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The ith p-affine surface area

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Abstract

About two decades ago Lutwak introduced the concept of p-affine surface area. More recently, the results of Lutwak have been generalized by Ma to the entire class of so-called ith p-affine surface areas. In this paper, we further research this new notion and give its integral representation. Affine isoperimetric and Blaschke-Santaló inequalities, which generalize the inequalities obtained by Lutwak, are established. Furthermore, we prove the ith p-affine area ratio of convex body K for the ith p-affine surface area, which does not exceed the generalized Santaló product of convex body K.

MSC: 52A30; 52A40

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1 Introduction

During the past three decades, the investigations of the classical affine surface area have received great attention from many articles (see articles [1–14] or books [15, 16]). Based on the classical affine surface area, Lutwak [17] introduced the notion of p-affine surface area and obtained some isoperimetric inequalities for p-affine surface area. Regarding the studies of p-affine surface area also see [18–25]. In particular, Ma [26] studied the ith p-geominimal surface area.

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n$. Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) and $\mathcal{K}_o^n$ denote the subset of $\mathcal{K}^n$ that contains the origin in their interiors in $\mathbb{R}^n$. Let $\mathcal{K}_c^n$ denote the set of convex bodies whose centroids lie at the origin. As usual, $V_i(K)$ denotes the i-dimensional volume (*i.e.*, Lebesgue measure) of a compact convex set K in $\mathbb{R}^n$. Instead of $V_n(K)$ we usually write $V(K)$. Let $\mathbb{S}^{n-1}$ denote the unit sphere with unit ball B_n, ω_n is the volume of B_n, and $\omega_i := V_i(B_n)$ denotes the i-dimensional intrinsic volume of B_n. For $K \in \mathcal{K}_o^n$, let K^* denote the polar body of K. Let $\mathcal{S}_o^n$ denote the set star bodies in $\mathbb{R}^n$ containing the origin in their interiors.

In [3], Leichtweiß defined the affine surface area $\Omega(K)$ by

$$n^{-\frac{1}{n}}\Omega(K)^{\frac{n+1}{n}} = \inf\bigl\{nV_1\bigl(K,Q^*\bigr)V(Q)^{\frac{1}{n}} : Q \in \mathcal{S}_o^n\bigr\}. \tag{1.1}$$

In [17], Lutwak generalized the affine surface area $\Omega(K)$ to the p-affine surface area $\Omega_p(K)$ by using the Brunn-Minkowski-Fiery theory as follows:

$$n^{-\frac{p}{n}}\Omega_p(K)^{\frac{n+p}{n}} = \inf\{nV_p(K,Q^*)V(Q)^{\frac{p}{n}} : Q \in \mathcal{S}_o^n\}. \tag{1.2}$$

Obviously, if $p = 1$, $\Omega_1(K)$ is just the classical affine surface area $\Omega(K)$.

Moreover, Lutwak proved the following inequalities for the p-affine surface area.

Theorem 1.1 *Let $K \in \mathcal{K}_c^n$ and $p \geq 1$. Then*

$$\Omega_p(K)^{n+p} \leq n^{n+p}\omega_n^{2p}V(K)^{n-p} \tag{1.3}$$

with equality if and only if K is an ellipsoid.

Theorem 1.2 *Let $K \in \mathcal{K}_c^n$ and $p \geq 1$. Then*

$$\Omega_p(K)\Omega_p(K^*) \leq (n\omega_n)^2 \tag{1.4}$$

with equality if and only if K is an ellipsoid.

For $K \in \mathcal{K}_o^n$, Lutwak also defined the p-affine area ratio of K by (see [17])

$$\left(\frac{\Omega_p(K)^{n+p}}{n^{n+p}V(K)^{n-p}}\right)^{\frac{1}{p}} \tag{1.5}$$

and proved (1.5) is monotone nondecreasing in p.

Theorem 1.3 *If $K \in \mathcal{F}_o^n$ and $1 \leq p \leq q$, then*

$$\left(\frac{\Omega_p(K)^{n+p}}{n^{n+p}V(K)^{n-p}}\right)^{\frac{1}{p}} \leq \left(\frac{\Omega_q(K)^{n+q}}{n^{n+q}V(K)^{n-q}}\right)^{\frac{1}{q}} \tag{1.6}$$

with equality if and only if $K \in \mathcal{E}^n$, where $\mathcal{E}^n = \{K \in \mathcal{F}_o^n : K^*$ and ΛK are dilates$\}$ and ΛK denotes the curvature image of K.

