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Splitting-midpoint method for zeros of the sum of accretive operator and μ-inversely strongly accretive operator in a q-uniformly smooth Banach space and its applications

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Abstract

Combining the implicit midpoint method and the splitting method, we present a new iterative algorithm with errors to solve the problems of finding zeros of the sum of m-accretive operators and μ-inversely strongly accretive operators in a real q-uniformly smooth and uniformly convex Banach space. We obtain some strong convergence theorems, which demonstrate the relationship between the zero of the sum of m-accretive operator and μ-inversely strongly accretive operator and the solution of one kind variational inequality. Moreover, the applications of the main results on the nonlinear problems with Neumann boundaries and Signorini boundaries are demonstrated.

MSC: 47H05; 47H09; 47H10

Keywords: μ-inversely strongly accretive operator; sum; zero; implicit midpoint method; splitting method; strong convergence; variational inequality

1 Introduction and preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* denote the dual space of E. We use '$\to$' and '$\rightharpoonup$' to denote strong and weak convergence either in E or in E^*, respectively. We denote the value of $f \in E^*$ at $x \in E$ by $\langle x, f\rangle$.

A Banach space E is said to be uniformly convex if, for each $\varepsilon \in (0,2]$, there exists $\delta > 0$ such that

$$\|x\| = \|y\| = 1, \qquad \|x-y\| \geq \varepsilon \quad \Rightarrow \quad \left\|\frac{x+y}{2}\right\| \leq 1-\delta.$$

A Banach space E is said to be smooth if

$$\lim_{t\to 0}\frac{\|x+ty\| - \|x\|}{t}$$

exists for each $x, y \in \{z \in E : \|z\| = 1\}$.

In addition, we define a function $\rho_E : [0, +\infty) \to [0, +\infty)$ called the modulus of smoothness of E as follows:

$$\rho_E(t) = \sup\left\{\frac{1}{2}\big(\|x+y\| + \|x-y\|\big) - 1 : x, y \in E, \|x\| = 1, \|y\| \le t\right\}.$$

It is well known that E is uniformly smooth if and only if $\frac{\rho_E(t)}{t} \to 0$, as $t \to 0$. Let $q > 1$ be a real number. A Banach space E is said to be q-uniformly smooth if there exists a positive constant C such that $\rho_E(t) \le Ct^q$. It is obvious that a q-uniformly smooth Banach space must be uniformly smooth.

The generalized duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q x := \big\{f \in E^* : \langle x, f\rangle = \|x\|^q, \|f\| = \|x\|^{q-1}\big\}, \quad x \in E.$$

In particular, $J \equiv J_2$ is called the normalized duality mapping and $J_q(x) = \|x\|^{q-2}J(x)$ for $x \neq 0$. If E is reduced to the Hilbert space H, then $J_q \equiv I$ is the identity mapping. It is well known that J is single-valued and norm-to-norm uniformly continuous on each bounded subset of E if E is a real smooth and uniformly convex Banach space, see [1]. Moreover, $J(cx) = cJx$ for all $x \in E$ and $c \in R^1$. In what follows, we still denote by J the single-valued normalized duality mapping. The normalized duality mapping J is said to be weakly sequentially continuous if $\{x_n\}$ is a sequence in E which converges weakly to x; it follows that $\{Jx_n\}$ converges in weak* to Jx. J is said to be weakly sequentially continuous at zero if $\{x_n\}$ is a sequence in E which converges weakly to 0; it follows that $\{Jx_n\}$ converges in weak* to 0.

For a mapping $T : E \to E$, we use $\mathrm{Fix}(T)$ to denote the fixed point set of it; that is, $\mathrm{Fix}(T) := \{x \in E : Tx = x\}$.

For an operator $A : D(A) \subset E \to 2^E$, we use $A^{-1}0$ to denote the set of zeros of it; that is, $A^{-1}0 := \{x \in D(A) : Ax = 0\}$.

Let $T : E \to E$ be a mapping. Then T is said to be

(1) nonexpansive if

$$\|Tx - Ty\| \le \|x - y\| \quad \text{for } \forall x, y \in E;$$

(2) k-Lipschitz if there exists $k > 0$ such that

$$\|Tx - Ty\| \le k\|x - y\| \quad \text{for } \forall x, y \in E;$$

in particular, if $0 < k < 1$, then T is called a contraction and if $k = 1$, then T reduces to a nonexpansive mapping;

(3) accretive if for all $x, y \in E$, there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\big\langle Tx - Ty, j_q(x - y)\big\rangle \ge 0;$$

(4) μ-inversely strongly accretive if for all $x, y \in E$, there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\big\langle Tx - Ty, j_q(x - y)\big\rangle \ge \mu\|Tx - Ty\|^q$$

for some $\mu > 0$;

(5) m-accretive if T is accretive and $R(I + \lambda T) = E$ for $\forall\lambda > 0$;
(6) strongly positive (see [2]) if E is a real smooth Banach space and there exists $\overline{\gamma} > 0$ such that

$$\langle Tx, Jx\rangle \geq \overline{\gamma}\|x\|^2 \quad \text{for } \forall x \in E;$$

in this case,

$$\|aI - bT\| = \sup_{\|x\|\leq 1} \left|\bigl\langle (aI - bT)x, J(x)\bigr\rangle\right|,$$

where I is the identity mapping and $a \in [0,1]$, $b \in [-1,1]$.

We denote by J_r^A (for $r > 0$) the resolvent of the accretive operator A; that is, $J_r^A := (I + rA)^{-1}$. It is well known that J_r^A is nonexpansive and $\operatorname{Fix}(J_r^A) = A^{-1}0$.

Many practical problems can be reduced to finding zeros of the sum of two accretive operators; that is, $0 \in (A + B)x$. Forward-backward splitting algorithms, which have recently received much attention from many mathematicians, were proposed by Lions and Mercier [3], by Passty [4], and, in a dual form for convex programming, by Han and Lou [5].

The classical forward-backward splitting algorithm is given in the following way:

$$x_{n+1} = (I + r_nB)^{-1}(I - r_nA)x_n, \quad n \geq 0. \tag{1}$$

Based on iterative algorithm (1), much work has been done for finding $x \in H$ such that $x \in (A + B)^{-1}0$, where A and B are μ-inversely strongly accretive operator and m-accretive operator defined in the Hilbert space H, respectively. However, most of the existing work is undertaken in the frame of Hilbert spaces, see [3–9], *etc.*

Recently, Qin *et al.*, presented the following iterative algorithm in the frame of q-uniformly smooth Banach spaces E in [10]:

$$x_0 \in E, \quad x_{n+1} = \alpha_n f(x_n) + \beta_n(I + r_nB)^{-1}\bigl[(I - r_nA)x_n + e_n\bigr] + \gamma_n f_n, \quad n \geq 0, \tag{2}$$

where $\{e_n\}$ is the error sequence, f is a contraction, A and B are μ-inversely strongly accretive operator and m-accretive operator, respectively. If $(A + B)^{-1}0 \neq \emptyset$, they proved that $\{x_n\}$ converges strongly to $x = \operatorname{Proj}_{(A+B)^{-1}0} f(x)$, where $\operatorname{Proj}_{(A+B)^{-1}0}$ is the unique sunny nonexpansive retraction of E onto $(A + B)^{-1}0$, under some conditions.

