\quad + \int_{t_l}^{t} p(s)h(s) e^{\int_s^t p(\xi)\,d\xi}, \end{aligned} \tag{2.25}$$

where $\mu_k = \beta_k + 1$ *and* $\nu_k = 1 + \frac{1}{\beta_k}$,

(ii) $\beta_k > \frac{1}{2}$ *for* $k = 1, 2, \ldots$,

$$\begin{aligned} m(t) &\leq h(t) + \Biggl\{ \sum_{t_0<t_k<t} \Biggl[\prod_{t_k<t_j<t} \Biggl\{ (1+\gamma_j) e^{\int_{t_{j-1}}^{t_j} p(\xi)\,d\xi} + \frac{\alpha_j}{\Gamma(\beta_j)} \biggl(\frac{e^{2t_j}}{2^{2\beta_j-1}} \Gamma(2\beta_j - 1) \biggr)^{\frac{1}{2}} \\ &\quad \times \biggl(\int_{t_{j-1}}^{t_j} e^{2(\int_{t_{j-1}}^{s} p(\xi)\,d\xi - s)}\,ds \biggr)^{\frac{1}{2}} \Biggr\} \Biggl\{ (1+\gamma_k) \int_{t_{k-1}}^{t_k} p(s)h(s) e^{\int_s^{t_k} p(\xi)\,d\xi}\,ds \\ &\quad + \frac{\alpha_k}{\Gamma(\beta_k)} \biggl(\frac{e^{2t_k}}{2^{2\beta_k-1}} \Gamma(2\beta_k - 1) \biggr)^{\frac{1}{2}} \\ &\quad \times \biggl(\int_{t_{k-1}}^{t_k} e^{-2s} \biggl\{ \int_{t_{k-1}}^{s} p(\nu)h(\nu) e^{\int_\nu^s p(\xi)\,d\xi}\,d\nu \biggr\}^{2} ds \biggr)^{\frac{1}{2}} + \gamma_k h(t_k) \\ &\quad + \frac{\alpha_k}{\Gamma(\beta_k)} \biggl(\frac{e^{2t_k}}{2^{2\beta_k-1}} \Gamma(2\beta_k - 1) \biggr)^{\frac{1}{2}} \biggl(\int_{t_{k-1}}^{t_k} e^{-2s} h^2(s)\,ds \biggr)^{\frac{1}{2}} \Biggr\} \Biggr] \Biggr\} e^{\int_{t_l}^{t} p(\xi)\,d\xi} \\ &\quad + \int_{t_l}^{t} p(s)h(s) e^{\int_s^t p(\xi)\,d\xi}. \end{aligned} \tag{2.26}$$

Proof Setting

$$g(t) = \int_{t_0}^{t} p(s)m(s)\,ds + \sum_{t_0<t_k<t} \gamma_k m(t_k) + \sum_{t_0<t_k<t} \frac{\alpha_k}{\Gamma(\beta_k)} \int_{t_{k-1}}^{t_k} (t_k - s)^{\beta_k-1} m(s)\,ds,$$

and using the fact that $m(t) \le h(t) + g(t)$, we have

$$\begin{cases} g'(t) \le p(t)g(t) + p(t)h(t), \quad t \neq t_k, \qquad g(t_0) = 0, \\ g(t_k^+) = (1+\gamma_k)g(t_k) + \frac{\alpha_k}{\Gamma(\beta_k)} \int_{t_{k-1}}^{t_k} (t_k - s)^{\beta_k-1} g(s)\,ds + \gamma_k h(t_k) \\ \qquad + \frac{\alpha_k}{\Gamma(\beta_k)} \int_{t_{k-1}}^{t_k} (t_k - s)^{\beta_k-1} h(s)\,ds. \end{cases}$$

Applying Theorem 2.2 for $0 < \beta_k \le 1/2$ and $\beta_k > 1/2$ together with $m(t) \le h(t) + g(t)$, we then obtain the estimates in (2.25) and (2.26), respectively. □

3 Applications to impulsive fractional integral jump conditions

In this section, two applications of impulsive differential and impulsive integral inequalities with Riemann-Liouville fractional integral jump conditions are given.

Definition 3.1 The Riemann-Liouville fractional integral of order $\beta > 0$ of a function $f : (t_0, \infty) \to \mathbb{R}$ is defined by

$$I_{t_0}^{\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-s)^{\beta-1} f(s)\,ds,$$

provided the right-hand side is point-wise defined on (t_0, ∞), where Γ is the Gamma function.