It is easily seen that the p-affine surface area belongs to the Brunn-Minkowski-Fiery theory. Recently, Ma [24] further extended the p-affine surface area $\Omega_p(K)$ to the ith p-affine surface area $\Omega_p^{(i)}(K)$ of $K \in \mathcal{K}_o^n$ (also called the $(i,0)$ type p-affine surface area, $i \in \{0,1,\ldots,n-1\}$) by using the Brunn-Minkowski-Fiery theory as follows:

$$n^{-\frac{p}{n-i}}\Omega_p^{(i)}(K)^{\frac{n+p-i}{n-i}} = \inf\{nW_{p,i}(K,Q^*)\widetilde{W}_i(Q)^{\frac{p}{n-i}} : Q \in \mathcal{S}_o^n\}. \tag{1.7}$$

It is the aim of this paper to establish several generalized forms of inequalities (1.3), (1.4), and (1.6). Our main results can be stated as follows.

Theorem 1.4 *If $p \geq 1$, $i \in \{0,1,\ldots,n-1\}$, and $K \in \mathcal{F}_{i,o}^n$, then the integral expressions of ith p-affine surface area $\Omega_p^{(i)}(K)$ are as follows:*

$$\Omega_p^{(i)}(K) = \int_{\mathbb{S}^{n-1}} f_{p,i}(K,u)^{\frac{n-i}{n+p-i}}\,dS(u), \tag{1.8}$$

where symbols $f_{p,i}(K,\cdot)$ and $\mathcal{F}_{i,o}^n$ are defined in Section 2.

Taking $i = 0$ in (1.8), the ith p-affine surface area reduces to Lutwak's p-affine surface area (see [17], Theorem 4.4):

$$\Omega_p(K) = \int_{\mathbb{S}^{n-1}} f_p(K,u)^{\frac{n}{n+p}}\, dS(u),$$

where $K \in \mathcal{F}_o^n \subset \mathcal{K}_o^n$ is a convex body with a positive continuous curvature function, $f_p(K,\cdot)$ denotes a p-curvature function of $K \in \mathcal{K}_o^n$.

Theorem 1.5 *Suppose $K \in \mathcal{K}_c^n$, $i \in \{0,1,\dots,n-1\}$ and $p \geq 1$. Then*

$$\Omega_p^{(i)}(K)^{n+p-i} \leq n^{n+p-i}\omega_n^{2p} W_i(K)^{n-i}\widetilde{W}_i(K)^{-p} \tag{1.9}$$

with equality for $i = 0$ if and only if K is an ellipsoid, and for $0 < i \leq n-1$ if and only if K is an n-ball centered at the origin.

Theorem 1.6 *Suppose $K \in \mathcal{K}_c^n$, $i \in \{0,1,\dots,n-1\}$ and $p \geq 1$. Then*

$$\Omega_p^{(i)}(K)^{n+p-i} \leq n^{n+p-i}(\omega_i\omega_{n-i})^{2p}\binom{n}{i}^{-2p} W_i(K)^{n-p-i} \tag{1.10}$$

with equality for $i = 0$ if and only if K is an ellipsoid, and for $0 < i \leq n-1$ if and only if all $(n-i)$-dimensional sub-convex bodies contained in K are $(n-i)$-ball centered at the origin.

Taking $i = 0$, inequality (1.10) reduces to Lutwak's result (see [17], this is also Theorem 1.1 in our article).

Theorem 1.7 *Suppose $K \in \mathcal{K}_o^n$, $i \in \{0,1,\dots,n-1\}$ and $p \geq 1$. Then*

$$\Omega_p^{(i)}(K)\Omega_p^{(i)}(K^*) \leq n^2 W_i(K)W_i(K^*) \tag{1.11}$$

with equality in inequality for $i = 0$ if and only if K is an ellipsoid centered at the origin, and for $0 < i \leq n-1$ if and only if $K$is a ball centered at the origin.

Corolloary 1.8 *Suppose $K \in \mathcal{K}_c^n$, $i \in \{0,1,\dots,n-1\}$ and $p \geq 1$. Then*

$$\Omega_p^{(i)}(K)\Omega_p^{(i)}(K^*) \leq (n\omega_i\omega_{n-i})^2\binom{n}{i}^{-2} \tag{1.12}$$

with equality in inequality for $i = 0$ if and only if K is an ellipsoid centered at the origin, and for $0 < i \leq n-1$ if and only if all $(n-i)$-dimensional sub-convex bodies contained in K are an $(n-i)$-ball centered at the origin.