On the other hand, there is some excellent work done on approximating fixed points of nonexpansive mappings. For example, in 2006, Marino and Xu presented the following iterative algorithm in the frame of Hilbert spaces in [11], which sets up the relationship between fixed point of a nonexpansive mapping and the solution of one kind variational inequality

$$x_0 \in C, \quad x_{n+1} = \alpha_n\gamma f(x_n) + (I - \alpha_nA)Tx_n, \quad n \geq 0, \tag{3}$$

where f is a contraction, A is a strongly positive linear bounded operator, and T is nonexpansive. If $\operatorname{Fix}(T) \neq \emptyset$, they proved that $\{x_n\}$ converges strongly to $p \in \operatorname{Fix}(T)$, which solves the variational inequality $\langle (\gamma f - A)p, z - p\rangle \leq 0$ for $\forall z \in \operatorname{Fix}(T)$ under some conditions.

The implicit midpoint rule (IMR) is one of the powerful numerical methods for solving ordinary differential equations, which is extensively studied recently by Alghamdi *et al.* They presented the following implicit midpoint rule for approximating fixed point of nonexpansive mapping in a Hilbert space in [12]:

$$x_0 \in H, \quad x_{n+1} = (1-\alpha_n)x_n + \alpha_n T\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 0, \tag{4}$$

where T is nonexpansive from H to H. If $\mathrm{Fix}(T) \neq \emptyset$, then $\{x_n\}$ converges weakly to $p_0 \in \mathrm{Fix}(T)$, under some conditions.

Inspired by the work in [10–12], we shall present the following iterative algorithm with errors in a real q-uniformly smooth and uniformly convex Banach space E:

$$\begin{cases} x_0 \in E, \\ y_n = \alpha_n x_n + (1-\alpha_n)J_{r_n}^A[\frac{x_n+y_n}{2} - r_n B(\frac{x_n+y_n}{2})], \\ x_{n+1} = \gamma_n \eta f(x_n) + (I - \gamma_n T)y_n + e_n, \quad n \geq 0, \end{cases} \tag{A}$$

where $\{e_n\}$ is the error sequence, $A : E \to E$ is an m-accretive operator and $B : E \to E$ is a μ-inversely strongly accretive operator. $T : E \to E$ is a strongly positive linear bounded operator with coefficient $\overline{\gamma}$ and $f : E \to E$ is a contraction with coefficient $k \in (0,1)$. $J_{r_n}^A = (I + r_n A)^{-1}$. More details of iterative algorithm (A) will be presented in Section 2. Then $\{x_n\}$ is proved to converge strongly to $p_0 \in (A+B)^{-1}0$, which is also a solution of the following variational inequality: $\forall z \in (A+B)^{-1}0$, $\langle (T - \eta f)p_0, J(p_0 - z)\rangle \leq 0$. In Section 3, we shall present two examples, one of which is the generalized p-Laplacian problems with Neumann boundaries and the other is Laplacian problems with Signorini boundaries, to demonstrate the applications of the main results in Section 2.

Our main contributions are:

(i) the iterative algorithm is new in the sense that it combines the idea of iterative algorithms (1)-(4);

(ii) the discussion is undertaken in the frame of a real q-uniformly smooth and uniformly convex Banach space, which is more general than that in a Hilbert space;

(iii) the assumption that 'the normalized duality mapping J is weakly sequentially continuous' in most of the existing related work is weakened to 'J is weakly sequentially continuous at zero';

(iv) a new path convergence theorem for nonexpansive mapping is proved, which extends the corresponding result in [11] from a Hilbert space to a real smooth and uniformly convex Banach space;

(v) compared to the work done in [12], strong convergence theorems are obtained instead of weak convergence theorems;

(vi) compared to the work done in [10], the connection between zeros of the sum of m-accretive operators and μ-inversely strongly accretive operators and the solution of one kind variational inequalities is being set up;

(vii) the applications of the main results on the nonlinear problems with Neumann boundaries and Signorini boundaries are demonstrated, from which we can see the connections among variational inequalities, nonlinear boundary value problems and iterative algorithms.

Next, we list some results we need in the sequel.

Lemma 1 (see [1]) *Let E be a Banach space and $f : E \to E$ be a contraction. Then f has a unique fixed point $u \in E$.*

Lemma 2 (see [13]) *Let E be a real uniformly convex Banach space, C be a nonempty, closed, and convex subset of E and $T : C \to E$ be a nonexpansive mapping such that* $\mathrm{Fix}(T) \neq \emptyset$, *then $I - T$ is demiclosed at zero.*

Lemma 3 (see [14]) *In a real Banach space E, the following inequality holds*:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \forall x, y \in E,$$

where $j(x + y) \in J(x + y)$.

Lemma 4 (see [15]) *Let $\{a_n\}$ and $\{c_n\}$ be two sequences of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 - t_n)a_n + b_n + c_n, \quad \forall n \geq 0,$$

where $\{t_n\} \subset (0,1)$ and $\{b_n\}$ is a number sequence. Assume that $\sum_{n=0}^{\infty} t_n = +\infty$, $\limsup_{n\to\infty} \frac{b_n}{t_n} \leq 0$, *and $\sum_{n=0}^{\infty} c_n < +\infty$. Then* $\lim_{n\to\infty} a_n = 0$.

Lemma 5 (see [16]) *Let E be a Banach space and let A be an m-accretive operator. For* $\lambda > 0$, $\mu > 0$, *and $x \in E$, one has*

$$J_\lambda^A x = J_\mu^A \left(\frac{\mu}{\lambda} x + \left(1 - \frac{\mu}{\lambda}\right) J_\lambda^A x \right),$$

where $J_\lambda^A = (I + \lambda A)^{-1}$ *and* $J_\mu^A = (I + \mu A)^{-1}$.

Lemma 6 (see [17]) *Let E be a real Banach space and let C be a nonempty, closed and convex subset of E. Suppose $A : C \to E$ is a single-valued operator and $B : E \to 2^E$ is m-accretive. Then*

$$\mathrm{Fix}\big((I + rB)^{-1}(I - rA)\big) = (A + B)^{-1}0 \quad \textit{for } \forall r > 0.$$

Lemma 7 (see [18]) *Assume T is a strongly positive bounded operator with coefficient $\overline{\gamma} > 0$ on a real smooth Banach space E and $0 < \rho \leq \|T\|^{-1}$. Then $\|I - \rho T\| \leq 1 - \rho\overline{\gamma}$.*

2 Strong convergence theorems

Lemma 8 *Let E be a real smooth and uniformly convex Banach space. Let $f : E \to E$ be a fixed contractive mapping with coefficient $k \in (0,1)$, $T : E \to E$ be a strongly positive linear bounded operator with coefficient $\overline{\gamma}$ and $U : E \to E$ be a nonexpansive mapping. Suppose that the duality mapping $J : E \to E^*$ is weakly sequentially continuous at zero, $0 < \eta < \frac{\overline{\gamma}}{2k}$ and* $\mathrm{Fix}(U) \neq \emptyset$. *If for each $t \in (0,1)$, define $T_t : E \to E$ by*

$$T_t x := t\eta f(x) + (I - tT)Ux, \tag{5}$$

then T_t has a fixed point x_t for each $0 < t \leq \|T\|^{-1}$, which is convergent strongly to the fixed point of U, as $t \to 0$. That is, $\lim_{t\to 0} x_t = p_0 \in \mathrm{Fix}(U)$. Moreover, p_0 satisfies the following variational inequality: for $\forall z \in \mathrm{Fix}(U)$,

$$\big\langle (T - \eta f)p_0, J(p_0 - z)\big\rangle \leq 0. \tag{6}$$

Proof Step 1. T_t is a contraction for $0 < t < \|T\|^{-1}$.

In fact, noticing Lemma 7, we have

$$\begin{aligned}\|T_t x - T_t y\| &\leq t\eta \big\|f(x) - f(y)\big\| + \big\|(I - tT)(Ux - Uy)\big\| \\ &\leq kt\eta\|x - y\| + (1 - t\overline{\gamma})\|x - y\| \\ &= \big[1 - t(\overline{\gamma} - k\eta)\big]\|x - y\|,\end{aligned}$$

which implies that T_t is a contraction since $0 < \eta < \frac{\overline{\gamma}}{2k}$.