We apply our results to work out the maximum principle of the impulsive differential inequality.

Proposition 3.2 *Assume that $x \in PC^1[J, \mathbb{R}]$ satisfies*

$$\begin{cases} x'(t) - Mx(t) + a(t) \le 0, \quad t \neq t_k, t \in J = [0, T], \\ x(t_k^+) \le c_k I_{t_{k-1}}^{\beta_k} x(t_k) - b_k, \quad k = 1, 2, \dots, n, \\ x(0) = x(T) + \lambda, \end{cases} \tag{3.1}$$

where $M > 0$, $a \in C[\mathbb{R}_+, \mathbb{R}_+]$, $0 = t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1} = T$, $b_k, c_k \ge 0$, $\beta_k > 0$, $k = 1, 2, \dots, n$, and λ are constants.

Suppose in addition that:

Case I: $M \neq 1$.

(i) $0 < \beta_k \le \frac{1}{2}$ *for* $k = 1, 2, \dots, n$,

$$(\mathrm{Q}_1) \qquad \prod_{k=1}^{n} \frac{c_k}{\Gamma(\beta_k)} \left(\frac{\Gamma(\beta_k^2)}{\mu_k^{\beta_k^2}} \right)^{\frac{1}{\mu_k}} \left(\frac{1 - e^{\nu_k (t_k - t_{k-1})(1-M)}}{\nu_k (M-1)} \right)^{\frac{1}{\nu_k}} < e^{-MT},$$

$$(\mathrm{Q}_2) \qquad \frac{c_k}{\Gamma(\beta_k)} \left(\frac{e^{\mu_k t_k} \Gamma(\beta_k^2)}{\mu_k^{\beta_k^2}} \right)^{\frac{1}{\mu_k}} \left(\int_{t_{k-1}}^{t_k} e^{\nu_k s(M-1)} \left\{ -\int_{t_{k-1}}^{s} a(v) e^{-Mv}\,dv \right\}^{\nu_k} ds \right)^{\frac{1}{\nu_k}} \le b_k,$$

$$(\mathrm{Q}_3)\qquad \lambda \le \int_{t_n}^{T} a(s)e^{M(T-s)}\,ds,$$

where $\mu_k = \beta_k + 1$ *and* $\nu_k = 1 + \frac{1}{\beta_k}$,

(ii) $\beta_k > \frac{1}{2}$ *for* $k = 1, 2, \ldots, n$,

$$(\mathrm{Q}_4)\qquad \prod_{k=1}^{n} \frac{c_k}{2^{\beta_k}\Gamma(\beta_k)}\left(\frac{\Gamma(2\beta_k - 1)}{M-1}\left[1 - e^{2(t_k - t_{k-1})(1-M)}\right]\right)^{\frac{1}{2}} < e^{-MT},$$

$$(\mathrm{Q}_5)\qquad \frac{c_k}{\Gamma(\beta_k)}\left(\frac{e^{2t_k}\Gamma(2\beta_k - 1)}{2^{2\beta_k - 1}}\right)^{\frac{1}{2}}\left(\int_{t_{k-1}}^{t_k} e^{2s(M-1)}\left\{-\int_{t_{k-1}}^{s} a(v)e^{-Mv}\,dv\right\}^2 ds\right)^{\frac{1}{2}} \le b_k,$$

and (Q_3) *holds.*

Case II: $M = 1$.

(i) $0 < \beta_k \le \frac{1}{2}$ *for* $k = 1, 2, \ldots, n$,

$$(\mathrm{Q}_6)\qquad \prod_{k=1}^{n} \frac{c_k}{\Gamma(\beta_k)}\left(\frac{\Gamma(\beta_k^2)}{\mu_k^{\beta_k^2}}\right)^{\frac{1}{\mu_k}} (t_k - t_{k-1})^{\frac{1}{\nu_k}} < e^{-T},$$

$$(\mathrm{Q}_7)\qquad \frac{c_k}{\Gamma(\beta_k)}\left(\frac{e^{\mu_k t_k}\Gamma(\beta_k^2)}{\mu_k^{\beta_k^2}}\right)^{\frac{1}{\mu_k}}\left(\int_{t_{k-1}}^{t_k}\left\{-\int_{t_{k-1}}^{s} a(v)e^{-v}\,dv\right\}^{\nu_k} ds\right)^{\frac{1}{\nu_k}} \le b_k,$$