Taking $i = 0$, inequality (1.12) reduces to Lutwak's result (see [17], this is also Theorem 1.2 in our article).

Theorem 1.9 *If $K \in \mathcal{F}_{i,o}^n$ and $i \in \{0,1,\dots,n-1\}$, then, for $1 \leq p \leq q$,*

$$\left(\frac{\Omega_p^{(i)}(K)^{n+p-i}}{n^{n+p-i}W_i(K)^{n-p-i}}\right)^{\frac{1}{p}} \leq \left(\frac{\Omega_q^{(i)}(K)^{n+q-i}}{n^{n+q-i}W_i(K)^{n-q-i}}\right)^{\frac{1}{q}} \tag{1.13}$$

with equality if and only if $K \in \mathcal{E}_i^n$, where symbol $\mathcal{E}_i^n$ is defined in (3.18).

Taking $i = 0$, inequality (1.13) reduces to Lutwak's result (see [17], this is also Theorem 1.3 in our article).

The paper is organized as follows. For the sake of convenience, in Section 2 we introduce the basic knowledge about the convex geometric analysis. In Section 3 we discuss some of the properties of the ith p-affine surface area $\Omega_p^{(i)}$ and ith p-curvature image $\Lambda_{p,i}$. Meanwhile, we prove Theorems 1.4-1.7 stated at the beginning of this paper. In Section 4 we establish the cyclic inequalities of ith p-affine surface area $\Omega_p^{(i)}(K)$ and the monotonicity of ith p-affine area ratio and ith p-curvature ratio; these results are a generalization of Lutwak's conclusions (see [17]). At the same time, we complete the proof of the monotonicity theorem (Theorem 1.9 stated at the beginning of this paper). In Section 5, we further define the concept of $\Omega_\infty^{(i)}$ and discuss its interesting properties. In addition, we give a daisy chain of inequalities for ith p-affine area ratio with monotone nondecreasing in p, which does not exceed the generalized Santaló product of convex body.

2 Notation and preliminaries

2.1 Support function, radial function, and polar of convex body

As usual, $GL(n)$ denotes a nonsingular linear transformation group in $\mathbb{R}^n$. For $\phi \in GL(n)$, let ϕ^t, ϕ^{-1}, and ϕ^{-t} denote the transpose, inverse, and inverse of the transpose of ϕ, respectively. For $K \in \mathcal{K}^n$, let $h(K,\cdot):\mathbb{R}^n \to \mathbb{R}$ denote the support function of $K \in \mathcal{K}^n$. Namely,

$$h(K,x) = h_K(x) := \max\{x\cdot y : y \in K\} \quad \text{for } x \in \mathbb{R}^n,$$

where $x \cdot y$ denotes the standard inner product of x and y in $\mathbb{R}^n$. For $\phi \in GL(n)$, then obviously $h(\phi K, x) = h(K, \phi^t x)$. For the sake of convenience, we write h_K rather than $h(K,\cdot)$ for the support function of K. Apparently, for $K, L \in \mathcal{K}^n$, $K \subseteq L$ if and only if $h_K \le h_L$. The set $\mathcal{K}^n$ will be viewed as equipped with the Hausdorff metric δ defined by $\delta(K,L) = \|h_K - h_L\|_\infty$ is the sup (or max) norm on the space of continuous functions on the unit sphere $C(S^{n-1})$.

For a compact subset L of $\mathbb{R}^n$, which is star-shaped with respect to the origin, we shall use $\rho(L,\cdot)$ to denote its radial function; *i.e.*, for $x \in \mathbb{R}^n\backslash\{0\}$,

$$\rho(L,x) = \rho_L(x) := \max\{\lambda > 0 : \lambda x \in L\}.$$

If $\rho(L,\cdot)$ is continuous and positive, L will be called a star body, and $\mathcal{S}_o^n$ will be used to denote the class of star bodies in $\mathbb{R}^n$ containing the origin in their interiors. Apparently, for $K, L \in \mathcal{S}_o^n$, $K \subseteq L$ if and only if $\rho_K \le \rho_L$. Two star bodies K and L are said to be dilates (of one another) if $\rho(K,u)/\rho(L,u)$ is independent of $u \in \mathbb{S}^{n-1}$. Let $\widetilde{\delta}$ denote the radial Hausdorff metric as follows: if $K, L \in \mathcal{S}_o^n$, then $\widetilde{\delta}(K,L) = \|\rho_K - \rho_L\|_\infty$.