Then Lemma 1 implies that T_t has a unique fixed point, denoted by x_t, which uniquely solves the fixed point equation $x_t = t\eta f(x_t) + (I - tT)Ux_t$.

Step 2. $\{x_t\}$ is bounded for $t \in (0, \|T\|^{-1})$.

For $p \in \mathrm{Fix}(U)$, then

$$\begin{aligned}\|x_t - p\| &= \big\|(I - tT)(Ux_t - p) + t\big(\eta f(x_t) - Tp\big)\big\| \\ &\leq (1 - t\overline{\gamma})\|x_t - p\| + t\big\|\eta f(x_t) - Tp\big\| \\ &= (1 - t\overline{\gamma})\|x_t - p\| + t\big\|\eta\big(f(x_t) - f(p)\big) + \big(\eta f(p) - Tp\big)\big\| \\ &\leq (1 - t\overline{\gamma})\|x_t - p\| + t\big(k\eta\|x_t - p\| + \big\|\eta f(p) - Tp\big\|\big) \\ &= \big[1 - t(\overline{\gamma} - k\eta)\big]\|x_t - p\| + t\big\|\eta f(p) - Tp\big\|.\end{aligned}$$

This ensures that

$$\|x_t - p\| \leq \frac{\|\eta f(p) - Tp\|}{\overline{\gamma} - k\eta}.$$

Thus $\{x_t\}$ is bounded, which implies that both $\{f(x_t)\}$ and $\{TUx_t\}$ are bounded.

Step 3. $x_t - Ux_t \to 0$, as $t \to 0$.

Noticing the result of Step 2, we have $\|x_t - Ux_t\| = t\|\eta f(x_t) - TUx_t\| \to 0$, as $t \to 0$.

Step 4. $\langle (T - \eta f)x - (T - \eta f)y, J(x - y)\rangle \geq (\overline{\gamma} - k\eta)\|x - y\|^2$ for $\forall x, y \in E$.

In fact,

$$\begin{aligned}&\big\langle (T - \eta f)x - (T - \eta f)y, J(x - y)\big\rangle \\ &\quad = \big\langle Tx - Ty, J(x - y)\big\rangle - \eta\big\langle f(x) - f(y), J(x - y)\big\rangle \\ &\quad \geq \overline{\gamma}\|x - y\|^2 - k\eta\|x - y\|^2 = (\overline{\gamma} - k\eta)\|x - y\|^2.\end{aligned}$$

Step 5. If the variational inequality (6) has solutions, then the solution must be unique.

Suppose both $u_0 \in \mathrm{Fix}(U)$ and $v_0 \in \mathrm{Fix}(U)$ are the solutions of the variational inequality (6). Then we have

$$\big\langle (T - \eta f)v_0, J(v_0 - u_0)\big\rangle \leq 0 \tag{7}$$

and

$$\langle (T-\eta f)u_0, J(u_0-v_0)\rangle \le 0. \tag{8}$$

Adding up (7) and (8), we obtain that

$$\langle (T-\eta f)u_0-(T-\eta f)v_0, J(u_0-v_0)\rangle \le 0.$$

In view of the result of Step 4, we have $u_0 = v_0$.

Step 6. $x_t \to p_0 \in \mathrm{Fix}(U)$, as $t \to 0$, which satisfies the variational inequality (6).

For $\forall z \in \mathrm{Fix}(U)$, $x_t - z = t(\eta f(x_t) - Tz) + (I - tT)(Ux_t - z)$. Thus Lemma 3 implies that

$$\begin{aligned}\|x_t-z\|^2 &\le \|I-tT\|^2\|Ux_t-Uz\|^2 + 2t\langle \eta f(x_t)-Tz, J(x_t-z)\rangle \\ &\le (1-t\overline{\gamma})\|x_t-z\|^2 + 2t\langle \eta f(x_t)-Tz, J(x_t-z)\rangle.\end{aligned}$$

Then

$$\begin{aligned}\|x_t-z\|^2 &\le \frac{2}{\overline{\gamma}}\langle \eta f(x_t)-Tz, J(x_t-z)\rangle \\ &= \frac{2}{\overline{\gamma}}\bigl[\eta\langle f(x_t)-f(z), J(x_t-z)\rangle + \langle \eta f(z)-T(z), J(x_t-z)\rangle\bigr] \\ &\le \frac{2}{\overline{\gamma}}\bigl[\eta k\|x_t-z\|^2 + \langle \eta f(z)-Tz, J(x_t-z)\rangle\bigr].\end{aligned}$$

Therefore, for $\forall z \in \mathrm{Fix}(U)$, we have

$$\|x_t-z\|^2 \le \frac{2}{\overline{\gamma}-2k\eta}\langle \eta f(z)-Tz, J(x_t-z)\rangle. \tag{9}$$

Since $\{x_t\}$ is bounded as $t \to 0^+$, then we can choose $\{t_n\} \subset (0,1)$ such that $t_n \to 0^+$ and $x_{t_n} \rightharpoonup p_0$. From Lemma 2 and the result of Step 3, we see that $p_0 = Up_0$. Thus $p_0 \in \mathrm{Fix}(U)$. Substituting z by p_0 in (9), then we can deduce that $x_{t_n} \to p_0$ since J is weakly sequentially continuous at zero. Next, we shall prove that p_0 solves the variational inequality (6).

Since $x_t = t\eta f(x_t) + (I - tT)Ux_t$, then

$$(T-\eta f)x_t = -\frac{1}{t}(I-tT)(I-U)x_t.$$

For $\forall z \in \mathrm{Fix}(U)$, since U is nonexpansive, then

$$\begin{aligned}&\langle (T-\eta f)x_t, J(x_t-z)\rangle \\ &\quad= -\frac{1}{t}\langle (I-tT)(I-U)x_t, J(x_t-z)\rangle \\ &\quad= -\frac{1}{t}\langle (I-U)x_t-(I-U)z, J(x_t-z)\rangle + \langle T(I-U)x_t, J(x_t-z)\rangle \\ &\quad= -\frac{1}{t}\bigl[\|x_t-z\|^2 - \langle Ux_t-Uz, J(x_t-z)\rangle\bigr] + \langle T(I-U)x_t, J(x_t-z)\rangle \\ &\quad\le \langle T(I-U)x_t, J(x_t-z)\rangle.\end{aligned} \tag{10}$$

Since $x_{t_n} \to p_0$, then $(I-U)x_{t_n} \to (I-U)p_0 = 0$, as $n \to \infty$. Since $\{x_{t_n}\}$ is bounded, $(T-\eta f)x_{t_n} \to (T-\eta f)p_0$ and J is uniformly continuous on each bounded subset of E, then taking limits on both sides of (10) we have $\langle (T-\eta f)p_0, J(p_0-z)\rangle \leq 0$ for $z \in \mathrm{Fix}(U)$. Thus p_0 satisfies (6).

In a summary, we infer that each cluster point of $\{x_t\}$ is equal to p_0, which is the unique solution of the variational inequality (6).