$$(\mathrm{Q}_8)\qquad \lambda \le \int_{t_n}^{T} a(s)e^{(T-s)}\,ds,$$

(ii) $\beta_k > \frac{1}{2}$ *for* $k = 1, 2, \ldots, n$,

$$(\mathrm{Q}_9)\qquad \prod_{k=1}^{n} \frac{c_k}{\Gamma(\beta_k)}\left(\frac{\Gamma(2\beta_k - 1)(t_k - t_{k-1})}{2^{2\beta_k - 1}}\right)^{\frac{1}{2}} < e^{-T},$$

$$(\mathrm{Q}_{10})\qquad \frac{c_k}{\Gamma(\beta_k)}\left(\frac{e^{2t_k}\Gamma(2\beta_k - 1)}{2^{2\beta_k - 1}}\right)^{\frac{1}{2}}\left(\int_{t_{k-1}}^{t_k}\left\{-\int_{t_{k-1}}^{s} a(v)e^{-v}\,dv\right\}^2 ds\right)^{\frac{1}{2}} \le b_k,$$

and (Q_8) *holds.*

Then $x(t) \le 0$ *for* $t \in [0, T]$.

Proof To prove Case I(i), applying Corollary 2.3 for $t \in [0, T]$, we have

$$x(t) \le x(0)\left(\prod_{t_0 < t_k < t} A_k^*\right)e^{Mt} + \sum_{t_0 < t_k < t}\left(\prod_{t_k < t_j < t} A_j^* B_k^* e^{M(t - t_k)}\right) - \int_{t_l}^{t} a(s)e^{M(t-s)}\,ds,$$

where

$$A_k^* = \frac{c_k}{\Gamma(\beta_k)}\left(\frac{\Gamma(\beta_k^2)}{\mu_k^{\beta_k^2}}\right)^{\frac{1}{\mu_k}}\left(\frac{1 - e^{\nu_k(t_k - t_{k-1})(1-M)}}{\nu_k(M-1)}\right)^{\frac{1}{\nu_k}},$$

$$B_k^* = \frac{c_k}{\Gamma(\beta_k)}\left(\frac{e^{\mu_k t_k}\Gamma(\beta_k^2)}{\mu_k^{\beta_k^2}}\right)^{\frac{1}{\mu_k}}\left(\int_{t_{k-1}}^{t_k} e^{\nu_k(M-1)s}\left\{-\int_{t_{k-1}}^{s} a(v)e^{-Mv}\,dv\right\}^{\nu_k} ds\right)^{\frac{1}{\nu_k}} - b_k.$$

It is easy to see that $A_k^* \geq 0$ for all $k = 1, 2, \dots, n$. The condition (Q_2) implies that $B_k^* \leq 0$ for all $k = 1, 2, \dots, n$. Then it is sufficient to show that $x(0) \leq 0$. For $t = T$, we have

$$x(T) \leq x(0)\left(\prod_{k=1}^{n} A_k^*\right)e^{MT} + \sum_{t_0<t_k<T}\left(\prod_{t_k<t_j<T} A_j^* B_k^* e^{M(T-t_k)}\right) - \int_{t_n}^{T} a(s)e^{M(T-s)}\,ds.$$

Using the conditions (Q_1) and (Q_3), we see that

$$\begin{aligned} x(0)\left[1-\left(\prod_{k=1}^{n} A_k^*\right)e^{MT}\right] &\leq \lambda + \sum_{t_0<t_k<T}\left(\prod_{t_k<t_j<T} A_j^* B_k^* e^{M(T-t_k)}\right) - \int_{t_n}^{T} a(s)e^{M(T-s)}\,ds \\ &\leq 0, \end{aligned}$$

which implies that $x(0) \leq 0$.

Using a similar method to prove Case I(i) with suitable conditions, we deduce that $x(0) \leq 0$. This completes the proof. □

The last application shows the boundedness of solution of impulsive differential equation with Riemann-Liouville fractional integral jump conditions.

Proposition 3.3 *Let $x \in PC^1[\mathbb{R}_+, \mathbb{R}]$ such that*

$$\begin{cases} x'(t) = f(t, x(t)), & t \neq t_k, t \in [t_0, \infty), \\ \Delta x(t_k) = Z_k(I_{t_{k-1}}^{\beta_k} x(t_k)), & k = 1, 2, \dots, \\ x(t_0) = x_0, \end{cases} \tag{3.2}$$

where $f \in C(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$, $Z_k \in C(\mathbb{R}, \mathbb{R})$, $0 \leq t_0 < t_1 < t_2 < \cdots$, $\lim_{k\to\infty} t_k = \infty$, $\Delta x(t_k) = x(t_k^+) - x(t_k)$, $\beta_k > 0$, $k = 1, 2, \dots$, and x_0 are constants.