For $K \in \mathcal{K}_o^n$, the polar body K^* of K is defined by

$$K^* = \bigl\{x \in \mathbb{R}^n : x\cdot y \le 1, y \in K\bigr\}.$$

Obviously, we have $(K^*)^* = K$. If $\lambda > 0$, then $(\lambda K)^* = \lambda^{-1}K^*$. More generally, if $\phi \in GL(n)$, then $(\phi K)^* = \phi^{-t}K^*$. For $K \in \mathcal{K}_o^n$, the support and radial function of the polar body K^* of K are defined respectively by (see [16, 27])

$$h_{K^*}(u) = \frac{1}{\rho_K(u)} \quad \text{and} \quad \rho_{K^*}(u) = \frac{1}{h_K(u)} \quad \text{for all } u \in \mathbb{S}^{n-1}. \tag{2.1}$$

Define the Santaló product of $K \in \mathcal{K}_o^n$ by $V(K)V(K^*)$. The Blaschke-Santaló inequality (see [16, 27]) is one of the fundamental affine isoperimetric inequalities. It states that if $K \in \mathcal{K}_c^n$ then

$$V(K)V(K^*) \leq \omega_n^2 \tag{2.2}$$

with equality if and only if K is an ellipsoid.

2.2 Mixed volumes, *p*-mixed quermassintegrals, and dual *p*-mixed quermassintegrals

We first introduce the following sum theorems for the mixed volumes and the mixed area measure of convex bodies (see [16], p.280).

There is a nonnegative symmetric function $V : (\mathcal{K}^n)^n \to \mathbb{R}$, the mixed volume such that, for $m \in \mathbb{N}$,

$$V_n(\lambda_1 K_1 + \lambda_m K_m) = \sum_{i_1,\ldots,i_n=1}^{m} \lambda_{i_1}\cdots\lambda_{i_n} V(K_{i_1},\ldots,K_{i_n})$$

for arbitrary convex bodies $K_1,\ldots,K_m \in \mathcal{K}^n$ and numbers $\lambda_1,\ldots,\lambda_m \geq 0$.

Further, there is a symmetric map S from $(\mathcal{K}^n)^{n-1}$ into the space of finite Borel measures on $\mathbb{S}^{n-1}$, the mixed area measure such that, for $m \in \mathbb{N}$,

$$S_{n-1}(\lambda_1 K_1 + \lambda_m K_m, \cdot) = \sum_{i_1,\ldots,i_{n-1}=1}^{m} \lambda_{i_1}\cdots\lambda_{i_{n-1}} S(K_{i_1},\ldots,K_{i_{n-1}}, \cdot)$$

for $K_1,\ldots,K_m \in \mathcal{K}^n$ and $\lambda_1,\ldots,\lambda_m \geq 0$ (where we write $S(K_1,\ldots,K_{n-1},\cdot) = S(K_1,\ldots,K_{n-1})(\cdot)$). Taking $K_1 = \cdots = K_{n-i-1} = K$ and $K_{n-i} = \cdots = K_{n-1} = B_n$ in $S(K_1,\ldots,K_{n-1},\cdot)$, we write $S_i(K,\cdot)$ for $S(K,\ldots,K,B_n,\ldots,B_n,\cdot)$.

The equality

$$V(K_1,\ldots,K_n) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h(K_1,u)\,dS(K_2,\ldots,K_n,u)$$

holds for $K_1,\ldots,K_n \in \mathcal{K}^n$.

For $K \in \mathcal{K}^n$ and $i \in \{0,1,\ldots,n-1\}$, the quermassintegral $W_i(K)$ of K is given by (see [28])

$$W_i(K) = V(\underbrace{K,\ldots,K}_{n-i},\underbrace{B_n,\ldots,B_n}_{i}) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h(K,u)\,dS_i(K,u). \tag{2.3}$$

It turns out that the ith surface area measure $S_i(K,\cdot)$ of K, $i \in \{0,1,\ldots,n-1\}$, on $\mathbb{S}^{n-1}$ is absolutely continuous with respect to the ordinary surface area measure $S(K,\cdot)$ of K and has the Radon-Nikodym derivative (see [29])

$$\frac{dS_i(K,\cdot)}{dS(K,\cdot)} = h(K,\cdot)^{-i}. \tag{2.4}$$

From (2.3), we easily see that $W_0(K) = V(K)$.