This completes the proof. □

Lemma 9 (see [10]) *Let E be a real q-uniformly smooth Banach space with constant K_q. Let $A : E \to E$ be a μ-inversely strongly accretive operator. Then for $\forall r \leq (\frac{q\mu}{K_q})^{\frac{1}{q-1}}$, $(I - rA)$ is nonexpansive.*

Theorem 10 *Let E be a real q-uniformly smooth Banach space with constant K_q and also be a uniformly convex Banach space. Let $f : E \to E$ be a fixed contractive mapping with coefficient $k \in (0,1)$, $T : E \to E$ be a strongly positive linear bounded operator with coefficient $\overline{\gamma}$. Suppose that the duality mapping $J : E \to E^*$ is weakly sequentially continuous at zero, and $0 < \eta < \frac{\overline{\gamma}}{2k}$. Let $A : E \to 2^E$ be an m-accretive operator and $B : E \to E$ be a μ-inversely strongly accretive operator. Let $\{x_n\}$ be generated by the iterative algorithm* (A). *Suppose $\{e_n\} \subset E$, $\{\alpha_n\}$ and $\{\gamma_n\}$ are two sequences in $(0,1)$ and $\{r_n\} \subset (0,+\infty)$ satisfying the following conditions*:

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$, $\gamma_n \to 0$, $\alpha_n \to 0$, *as* $n \to \infty$;

(ii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < +\infty$, $\sum_{n=0}^{\infty} |\gamma_{n+1} - \gamma_n| < +\infty$;

(iii) $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < +\infty$, $0 < \varepsilon \leq r_n \leq (\frac{q\mu}{K_q})^{\frac{1}{q-1}}$ *for* $n \geq 0$;

(iv) $\sum_{n=0}^{\infty} \|e_n\| < +\infty$.

If $(A+B)^{-1}0 \neq \emptyset$, then $\{x_n\}$ converges strongly to a point $p_0 \in (A+B)^{-1}0$, which is the unique solution of the following variational inequality: *for $\forall z \in (A+B)^{-1}0$,*

$$\big\langle (T-\eta f)p_0, J(p_0 - z)\big\rangle \leq 0. \tag{11}$$

Proof Let $u_n = (I - r_n B)(\frac{x_n + y_n}{2})$ for $n \geq 0$.

We shall split the proof into six steps.

Step 1. $\{y_n\}$ is well defined.

Define $W_t : E \to E$ by $W_t x := tu + (1-t)W(\frac{u+x}{2})$, where $W : E \to E$ is nonexpansive for $x, u \in E$, then W_t is a contraction for $0 \leq t < 1$.

In fact,

$$\begin{aligned} \|W_t x - W_t y\| &\leq (1-t)\left\| \frac{u+x}{2} - \frac{u+y}{2} \right\| \\ &\leq \frac{1-t}{2}\|x-y\|, \end{aligned}$$

which implies that W_t is a contraction. Thus there exists x_t such that $W_t x_t = x_t$. That is, $x_t = tu + (1-t)W(\frac{u+x}{2})$ for $0 \leq t < 1$.

From Lemma 9 we know that $J_{r_n}^A (I - r_n B)$ is nonexpansive, therefore $\{y_n\}$ is well defined.

Step 2. $\{x_n\}$, $\{u_n\}$, $\{y_n\}$ are all bounded.

In view of Lemmas 6 and 9, we have, for $\forall p \in (A+B)^{-1}0$,

$$\begin{aligned} \|y_n - p\| &\le \alpha_n \|x_n - p\| + (1-\alpha_n)\left\| \frac{x_n + y_n}{2} - p \right\| \\ &\le \alpha_n \|x_n - p\| + \frac{1-\alpha_n}{2}\|x_n - p\| + \frac{1-\alpha_n}{2}\|y_n - p\|, \end{aligned} \tag{12}$$

which implies that $\|y_n - p\| \le \|x_n - p\|$.

Using Lemma 7 and (12), we have, for $p \in (A+B)^{-1}0$ and $n \ge 0$,

$$\begin{aligned} \|x_{n+1} - p\| &\le \gamma_n \big\| \eta f(x_n) - Tp \big\| + \big\| (I - \gamma_n T)(y_n - p) \big\| + \|e_n\| \\ &\le \gamma_n \eta k \|x_n - p\| + \gamma_n \big\| \eta f(p) - Tp \big\| + (1 - \gamma_n \overline{\gamma})\|y_n - p\| + \|e_n\| \\ &\le \big[1 - \gamma_n(\overline{\gamma} - k\eta)\big]\|x_n - p\| + \gamma_n(\overline{\gamma} - k\eta)\frac{\|\eta f(p) - Tp\|}{\overline{\gamma} - k\eta} + \|e_n\| \\ &\le \max\left\{ \|x_n - p\|, \frac{\|\eta f(p) - Tp\|}{\overline{\gamma} - k\eta} \right\} + \|e_n\|. \end{aligned} \tag{13}$$

By using the inductive method, we can easily get the following result from (13):

$$\|x_{n+1} - p\| \le \max\left\{ \|x_0 - p\|, \frac{\|\eta f(p) - Tp\|}{\overline{\gamma} - k\eta} \right\} + \sum_{k=0}^{n} \|e_k\|,$$

which implies that $\{x_n\}$ is bounded. Then (12) implies that $\{y_n\}$ is bounded.

Since $J_{r_n}^A$ and $(I - r_n B)$ are nonexpansive, f is a contraction and T is bounded, then $\{u_n\}$, $\{f(x_n)\}$, $\{J_{r_n}^A u_n\}$, $\{B(\frac{x_n+y_n}{2})\}$ and $\{Ty_n\}$ are all bounded.

Set $M_1 = \sup\{\|u_n\|, \|J_{r_n}^A u_n\|, \|B(\frac{x_n+y_n}{2})\|, \|x_n\|, \eta\|f(x_n)\|, \|Ty_n\| : n \ge 0\}$.

Step 3. $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

First, we shall discuss $\|J_{r_n}^A u_n - J_{r_{n-1}}^A u_{n-1}\|$ for $n \ge 1$.

If $r_{n-1} \le r_n$, then by using Lemma 5, we have

$$\begin{aligned} &\big\|J_{r_n}^A u_n - J_{r_{n-1}}^A u_{n-1}\big\| \\ &\quad = \left\| J_{r_{n-1}}^A\left(\frac{r_{n-1}}{r_n}u_n + \left(1 - \frac{r_{n-1}}{r_n}\right)J_{r_n}^A u_n\right) - J_{r_{n-1}}^A u_{n-1} \right\| \\ &\quad \le \left\| \frac{r_{n-1}}{r_n}u_n + \left(1 - \frac{r_{n-1}}{r_n}\right)J_{r_n}^A u_n - u_{n-1} \right\| \\ &\quad \le \frac{r_{n-1}}{r_n}\|u_n - u_{n-1}\| + \left(1 - \frac{r_{n-1}}{r_n}\right)\big\|J_{r_n}^A u_n - u_{n-1}\big\| \\ &\quad \le \|u_n - u_{n-1}\| + \frac{r_n - r_{n-1}}{\varepsilon}\big\|J_{r_n}^A u_n - u_{n-1}\big\|. \end{aligned} \tag{14}$$

If $r_n \le r_{n-1}$, then imitating the proof of (14), we have

$$\big\|J_{r_n}^A u_n - J_{r_{n-1}}^A u_{n-1}\big\| \le \|u_n - u_{n-1}\| + \frac{r_{n-1} - r_n}{\varepsilon}\big\|J_{r_n}^A u_n - u_{n-1}\big\|. \tag{15}$$

Combining (14) and (15), we have, for $n \ge 1$,

$$\big\|J_{r_n}^A u_n - J_{r_{n-1}}^A u_{n-1}\big\| \le \|u_n - u_{n-1}\| + \frac{|r_{n-1} - r_n|}{\varepsilon}\big\|J_{r_n}^A u_n - u_{n-1}\big\|. \tag{16}$$