Assume that:

(Q_{11}) *there exists a constant $N > 0$, such that*

$$\left|f\left(t, x(t)\right)\right| \leq N\left|x(t)\right| \quad \text{for } t \geq t_0,$$

(Q_{12}) *there exist constants $L_k \geq 0$ such that*

$$\left|Z_k(x)\right| \leq L_k|x|, \quad x \in \mathbb{R}, k = 1, 2, \dots.$$

Then, for $t \geq t_0$, the following inequalities hold:

Case I: $N \neq 1$.

(i) $0 < \beta_k \leq \frac{1}{2}$ *for $k = 1, 2, \dots$,*

$$\begin{aligned} \left|x(t)\right| \leq{}& |x_0| \prod_{t_0<t_k<t}\left\{1 + \frac{L_k}{\Gamma(\beta_k)}\left(\frac{\Gamma(\beta_k^2)}{\mu_k^{\beta_k^2}}\right)^{\frac{1}{\mu_k}}\left(\frac{1-e^{\nu_k(t_k-t_{k-1})(1-N)}}{\nu_k(N-1)}\right)^{\frac{1}{\nu_k}}\right\} \\ &\times e^{N(t-t_0)}, \end{aligned} \tag{3.3}$$

where $\mu_k = \beta_k + 1$ and $\nu_k = 1 + \frac{1}{\beta_k}$,

(ii) $\beta_k > \frac{1}{2}$ *for* $k = 1, 2, \ldots,$

$$\bigl|x(t)\bigr| \le |x_0| \prod_{t_0<t_k<t} \left\{1 + \frac{L_k}{2^{\beta_k}\Gamma(\beta_k)}\left(\frac{\Gamma(2\beta_k - 1)}{N-1}\bigl[1 - e^{2(t_k-t_{k-1})(1-N)}\bigr]\right)^{\frac{1}{2}}\right\} \times e^{N(t-t_0)}. \tag{3.4}$$

Case II: $N = 1$.

(i) $0 < \beta_k \le \frac{1}{2}$ *for* $k = 1, 2, \ldots,$

$$\bigl|x(t)\bigr| \le |x_0| \prod_{t_0<t_k<t} \left\{1 + \frac{L_k}{\Gamma(\beta_k)}\left(\frac{\Gamma(\beta_k^2)}{\mu_k^{\beta_k^2}}\right)^{\frac{1}{\mu_k}} (t_k - t_{k-1})^{\frac{1}{\nu_k}}\right\} e^{(t-t_0)}, \tag{3.5}$$

(ii) $\beta_k > \frac{1}{2}$ *for* $k = 1, 2, \ldots,$

$$\bigl|x(t)\bigr| \le |x_0| \prod_{t_0<t_k<t} \left\{1 + \frac{L_k}{\Gamma(\beta_k)}\left(\frac{\Gamma(2\beta_k - 1)(t_k - t_{k-1})}{2^{2\beta_k-1}}\right)^{\frac{1}{2}}\right\} e^{(t-t_0)}. \tag{3.6}$$

Proof The solution $x(t)$ of problem (3.2) satisfies the impulsive integral equation

$$x(t) = x(t_0) + \int_{t_0}^{t} f\bigl(s, x(s)\bigr)\,ds + \sum_{t_0<t_k<t} Z_k\bigl(I_{t_{k-1}}^{\beta_k} x(t_k)\bigr).$$

From conditions (Q_{11})-(Q_{12}), it follows for $t \ge t_0$ that

$$\begin{aligned}\bigl|x(t)\bigr| &\le |x_0| + \int_{t_0}^{t} \bigl|f\bigl(s, x(s)\bigr)\bigr|\,ds + \sum_{t_0<t_k<t} \bigl|Z_k\bigl(I_{t_{k-1}}^{\beta_k} x(t_k)\bigr)\bigr| \\ &\le |x_0| + \int_{t_0}^{t} N\bigl|x(s)\bigr|\,ds + \sum_{t_0<t_k<t} L_k I_{t_{k-1}}^{\beta_k} |x|(t_k).\end{aligned}$$