The definition of $W_i(K)$ is the classical Steiner formula, which we write in the two forms (see [16], pp.213, 286):

$$V_n(K + \lambda B_n) = \sum_{i=0}^{n} \lambda^i \binom{n}{i} W_i(K) = \sum_{i=0}^{n} \lambda^{n-i} \omega_{n-i} V_i(K).$$

From the above definition of $W_i(K)$ and the definition of $V(K_{i_1}, \dots, K_{i_n})$, it follows that (see [16], pp.213, 286)

$$W_i(K) = V(\underbrace{K, \dots, K}_{n-i}, \underbrace{B_n, \dots, B_n}_{i}) = \frac{\omega_i}{\binom{n}{i}} V_{n-i}(K), \quad i \in \{0, 1, \dots, n\}. \tag{2.5}$$

For real $p \geq 1$, $K, L \in \mathcal{K}_o^n$, and $\alpha, \beta \geq 0$ (not both zero), the Firey p-linear combination $\alpha \circ K +_p \beta \circ L$, is defined by (see [30])

$$h(\alpha \circ K +_p \beta \circ L, \cdot)^p = \alpha h(K, \cdot)^p + \beta h(L, \cdot)^p.$$

For $K, L \in \mathcal{K}_o^n$, $\varepsilon > 0$, and real $p \geq 1$, the p-mixed quermassintegrals $W_{p,i}(K, L)$ of K and L, $i \in \{0, 1, \dots, n-1\}$ are defined by (see [28])

$$\frac{n-i}{p} W_{p,i}(K, L) = \lim_{\varepsilon \to 0^+} \frac{W_i(K +_p \varepsilon \circ L) - W_i(K)}{\varepsilon}.$$

Obviously, for $p = 1$, $W_{1,i}(K, L)$ is just the classical mixed quermassintegral $W_i(K, L)$. For $i = 0$, the p-mixed quermassintegral $W_{p,0}(K, L)$ is just the p-mixed volume $V_p(K, L)$.

For $p \geq 1$, $i \in \{0, 1, \dots, n-1\}$, and each $K \in \mathcal{K}_o^n$, there exists a positive Borel measure $S_{p,i}(K, \cdot)$ on $\mathbb{S}^{n-1}$ such that the p-mixed quermassintegral $W_{p,i}(K, L)$ has the following integral representation (see [28]):

$$W_{p,i}(K, L) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_L^p(v) \, dS_{p,i}(K, v) \tag{2.6}$$

for all $L \in \mathcal{K}_o^n$. It turns out that the measure $S_{p,i}(K, \cdot)$, $i \in \{0, 1, \dots, n-1\}$, on $\mathbb{S}^{n-1}$ is absolutely continuous with respect to $S_i(K, \cdot)$ and has the Radon-Nikodym derivative

$$\frac{dS_{p,i}(K, \cdot)}{dS_i(K, \cdot)} = h(K, \cdot)^{1-p}. \tag{2.7}$$

Together with (2.3) and (2.6), for $K \in \mathcal{K}_o^n$, $p \geq 1$, we have $W_{p,i}(K, K) = W_i(K)$.

For $K \in \mathcal{S}_o^n$ and any real i, the ith dual quermassintegral $\widetilde{W}_i(K)$ of K is defined by (see [16, 27])

$$\widetilde{W}_i(K) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} \rho_K^{n-i}(u) \, dS(u). \tag{2.8}$$

Obviously, $\widetilde{W}_0(K) = V(K)$.

For $K, L \in \mathcal{S}_o^n$, $p \geq 1$, and $\lambda, \mu \geq 0$ (not both zero), the p-harmonic radial combination $\lambda * K +_{-p} \mu * L \in \mathcal{S}_o^n$ is defined by (see [17])

$$\rho(\lambda * K +_{-p} \mu * L, \cdot)^{-p} = \lambda \rho(K, \cdot)^{-p} + \mu \rho(L, \cdot)^{-p}.$$

Note that here '$\varepsilon * L$' is different from '$\varepsilon \circ L$' in Firey p-linear combination.