Note that

$$\begin{aligned}\|u_n - u_{n-1}\| &\le \left\|(I - r_n B)\left(\frac{x_n + y_n}{2} - \frac{x_{n-1} + y_{n-1}}{2}\right)\right\| + |r_{n-1} - r_n|\left\|B\left(\frac{x_{n-1} + y_{n-1}}{2}\right)\right\| \\ &\le \frac{\|x_n - x_{n-1}\|}{2} + \frac{\|y_n - y_{n-1}\|}{2} + |r_{n-1} - r_n|\left\|B\left(\frac{x_{n-1} + y_{n-1}}{2}\right)\right\|. \end{aligned} \tag{17}$$

Using (16) and (17), for $n \ge 1$, we have

$$\begin{aligned}\|y_n - y_{n-1}\| &\le \alpha_n\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|x_{n-1}\| \\ &\quad + (1 - \alpha_n)\left\|J_{r_n}^A u_n - J_{r_{n-1}}^A u_{n-1}\right\| + |\alpha_n - \alpha_{n-1}|\left\|J_{r_{n-1}}^A u_{n-1}\right\| \\ &\le \alpha_n\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|x_{n-1}\| + (1 - \alpha_n)\|u_n - u_{n-1}\| \\ &\quad + \frac{|r_n - r_{n-1}|}{\varepsilon}\left\|J_{r_n}^A u_n - u_{n-1}\right\| + |\alpha_n - \alpha_{n-1}|\left\|J_{r_{n-1}}^A u_{n-1}\right\| \\ &\le \frac{1 + \alpha_n}{2}\|x_n - x_{n-1}\| + \frac{1 - \alpha_n}{2}\|y_n - y_{n-1}\| \\ &\quad + 2M_1|\alpha_n - \alpha_{n-1}| + \left(1 + \frac{2}{\varepsilon}\right)M_1|r_n - r_{n-1}|. \end{aligned} \tag{18}$$

From (18), we know that for $n \ge 1$,

$$\|y_n - y_{n-1}\| \le \|x_n - x_{n-1}\| + 4M_1|\alpha_n - \alpha_{n-1}| + 2M_1\left(1 + \frac{2}{\varepsilon}\right)|r_n - r_{n-1}|. \tag{19}$$

Using (19), we have for $n \ge 1$,

$$\begin{aligned}\|x_{n+1} - x_n\| &\le \gamma_n\eta k\|x_n - x_{n-1}\| + \eta|\gamma_n - \gamma_{n-1}|\left\|f(x_{n-1})\right\| + (1 - \gamma_n\overline{\gamma})\|y_n - y_{n-1}\| \\ &\quad + |\gamma_n - \gamma_{n-1}|\|Ty_{n-1}\| + \|e_{n+1} - e_n\| \\ &\le \left[1 - \gamma_n(\overline{\gamma} - \eta k)\right]\|x_n - x_{n-1}\| \\ &\quad + 2M_1\left[|\gamma_n - \gamma_{n-1}| + 2|\alpha_n - \alpha_{n-1}| + \left(1 + \frac{2}{\varepsilon}\right)|r_n - r_{n-1}|\right] \\ &\quad + \|e_{n+1} - e_n\|. \end{aligned} \tag{20}$$

From the assumptions on $\{e_n\}$, $\{\alpha_n\}$, $\{\gamma_n\}$ and $\{r_n\}$, in view of (20) and Lemma 4, we have $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

Step 4. Set $W_n = J_{r_n}^A(I - r_n B)$, then $W_n y_n - y_n \to 0$, as $n \to \infty$.

It is obvious that W_n is nonexpansive and $(A + B)^{-1}0 = \mathrm{Fix}(W_n)$.

Since both $\{x_n\}$ and $\{W_n(\frac{x_n+y_n}{2})\}$ are bounded and $\alpha_n \to 0$, as $n \to +\infty$, then

$$y_n - W_n\left(\frac{x_n + y_n}{2}\right) = \alpha_n\left[x_n - W_n\left(\frac{x_n + y_n}{2}\right)\right] \to 0, \quad \text{as } n \to +\infty.$$

Since both $\{f(x_n)\}$ and $\{Ty_n\}$ are bounded and $\gamma_n \to 0$, as $n \to +\infty$, then

$$x_{n+1} - y_n = \gamma_n\left[\eta f(x_n) - Ty_n\right] + e_n \to 0, \quad \text{as } n \to +\infty.$$

Therefore, in view of the result of Step 3,

$$\begin{aligned}\|W_ny_n - y_n\| &\le \left\|W_ny_n - W_n\left(\frac{x_n + y_n}{2}\right)\right\| + \left\|W_n\left(\frac{x_n + y_n}{2}\right) - y_n\right\| \\ &\le \frac{\|x_n - y_n\|}{2} + \alpha_n\left\|x_n - W_n\left(\frac{x_n + y_n}{2}\right)\right\| \to 0, \quad \text{as } n \to \infty.\end{aligned}$$

Step 5. $\limsup_{n\to+\infty}\langle \eta f(p_0) - Tp_0, J(x_{n+1} - p_0)\rangle \le 0$, where $p_0 \in (A + B)^{-1}0$, which is the unique solution of the variational inequality (11).

Since W_n is nonexpansive, then Lemma 8 implies that there exists z_t such that $z_t = t\eta f(z_t) + (I - tT)W_nz_t$ for $t \in (0,1)$. Moreover, $z_t \to p_0 \in \mathrm{Fix}(W_n) = (A + B)^{-1}0$, as $t \to 0$; and p_0 is the unique solution of the variational inequality (11).

Since $\|z_t\| \le \|z_t - p_0\| + \|p_0\|$, then $\{z_t\}$ is bounded, as $t \to 0$. Using Lemma 3 repeatedly, we have

$$\begin{aligned}&\|z_t - y_n\|^2 \\ &\quad = \|z_t - W_ny_n + W_ny_n - y_n\|^2 \\ &\quad \le \|z_t - W_ny_n\|^2 + 2\big\langle W_ny_n - y_n, J(z_t - y_n)\big\rangle \\ &\quad = \big\|t\eta f(z_t) + (I - tT)W_nz_t - W_ny_n\big\|^2 + 2\big\langle W_ny_n - y_n, J(z_t - y_n)\big\rangle \\ &\quad \le \|W_nz_t - W_ny_n\|^2 + 2t\big\langle \eta f(z_t) - TW_nz_t, J(z_t - W_ny_n)\big\rangle + 2\big\langle W_ny_n - y_n, J(z_t - y_n)\big\rangle \\ &\quad \le \|z_t - y_n\|^2 + 2t\big\langle \eta f(z_t) - TW_nz_t, J(z_t - W_ny_n)\big\rangle + 2\|W_ny_n - y_n\|\|z_t - y_n\|,\end{aligned}$$

which implies that

$$t\big\langle TW_nz_t - \eta f(z_t), J(z_t - W_ny_n)\big\rangle \le \|W_ny_n - y_n\|\|z_t - y_n\|.$$

So, $\lim_{t\to0}\limsup_{n\to+\infty}\langle TW_nz_t - \eta f(z_t), J(z_t - W_ny_n)\rangle \le 0$ in view of Step 4.

Since $z_t \to p_0$, then $W_nz_t \to W_np_0 = p_0$, as $t \to 0$. Noticing the following fact that

$$\begin{aligned}&\big\langle Tp_0 - \eta f(p_0), J(p_0 - W_ny_n)\big\rangle \\ &\quad = \big\langle Tp_0 - \eta f(p_0), J(p_0 - W_ny_n) - J(z_t - W_ny_n)\big\rangle + \big\langle Tp_0 - \eta f(p_0), J(z_t - W_ny_n)\big\rangle \\ &\quad = \big\langle Tp_0 - \eta f(p_0), J(p_0 - W_ny_n) - J(z_t - W_ny_n)\big\rangle \\ &\qquad + \big\langle Tp_0 - \eta f(p_0) - TW_nz_t + \eta f(z_t), J(z_t - W_ny_n)\big\rangle \\ &\qquad + \big\langle TW_nz_t - \eta f(z_t), J(z_t - W_ny_n)\big\rangle,\end{aligned}$$

we have $\limsup_{n\to+\infty}\langle Tp_0 - \eta f(p_0), J(p_0 - W_ny_n)\rangle \le 0$.