Hence Theorem 2.5 yields the estimate

$$\begin{aligned}\bigl|x(t)\bigr| &\le |x_0| \prod_{t_0<t_k<t} \left\{e^{N(t_k-t_{k-1})} + \frac{L_k}{\Gamma(\beta_k)}\left(\frac{e^{\mu_k t_k}\Gamma(\beta_k^2)}{\mu_k^{\beta_k^2}}\right)^{\frac{1}{\mu_k}} \right. \\ &\qquad \left. \times \left(\int_{t_{k-1}}^{t_k} e^{\nu_k(N(s-t_{k-1})-s)}\,ds\right)^{\frac{1}{\nu_k}}\right\} e^{N(t-t_l)} \\ &= |x_0| \prod_{k=1}^{l} \left\{e^{N(t_k-t_{k-1})} + \frac{L_k}{\Gamma(\beta_k)}\left(\frac{e^{\mu_k t_k}\Gamma(\beta_k^2)}{\mu_k^{\beta_k^2}}\right)^{\frac{1}{\mu_k}} \right. \\ &\qquad \left. \times \left(\frac{e^{\nu_k(N(t_k-t_{k-1})-t_k)} - e^{-\nu_k t_{k-1}}}{\nu_k(N-1)}\right)^{\frac{1}{\nu_k}}\right\} e^{N(t-t_l)} \\ &= |x_0| \prod_{k=1}^{l} \left\{e^{N(t_k-t_{k-1})} + \frac{L_k}{\Gamma(\beta_k)}\left(\frac{e^{\mu_k t_k}\Gamma(\beta_k^2)}{\mu_k^{\beta_k^2}}\right)^{\frac{1}{\mu_k}} \right. \\ &\qquad \left. \times \left(\frac{e^{\nu_k(N(t_k-t_{k-1})-t_k)}(1 - e^{\nu_k(t_k-t_{k-1})(1-N)})}{\nu_k(N-1)}\right)^{\frac{1}{\nu_k}}\right\} e^{N(t-t_l)}\end{aligned}$$

$$= |x_0| \prod_{k=1}^{l} \left\{ e^{N(t_k - t_{k-1})} + e^{N(t_k - t_{k-1})} \frac{L_k}{\Gamma(\beta_k)} \left(\frac{\Gamma(\beta_k^2)}{\mu_k^{\beta_k^2}} \right)^{\frac{1}{\mu_k}} \right.$$
$$\left. \times \left(\frac{1 - e^{\nu_k (t_k - t_{k-1})(1-N)}}{\nu_k (N-1)} \right)^{\frac{1}{\nu_k}} \right\} e^{N(t - t_l)}, \quad t \geq t_0.$$

Therefore, inequality (3.3) holds for $t \geq t_0$. For the other cases the proofs are similar and thus omitted. The proof is completed. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this article. They read and approved the final manuscript.

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References

1. Lakshmikantham, V, Bainov, DD, Simeonov, PS: Theory of Impulsive Differential Equations. World Scientific, Singapore (1989)
2. Bainov, DD, Simeonov, PS: Impulsive Differential Equations: Periodic Solutions and Applications. Longman, Harlow (1993)
3. Bainov, DD, Simeonov, PS: Impulsive Differential Equations: Asymptotic Properties of the Solutions. World Scientific, Singapore (1995)
4. Samoilenko, AM, Perestyuk, NA: Impulsive Differential Equations. World Scientific, Singapore (1995)
5. Benchohra, M, Henderson, J, Ntouyas, SK: Impulsive Differential Equations and Inclusions, vol. 2. Hindawi Publishing Corporation, New York (2006)
6. Bainov, DD, Hristova, SG: The method of quasilinearization for the periodic boundary value problem for systems of impulsive differential equations. Appl. Math. Comput. **117**, 73-85 (2001)
7. Bonotto, EM, Gimenes, LP, Federson, M: Oscillation for a second-order neutral equation with impulses. Appl. Math. Comput. **215**, 1-15 (2009)
8. Cui, BT, Han, M, Yang, H: Some sufficient conditions for oscillation of impulsive delay hyperbolic systems with Robin boundary conditions. J. Comput. Appl. Math. **180**, 365-375 (2005)
9. Ding, W, Xing, Y, Han, M: Anti-periodic boundary value problems for first order impulsive functional differential equations. Appl. Math. Comput. **186**, 45-53 (2007)
10. Franco, D, Nieto, JJ: Maximum principles for periodic impulsive first order problems. J. Comput. Appl. Math. **88**, 149-159 (1998)
11. Fu, X, Zhang, L: Forced oscillation for impulsive hyperbolic boundary value problems with delay. Appl. Math. Comput. **158**, 761-780 (2004)
12. Gimenes, LP, Federson, M: Oscillation by impulses for a second-order delay differential equation. Comput. Math. Appl. **52**, 819-828 (2006)
13. He, Z, Ge, W: Oscillations of second-order nonlinear impulsive ordinary differential equations. J. Comput. Appl. Math. **158**, 397-406 (2003)
14. He, Z, He, X: Periodic boundary value problems for first order impulsive integro-differential equations of mixed type. J. Math. Anal. Appl. **296**, 8-20 (2004)
15. Huang, M: Oscillation criteria for second order nonlinear dynamic equations with impulses. Comput. Math. Appl. **59**, 31-41 (2010)
16. Jiao, J, Chen, L, Li, L: Asymptotic behavior of solutions of second-order nonlinear impulsive differential equations. J. Math. Anal. Appl. **337**, 458-463 (2008)
17. Li, J: Periodic boundary value problems for second-order impulsive integro-differential equations. Appl. Math. Comput. **198**, 317-325 (2008)
18. Li, Q, Liang, H, Zhang, Z, Yu, Y: Oscillation of second order self-conjugate differential equation with impulses. J. Comput. Appl. Math. **197**, 78-88 (2006)
19. Li, J, Shen, J: Periodic boundary value problems for delay differential equations with impulses. J. Comput. Appl. Math. **193**, 563-573 (2006)
20. Liu, H, Li, Q: Asymptotic behavior of second-order impulsive differential equations. Electron. J. Differ. Equ. **2011**, 33 (2011)