For $K, L \in \mathcal{S}_o^n$, $\varepsilon > 0$, $p \geq 1$, and real $i \neq n$, the dual p-mixed quermassintegral $\widetilde{W}_{-p,i}(K, L)$ of K and L is defined by (see [31])

$$\frac{n-i}{-p}\widetilde{W}_{-p,i}(K,L) = \lim_{\varepsilon \to 0^+} \frac{\widetilde{W}_i(K +_{-p} \varepsilon * L) - \widetilde{W}_i(K)}{\varepsilon}. \tag{2.9}$$

If $i = 0$, we easily see that (2.9) is just the definition of dual p-mixed volume, *i.e.*, $\widetilde{W}_{-p,0}(K, L) = \widetilde{V}_{-p}(K, L)$.

From (2.9), the integral representation of the dual p-mixed quermassintegrals is given by Wang and Leng [31]: If $K, L \in \mathcal{S}_o^n$, $p \geq 1$, and real $i \neq n$, $i \neq n + p$, then

$$\widetilde{W}_{-p,i}(K,L) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} \rho_K^{n+p-i}(u)\rho_L^{-p}(u)\,dS(u). \tag{2.10}$$

Together with (2.8) and (2.10), for $K \in \mathcal{S}_o^n$, $p \geq 1$, and $i \neq n, n + p$, it follows that $\widetilde{W}_{-p,i}(K, K) = \widetilde{W}_i(K)$.

Further, Wang and Leng [31] proved the following analog of the Minkowski inequality for the dual p-mixed quermassintegrals: If $K, L \in \mathcal{S}_o^n$, $p \geq 1$, then, for $i < n$ or $i > n + p$,

$$\widetilde{W}_{-p,i}(K,L)^{n-i} \geq \widetilde{W}_i(K)^{n+p-i}\widetilde{W}_i(L)^{-p}, \tag{2.11}$$

and for $n < i < n + p$, inequality (2.11) is reverse, with equality in every inequality if and only if K and L are dilates of each other.

Another consequence of inequality (2.11) will be needed (see [32]): Suppose $K, L \in \mathcal{S}_o^n$, $p \geq 1$ and $\lambda, \mu > 0$. If real $i < n$ or $n < i < n + p$, then

$$\widetilde{W}_i(\lambda * K \hat{+}_{-p} \mu * L)^{-p/(n-i)} \geq \lambda\widetilde{W}_i(K)^{-p/(n-i)} + \mu\widetilde{W}_i(L)^{-p/(n-i)}, \tag{2.12}$$

with equality in every inequality if and only if K and L are dilates of each other, and for $n > n + p$ inequality (2.12) is reverse.

The following result will be needed.

Lemma 2.1 *If $p \geq 1$, $i \in \mathbb{R}$, $\mathcal{M} \subset \mathcal{S}_o^n$ is a class of bodies such that $K, L \in \mathcal{M}$. If*

$$\widetilde{W}_{-p,i}(K,Q)/\widetilde{W}_i(K) = \widetilde{W}_{-p,i}(L,Q)/\widetilde{W}_i(L) \quad \textit{for all } Q \in \mathcal{M}, \tag{2.13}$$

then $K = L$.

Proof Taking $Q = L$ gives $\widetilde{W}_{-p,i}(K,L)/\widetilde{W}_i(K) = \widetilde{W}_{-p,i}(L,L)/\widetilde{W}_i(L) = 1$. Now inequality (2.11) gives $\widetilde{W}_i(L) \geq \widetilde{W}_i(K)$ with equality if and only if K and L are dilates. Take $Q = K$ and get $\widetilde{W}_i(K) \geq \widetilde{W}_i(L)$ with equality if and only if L and K are dilates. Hence, $\widetilde{W}_i(K) = \widetilde{W}_i(L)$, and K and L must be dilates. Thus, $K = L$. □

By inequality (2.12), for convex bodies, we introduce the following definition: Suppose $K \in \mathcal{K}_o^n$ and $L \in \mathcal{S}_o^n$. For $p \geq 1$ and $i \in \{0, 1, \ldots, n-1\}$, define $W_{p,i}(K, L^*)$ by

$$W_{p,i}(K,L^*) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} \rho_L(u)^{-p}\,dS_{p,i}(K,u). \tag{2.14}$$

Since $h_{Q^*} = 1/\rho_Q$ for $Q \in \mathcal{K}_o^n$, it follows from the integral representation (2.6) that, if L happens to belong to $\mathcal{K}_o^n$ (rather than just $\mathcal{S}_o^n$), the new definition of $W_{p,i}(K, L^*)$ agrees with the old definition.