Since $\langle Tp_0 - \eta f(p_0), J(p_0 - x_{n+1})\rangle = \langle Tp_0 - \eta f(p_0), J(p_0 - x_{n+1}) - J(p_0 - W_ny_n)\rangle + \langle Tp_0 - \eta f(p_0), J(p_0 - W_ny_n)\rangle$ and $x_{n+1} - W_ny_n \to 0$, then $\limsup_{n\to\infty}\langle \eta f(p_0) - Tp_0, J(x_{n+1} - p_0)\rangle \le 0$.

Step 6. $x_n \to p_0$, as $n \to +\infty$, where $p_0 \in (A + B)^{-1}0$ is the same as that in Step 5.

Since

$$\|y_n - p_0\| \le \alpha_n\|x_n - p_0\| + (1 - \alpha_n)\left\|\frac{x_n + y_n}{2} - p_0\right\| \le \frac{1 + \alpha_n}{2}\|x_n - p_0\| + \frac{1 - \alpha_n}{2}\|y_n - p_0\|,$$

then $\|y_n - p_0\| \le \|x_n - p_0\|$.

Using Lemma 3 and letting $M_2 = \max\{M_1, \|p_0\|\}$, we have for $n \geq 0$,

$$\begin{aligned}
&\|x_{n+1} - p_0\|^2 \\
&\quad = \big\| \gamma_n\big(\eta f(x_n) - Tp_0\big) + (I - \gamma_n T)(y_n - p_0) + e_n \big\|^2 \\
&\quad \leq (1 - \gamma_n\overline{\gamma})^2 \|y_n - p_0\|^2 + 2\gamma_n\big\langle \eta f(x_n) - Tp_0, J(x_{n+1} - p_0)\big\rangle \\
&\qquad + 2\big\langle e_n, J(x_{n+1} - p_0)\big\rangle \\
&\quad \leq (1 - \gamma_n\overline{\gamma})\|x_n - p_0\|^2 + 2\gamma_n\eta\big\langle f(x_n) - f(p_0), J(x_{n+1} - p_0) - J(x_n - p_0)\big\rangle \\
&\qquad + 2\gamma_n\eta\big\langle f(x_n) - f(p_0), J(x_n - p_0)\big\rangle \\
&\qquad + 2\gamma_n\big\langle \eta f(p_0) - Tp_0, J(x_{n+1} - p_0)\big\rangle + 2\|e_n\|\|x_{n+1} - p_0\| \\
&\quad \leq \big[1 - \gamma_n(\overline{\gamma} - 2\eta k)\big]\|x_n - p_0\|^2 \\
&\qquad + \gamma_n\big[2\big\langle \eta f(p_0) - Tp_0, J(x_{n+1} - p_0)\big\rangle + 2\eta\|x_n - p_0\|\|x_{n+1} - x_n\|\big] + 4M_2\|e_n\|.
\end{aligned} \tag{21}$$

Let $\delta_n^{(1)} = \gamma_n(\overline{\gamma} - 2\eta k)$, $\delta_n^{(2)} = \gamma_n[2\langle \eta f(p_0) - Tp_0, J(x_{n+1} - p_0)\rangle + 2\eta\|x_n - p_0\|\|x_{n+1} - x_n\|]$ and $\delta_n^{(3)} = 4M_2\|e_n\|$. Then (21) can be simplified as $\|x_{n+1} - p_0\|^2 \leq (1 - \delta_n^{(1)})\|x_n - p_0\|^2 + \delta_n^{(2)} + \delta_n^{(3)}$.

Using the assumptions, the results of Steps 2, 3 and 5 and by using Lemma 4, we know that $x_n \to p_0$, as $n \to +\infty$.

This completes the proof. □

Theorem 11 *If $e_n \equiv 0$, then iterative algorithm* (A) *becomes the following accurate iterative algorithm*:

$$\begin{cases} x_0 \in E, \\ y_n = \alpha_n x_n + \beta_n J_{r_n}^A[\frac{x_n + y_n}{2} - r_n B(\frac{x_n + y_n}{2})], \quad n \geq 0, \\ x_{n+1} = \gamma_n \eta f(x_n) + (I - \gamma_n T)y_n, \quad n \geq 0. \end{cases} \tag{B}$$

If $(A+B)^{-1}0 \neq \emptyset$, then under the assumptions except on $\{e_n\}$ of Theorem 10, *$\{x_n\}$ generated by the iterative algorithm* (B) *converges strongly to $p_0 \in (A+B)^{-1}0$, which is the unique solution of the variational inequality* (11).

3 Applications

In this section, we shall demonstrate the applications of Theorem 10 to the nonlinear problems with Neumann boundaries and Signorini boundaries, respectively.

Example 1 Now, we shall present an example of nonlinear Neumann boundary value problem involving the generalized p-Laplacian, which comes from [19]:

$$\begin{cases} -\operatorname{div}[(C(x) + |\nabla u|^2)^{\frac{p-2}{2}}\nabla u] + \varepsilon|u|^{q-2}u + g(x, u(x), \nabla u(x)) = h(x), \quad \text{a.e. } x \in \Omega, \\ -\langle \vartheta, (C(x) + |\nabla u|^2)^{\frac{p-2}{2}}\nabla u\rangle \in \beta_x(u(x)), \quad \text{a.e. } x \in \Gamma. \end{cases} \tag{C}$$

In (C), Ω is a bounded conical domain of a Euclidean space R^N with its boundary $\Gamma \in C^1$ (see [20]). $h(x) \in L^2(\Omega)$ is a given function. ε is a nonnegative constant and ϑ denotes the exterior normal derivative of Γ, $0 \leq C(x) \in L^p(\Omega)$.

Let $\varphi : \Gamma \times R \to R$ be a given function such that, for each $x \in \Gamma$, $\varphi_x = \varphi(x,\cdot) : R \to R$ is a proper, convex and lower-semi-continuous function with $\varphi_x(0) = 0$. Let β_x be the subdifferential of φ_x, *i.e.*, $\beta_x \equiv \partial\varphi_x$. Suppose that $0 \in \beta_x(0)$ and for each $t \in R$, the function $x \in \Gamma \to (I + \lambda\beta_x)^{-1}(t) \in R$ is measurable for $\lambda > 0$.

Suppose that $g : \Omega \times R^{N+1} \to R$ is a given function satisfying the following conditions:

(a) Carathéodory's conditions:

$$x \to g(x,r) \quad \text{is measurable on } \Omega \text{ for all } r \in R^{N+1};$$

$$r \to g(x,r) \quad \text{is continuous on } R^{N+1} \text{ for almost all } x \in \Omega.$$

(b) Nonexpansive with respect to r_1, *i.e.*,

$$\bigl|g(x,r_1,\ldots,r_{N+1}) - g(x,t_1,\ldots,t_{N+1})\bigr| \le |r_1 - t_1|,$$

where $(r_1,r_2,\ldots,r_{N+1}),(t_1,\ldots,t_{N+1}) \in R^{N+1}$.

(c) Monotone with respect to r_1, *i.e.*,

$$\bigl(g(x,r_1,\ldots,r_{N+1}) - g(x,t_1,\ldots,t_{N+1})\bigr)(r_1 - t_1) \ge 0$$

for all $x \in \Omega$ and $(r_1,\ldots,r_{N+1}),(t_1,\ldots,t_{N+1}) \in R^{N+1}$.