21. Luo, J: Oscillation of hyperbolic partial differential equations with impulses. Appl. Math. Comput. **133**, 309-318 (2002)
22. Nieto, JJ, Rodriguez-Lopez, R: New comparison results for impulsive integro-differential equations and applications. J. Math. Anal. Appl. **328**, 1343-1368 (2007)
23. Peng, M: Oscillation theorems of second-order nonlinear neutral delay difference equations with impulses. Comput. Math. Appl. **44**, 741-748 (2002)
24. Shen, J: New maximum principles for first-order impulsive boundary value problems. Appl. Math. Lett. **16**, 105-112 (2003)
25. Stamova, IM: Lyapunov method for boundedness of solutions of nonlinear impulsive functional differential equations. Appl. Math. Comput. **177**, 714-719 (2006)
26. Zhang, C, Feng, W, Yang, J, Huang, M: Oscillations of second order impulses nonlinear FDE with forcing term. Appl. Math. Comput. **198**, 271-279 (2008)
27. Akça, H, Covachev, V, Covacheva, Z: Improved stability estimates for impulsive delay reaction-diffusion Cohen-Grossberg neural networks. Tatra Mt. Math. Publ. **54**, 1-18 (2013)
28. Deng, S, Prather, C: Generalization of an impulsive nonlinear singular Gronwall-Bihari inequality with delay. J. Inequal. Pure Appl. Math. **9**(2), Article ID 34 (2008)
29. Gallo, A, Piccirillo, AM: Multidimensional impulse inequalities and general Bahari type inequalities for discontinuous functions with delay. Nonlinear Stud. **19**(1), 13-24 (2012)
30. Hristova, SG: Nonlinear delay integral inequalities for piecewise continuous functions and applications. J. Inequal. Pure Appl. Math. **5**(4), Article ID 88 (2004)
31. Li, J: On some new impulsive integral inequalities. J. Inequal. Appl. **2008**, Article ID 312395 (2008)
32. Mi, Y: Some generalized Gronwall-Bellman type impulsive integral inequalities and their applications. J. Appl. Math. **2014**, Article ID 353210 (2014)
33. Peng, Y, Kang, Y, Yuan, M, Huang, R, Yang, L: Gronwall-type integral inequalities with impulses on time scales. Adv. Differ. Equ. **2011**, 26 (2011)
34. Tatar, NE: An impulsive nonlinear singular version of the Gronwall-Bihari inequality. J. Inequal. Appl. **2006**, Article ID 84561 (2006)
35. Wang, H, Ding, C: A new nonlinear impulsive delay differential inequality and its applications. J. Inequal. Appl. **2011**, 11 (2011)
36. Wang, X: A generalized Halanay inequality with impulse and delay. Adv. Devel. Technol. Inter. **1**(2), 9-23 (2012)
37. Thiramanus, P, Tariboon, J: Impulsive differential and impulsive integral inequalities with integral jump conditions. J. Inequal. Appl. **2012**, 25 (2012)
38. Herrmann, R: Fractional Calculus: An Introduction for Physicists. World Scientific, Singapore (2011)