2.3 The *i*th *p*-curvature function and *i*th *p*-curvature image

A convex body $K \in \mathcal{K}^n$ is said to have a continuous ith curvature function $f_i(K,\cdot) : \mathbb{S}^{n-1} \to \mathbb{R}$ if its mixed surface area measure $S_i(K,\cdot)$ is absolutely continuous with respect to the spherical Lebesgue measure S and has the Radon-Nikodym derivative (see [28])

$$\frac{dS_i(K,\cdot)}{dS} = f_i(K,\cdot), \quad \text{for } i \in \{0,1,\ldots,n-1\}. \tag{2.15}$$

Let $\mathcal{F}_i^n$, $\mathcal{F}_{i,o}^n$, $\mathcal{F}_{i,c}^n$ denote a set of all bodies in $\mathcal{K}^n$, $\mathcal{K}_o^n$, $\mathcal{K}_c^n$, respectively, that have an ith positive continuous curvature function. In particular, $\mathcal{F}_0^n := \mathcal{F}^n$, $\mathcal{F}_{0,o}^n := \mathcal{F}_o^n$, $\mathcal{F}_{0,c}^n := \mathcal{F}_c^n$.

A convex body $K \in \mathcal{K}_o^n$ is said to have a p-curvature function $f_p(K,\cdot) : \mathbb{S}^{n-1} \to \mathbb{R}$ if its p-surface area measure $S_p(K,\cdot)$ is absolutely continuous with respect to the spherical Lebesgue measure S and has the Radon-Nikodym derivative (see [17])

$$\frac{dS_p(K,\cdot)}{dS} = f_p(K,\cdot). \tag{2.16}$$

Lutwak [17] showed the notion of p-curvature image as follows: For each $K \in \mathcal{F}_o^n$ and $p \geq 1$, define $\Lambda_p K \in \mathcal{S}_o^n$, the p-curvature image of K, by

$$\rho(\Lambda_p K,\cdot)^{n+p} = \frac{V(\Lambda_p K)}{\omega_n} f_p(K,\cdot). \tag{2.17}$$

It should be noted that, for $p = 1$, this definition of curvature image differs from the definition used by the author in [8, 10], and [33].

Recently, Liu *et al.* [34], Lu and Wang [35], as well as Ma and Liu [24, 36, 37] independently introduced the concept of ith p-curvature function of $K \in \mathcal{K}_o^n$ as follows: Let $p \geq 1$, $i \in \{0,1,\ldots,n-1\}$, a convex body $K \in \mathcal{K}_o^n$ is said to have an ith p-curvature function $f_{p,i}(K,\cdot) : \mathbb{S}^{n-1} \to \mathbb{R}$ if its ith p-surface area measure $S_{p,i}(K,\cdot)$ is absolutely continuous with respect to the spherical Lebesgue measure S and has the Radon-Nikodym derivative

$$\frac{dS_{p,i}(K,\cdot)}{dS} = f_{p,i}(K,\cdot). \tag{2.18}$$

If the ith surface area measure $S_i(K,\cdot)$ is absolutely continuous with respect to the spherical Lebesgue measure S, we have

$$f_{p,i}(K,\cdot) = h(K,\cdot)^{1-p} f_i(K,\cdot). \tag{2.19}$$

Together with (2.15) and (2.19), we easily get that, for $K \in \mathcal{K}_o^n$ and real $\lambda > 0$,

$$f_{p,i}(\lambda K,\cdot) = \lambda^{n-p-i} f_{p,i}(K,\cdot). \tag{2.20}$$

According to the concept of ith p-curvature function of convex body, Lu and Wang [35] as well as Ma [24] introduced independently the concept of ith p-curvature image of convex body as follows: For each $K \in \mathcal{F}_{i,o}^n$, $i \in \{0,1,\ldots,n-1\}$, and real $p \geq 1$, define $\Lambda_{p,i} K \in \mathcal{S}_o^n$, the ith p-curvature image of K, by

$$\rho(\Lambda_{p,i}K,\cdot)^{n+p-i}=\frac{\widetilde{W}_i(\Lambda_{p,i}K)}{\omega_n}f_{p,i}(K,\cdot).\tag{2.21}$$