Assume $\frac{2N}{N+1} < p < +\infty$, $\frac{2N}{N+1} < q < +\infty$, where $N \ge 1$. Let $\frac{1}{p} + \frac{1}{p'} = 1$. We use $\|\cdot\|_2$ to denote the norm of $L^2(\Omega)$ and $\langle\cdot,\cdot\rangle$ to denote the inner product in R^N, respectively.

Lemma 12 ([19]) *Define the mapping $B_{p,q} : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ by*

$$\langle v, B_{p,q}u\rangle = \int_\Omega \bigl\langle \bigl(C(x) + |\nabla u|^2\bigr)^{\frac{p-2}{2}} \nabla u, \nabla v\bigr\rangle\,dx + \varepsilon \int_\Omega \bigl|u(x)\bigr|^{q-2} u(x)v(x)\,dx$$

for any $u, v \in W^{1,p}(\Omega)$. Then $B_{p,q}$ is everywhere defined, strictly monotone, hemi-continuous and coercive.

Lemma 13 ([19]) *The mapping $\Phi_p : W^{1,p}(\Omega) \to R$ defined by $\Phi_p(u) = \int_\Gamma \varphi_x(u|_\Gamma(x))\,d\Gamma(x)$ for any $u \in W^{1,p}(\Omega)$ is proper convex and lower-semi-continuous on $W^{1,p}(\Omega)$.*

Lemma 14 ([19]) *The mapping $A : L^2(\Omega) \to 2^{L^2(\Omega)}$ defined by*

$$Au = \bigl\{f \in L^2(\Omega) : f \in B_{p,q}u + \partial\Phi_p(u)\bigr\} \quad \text{for } \forall u \in D(A),$$

where

$$D(A) = \bigl\{u \in L^2(\Omega) : \text{there exists } f \in L^2(\Omega) \text{ such that } f \in B_{p,q}u + \partial\Phi_p(u)\bigr\}$$

is m-accretive.

Lemma 15 *Define $S : D(A) \subset L^2(\Omega) \to L^2(\Omega)$ by*

$$Su(x) = g\bigl(x, u(x), \nabla u(x)\bigr) - h(x)$$

for $u(x) \in D(S)$, where $h(x)$ is the same as that in (C). *Then S is inversely strongly accretive.*

Proof From assumptions (c) and (b) on g, we know that if $r_1 \le t_1$, then

$$\begin{aligned} g(x, t_1, \dots, t_{N+1}) - g(x, r_1, \dots, r_{N+1}) &= \bigl|g(x, r_1, \dots, r_{N+1}) - g(x, t_1, \dots, t_{N+1})\bigr| \\ &\le |r_1 - t_1| = t_1 - r_1; \end{aligned}$$

if $r_1 \ge t_1$, then

$$\begin{aligned} g(x, r_1, \dots, r_{N+1}) - g(x, t_1, \dots, t_{N+1}) &= \bigl|g(x, r_1, \dots, r_{N+1}) - g(x, t_1, \dots, t_{N+1})\bigr| \\ &\le |r_1 - t_1| = r_1 - t_1. \end{aligned}$$

Thus

$$\bigl[g(x, r_1, \dots, r_{N+1}) - g(x, t_1, \dots, t_{N+1})\bigr]^2 \le \bigl[g(x, r_1, \dots, r_{N+1}) - g(x, t_1, \dots, t_{N+1})\bigr](r_1 - t_1),$$

which implies that

$$\begin{aligned} \langle Su - Sv, u - v\rangle &= \int_\Omega \bigl[g\bigl(x, u(x), \nabla u(x)\bigr) - g\bigl(x, v(x), \nabla v(x)\bigr)\bigr](u - v)\,dx \\ &\ge \int_\Omega \bigl[g\bigl(x, u(x), \nabla u(x)\bigr) - g\bigl(x, v(x), \nabla v(x)\bigr)\bigr]^2\,dx = \|Su - Sv\|_2^2. \end{aligned}$$

Then S is inversely strongly monotone.

This completes the proof. □

Lemma 16 ([19]) *For $h(x) \in L^2(\Omega)$, nonlinear boundary value problem* (C) *has a unique solution in $L^2(\Omega)$.*

Lemma 17 ([20]) *For $\forall \varphi \in C_0^\infty(\Omega)$, $\langle \varphi, \partial\Phi_p(u)\rangle = 0$, $u \in W^{1,p}(\Omega)$.*

Lemma 18 *$u(x) \in L^2(\Omega)$ is the solution of* (C) *if and only if $u(x) \in (A + S)^{-1}0$.*

Proof If $u(x)$ is the solution of (C), then

$$-\operatorname{div}\bigl[\bigl(C(x) + |\nabla u|^2\bigr)^{\frac{p-2}{2}} \nabla u\bigr] + \varepsilon|u|^{q-2}u + g\bigl(x, u(x), \nabla u(x)\bigr) = h(x), \quad \text{a.e. } x \in \Omega.$$

Thus, for $\forall \varphi \in C_0^\infty(\Omega)$, by using the property of generalized function and Lemma 17, we have

$$\begin{aligned} 0 &= \int_\Omega -\operatorname{div}\bigl[\bigl(C(x) + |\nabla u|^2\bigr)^{\frac{p-2}{2}} \nabla u\bigr]\varphi\,dx + \varepsilon\int_\Omega |u|^{q-2}u\varphi\,dx \\ &\quad + \int_\Omega g\bigl(x, u(x), \nabla u(x)\bigr)\varphi\,dx - \int_\Omega h\varphi\,dx \\ &= \int_\Omega \bigl\langle\bigl(C(x) + |\nabla u|^2\bigr)^{\frac{p-2}{2}} \nabla u, \nabla\varphi\bigr\rangle\,dx + \int_\Omega \varepsilon|u|^{q-2}u\varphi\,dx \end{aligned}$$

$$+\int_\Omega g\big(x,u(x),\nabla u(x)\big)\varphi\,dx-\int_\Omega h\varphi\,dx$$
$$=\langle\varphi,B_{p,q}u+Su\rangle=\big\langle\varphi,B_{p,q}u+\partial\Phi_p(u)+Su\big\rangle=\langle\varphi,Au+Su\rangle.$$

Then $u(x)\in(A+S)^{-1}0$.

On the other hand, if $u(x)\in(A+S)^{-1}0$, then for $\forall\varphi\in C_0^\infty(\Omega)$,

$$\begin{aligned}0&=\langle\varphi,Au+Su\rangle=\big\langle\varphi,B_{p,q}u+\partial\Phi_p(u)+Su\big\rangle\\&=\int_\Omega\big\langle\big(C(x)+|\nabla u|^2\big)^{\frac{p-2}{2}}\nabla u,\nabla\varphi\big\rangle\,dx+\varepsilon\int_\Omega|u|^{q-2}u\varphi\,dx\\&\quad+\int_\Omega g\big(x,u(x),\nabla u(x)\big)\varphi\,dx-\int_\Omega h\varphi\,dx\\&=\int_\Omega\big\{-\operatorname{div}\big[\big(C(x)+|\nabla u|^2\big)^{\frac{p-2}{2}}\nabla u\big]+\varepsilon|u|^{q-2}u+g\big(x,u(x),\nabla u(x)\big)-h\big\}\varphi\,dx.\end{aligned}$$

Therefore, $-\operatorname{div}[(C(x)+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u]+\varepsilon|u|^{q-2}u+g(x,u(x),\nabla u(x))=h(x)$, a.e. $x\in\Omega$. By using Green's formula, we know that for any $v\in W^{1,p}(\Omega)$,

$$\begin{aligned}&\int_\Gamma\big\langle\vartheta,\big(C(x)+|\nabla u|^2\big)^{\frac{p-2}{2}}\nabla u\big\rangle v|_\Gamma\,d\Gamma(x)\\&\quad=\int_\Omega\operatorname{div}\big[\big(C(x)+|\nabla u|^2\big)^{\frac{p-2}{2}}\nabla u\big]v\,dx+\int_\Omega\big\langle\big(C(x)+|\nabla u|^2\big)^{\frac{p-2}{2}}\nabla u,\nabla v\big\rangle\,dx\\&\quad=\int_\Omega\big[\varepsilon|u|^{q-2}u+g\big(x,u(x),\nabla u(x)\big)-h(x)\big]v(x)\,dx+\langle v,B_{p,q}u\rangle-\int_\Omega\varepsilon|u|^{q-2}uv\,dx\\&\quad=\big\langle v,Au+Su-\partial\Phi_p(u)\big\rangle=\big\langle v,-\partial\Phi_p(u)\big\rangle\\&\quad=-\int_\Gamma\beta_x(u)v|_\Gamma\,d\Gamma(x).\end{aligned}$$

Thus $-\langle\vartheta,(C(x)+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u\rangle\in\beta_x(u)$, a.e. on Γ.

This completes the proof. □

Theorem 19 *Let A and S be the same as those in Lemma* 14 *and Lemma* 15, *respectively. Let $f:L^2(\Omega)\to L^2(\Omega)$ be a fixed contractive mapping with coefficient $k\in(0,1)$ and $T:L^2(\Omega)\to L^2(\Omega)$ be a strongly positive linear bounded operator with coefficient $\overline{\gamma}$. Suppose that $0<\eta<\frac{\overline{\gamma}}{2k}$. Let $\{u_n\}$ be generated by the iterative algorithm* (D)

$$\begin{cases}u_0\in L^2(\Omega),\\ y_n=\alpha_nu_n+(1-\alpha_n)(I+r_nA)^{-1}[\frac{u_n+y_n}{2}-r_nS(\frac{u_n+y_n}{2})],\\ u_{n+1}=\gamma_n\eta f(u_n)+(I-\gamma_nT)y_n+e_n,\quad n\geq0.\end{cases}\tag{D}$$

Suppose $\{e_n\}\subset L^2(\Omega)$, $\{\alpha_n\}$ and $\{\gamma_n\}$ are two sequences in $(0,1)$ and $\{r_n\}\subset(0,+\infty)$ satisfies the conditions presented in Theorem 10. *Then $\{u_n(x)\}$ converges strongly to a point $p_0(x)\in(A+S)^{-1}0$, which is the common solution of nonlinear boundary value problem* (C) *and the following variational inequality*: *for $\forall z\in(A+S)^{-1}0$,*

$$\big\langle(T-\eta f)p_0,p_0-z\big\rangle\leq0.\tag{22}$$

Example 2 Next, we shall consider the following Laplacian equation with Signorini boundary value conditions, which can be found in [21]:

$$\begin{cases} -\Delta u + u = h(x), \quad x \in \Omega, \\ u \geq Mu, \frac{\partial u}{\partial \vartheta} \geq 0, \quad (u - Mu)\frac{\partial u}{\partial \vartheta} = 0, \quad \text{a.e. } x \in \Gamma, \end{cases} \tag{E}$$

where Ω is a bounded domain of R^N $(N \geq 1)$ with its boundary Γ sufficiently smooth, ϑ is the exterior normal derivative of Γ. $h(x) \in L^2(\Omega)$, $\overline{h} \in H^{\frac{1}{2}}(\Gamma)$, $\phi \in H^{\frac{1}{2}}(\Gamma)$ are given functions. $Mu = \overline{h}(x) - \int_\Gamma \phi \frac{\partial u}{\partial \vartheta} d\Gamma(x)$ for $x \in \Gamma$ and ϕ is a nonnegative function defined on Γ.

From [21], we know that (E) can be expressed in the form of the following quasi-variational inequality:

$$\begin{cases} u \in H_L^1(\Omega), \qquad u \in Q(u), \\ \langle Lu, u - w\rangle \leq \langle h(x), u - w\rangle, \quad \forall w \in Q(u), \end{cases} \tag{F}$$

where $H_L^1(\Omega) := \{u \in H^1(\Omega) : Lu \in L^2(\Omega)\}$ with norm $\|\cdot\|_{H_L^1(\Omega)} = (\|\cdot\|_{H^1(\Omega)}^2 + \|\cdot\|_{L^2(\Omega)}^2)^{\frac{1}{2}}$ for $u \in H_L^1(\Omega)$, $L = L_1 + L_2$ and $L_1 u = -\Delta u$, $L_2 u = u$ for $u \in H_L^1(\Omega)$. $Q(u) := \{v \in H^1(\Omega) : v \geq \overline{h}(x) - \langle \phi, \frac{\partial u}{\partial \vartheta}\rangle_\Gamma, \text{a.e. } x \in \Gamma\}$, here $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between $H^{\frac{1}{2}}(\Gamma)$ and $H^{-\frac{1}{2}}(\Gamma)$.

Lemma 20 ([21]) *Quasi-variational inequality* (F) *has a solution, which implies that* (E) *has a solution* $u(x) \in H_L^1(\Omega)$.

Lemma 21 *If* $u(x) \in (L_1 + L_3)^{-1}0$, *where* $L_3 u = L_2 u - h(x)$, *then* $u(x) \in H_L^1(\Omega)$ *is the solution of* (E).

Proof It is easy to check that if $u(x) \in (L_1 + L_3)^{-1}0$, then $u(x)$ satisfies (F), which implies that the result is true in view of Lemma 20.

This completes the proof. □

Theorem 22 *Let* L_1 *and* L_3 *be the same as those above. Let* $f : H_L^1(\Omega) \to H_L^1(\Omega)$ *be a fixed contractive mapping with coefficient* $k \in (0,1)$, $T : H_L^1(\Omega) \to H_L^1(\Omega)$ *be a strongly positive linear bounded operator with coefficient* $\overline{\gamma}$. *Suppose that* $0 < \eta < \frac{\overline{\gamma}}{2k}$. *Let* $\{u_n(x)\}$ *be generated by the iterative algorithm* (G):

$$\begin{cases} u_0 \in H_L^1(\Omega), \\ y_n = \alpha_n u_n + (1 - \alpha_n)(I + r_n L_1)^{-1}[\frac{u_n + y_n}{2} - r_n L_3(\frac{u_n + y_n}{2})], \\ u_{n+1} = \gamma_n \eta f(u_n) + (I - \gamma_n T)y_n + e_n, \quad n \geq 0. \end{cases} \tag{G}$$

Suppose $\{e_n\} \subset H_L^1(\Omega)$, $\{\alpha_n\}$ *and* $\{\gamma_n\}$ *are two sequences in* $(0,1)$ *and* $\{r_n\} \subset (0,+\infty)$ *satisfies the conditions presented in Theorem* 10. *Then* $\{u_n(x)\}$ *converges strongly to a point* $p_0(x) \in (L_1 + L_3)^{-1}0$, *which is the common solution of the Laplacian equation with Signorini boundary value condition* (E) *and the following variational inequality*: *for* $\forall z \in (L_1 + L_3)^{-1}0$,

$$\langle (T - \eta f)p_0, p_0 - z\rangle \leq 0. \tag{23}$$

Competing interests
The authors declare that there is no conflict of interests regarding the publication of this article.

Authors' contributions
All authors contributed equally to this manuscript. All authors read and approved the final manuscript.

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