For the case $p=1$ or $i=0$, the subscript p or i in $\Lambda_{p,i}$ will often be suppressed. If $\Lambda_{p,i}K\in\mathcal{K}_o^n$, write $\Lambda_{p,i}^*K$ for $(\Lambda_{p,i}K)^*$. The unusual normalization of definition (2.21) is chosen so that, for the unit ball B_n, it follows that $\Lambda_{p,i}B_n=B_n$. From definitions (2.17), (2.19), and (2.21), if $i=0$, then $\Lambda_{p,0}K=\Lambda_pK$. In particular, we note that if $p=1$ in (2.21), then

$$\rho(\Lambda_iK,\cdot)^{n+1-i}=\frac{\widetilde{W}_i(\Lambda_iK)}{\omega_n}f_i(K,\cdot).\tag{2.22}$$

An immediate consequence of the definition of the ith p-curvature image and the integral representations of $W_{p,i}$ and $\widetilde{W}_{-p,i}$ is the following results.

Proposition 2.2 *If $p\geq 1$, $i\in\{0,1,\ldots,n-1\}$, and $K\in\mathcal{F}_{i,o}^n$, then, for all $Q\in\mathcal{S}_o^n$,*

$$W_{p,i}\bigl(K,Q^*\bigr)=\omega_n\widetilde{W}_{-p,i}(\Lambda_{p,i}K,Q)/\widetilde{W}_i(\Lambda_{p,i}K).\tag{2.23}$$

The following characterization follows directly from Proposition 2.2 and Lemma 2.1.

Proposition 2.3 *Suppose $K\in\mathcal{F}_{i,o}^n$ and $L\in\mathcal{S}_o^n$. If $p\geq 1$, $i\in\{0,1,\ldots,n-1\}$ and if*

$$W_{p,i}\bigl(K,Q^*\bigr)=\omega_n\widetilde{W}_{-p,i}(L,Q)/\widetilde{W}_i(L)\quad\text{for all } Q\in\mathcal{S}_o^n,\tag{2.24}$$

then $L=\Lambda_{p,i}K$.

3 The *i*th *p*-affine surface area

Let $O(n)$ denotes an orthogonal transformation group in $\mathbb{R}^n$.

Lemma 3.1 (see [28]) *Suppose $K,L\in\mathcal{K}_o^n$, $p\geq 1$, and $i\in\{0,1,\ldots,n-1\}$. Then, for any $\phi\in O(n)$,*

$$W_{p,i}(\phi K,\phi L)=W_{p,i}(K,L).$$

Lemma 3.2 (see [38]) *Suppose $K,L\in\mathcal{S}_o^n$, $p\geq 1$, and real $i\in\mathbb{R}$ as well as $i\neq n$, $i\neq n+p$. Then, for any $\phi\in O(n)$,*

$$\widetilde{W}_{-p,i}(\phi K,\phi L)=\widetilde{W}_{-p,i}(K,L).$$

An immediate consequence of the definition of $\Omega_p^{(i)}$ and Lemma 3.1 as well as Lemma 3.2 is the following.

Proposition 3.3 *If $p\geq 1$, $i\in\{0,1,\ldots n-1\}$, and $K\in\mathcal{K}_o^n$, then, for all $\phi\in O(n)$,*

$$\Omega_p^{(i)}(\phi K)=\Omega_p^{(i)}(K).$$

The ordinary surface area measure of a polytope is concentrated on a finite set of points of $\mathbb{S}^{n-1}$ (see, for example, Lutwak [17]). From this, (2.4) and (2.7), it follows that the ith p-

surface area measure $S_{p,i}(P,\cdot)$ of a polytope $P \in \mathcal{K}_o^n$ is concentrated on a finite set of points of $\mathbb{S}^{n-1}$. A direct consequence of this fact and the definition of ith p-affine surface area is as follows.

Proposition 3.4 *If $p \geq 1$, and $P \in \mathcal{K}_o^n$ is a polytope, then $\Omega_p^{(i)}(P) = 0$ for any $i \in \{0,1,\dots, n-1\}$.*

If $i = 0$, Proposition 3.4 reduces to the isotropy of the p-surface area measures, which was essentially proved in [17] by Lutwak.

Lemma 3.5 *If $p \geq 1$ and K_j is a sequence of convex bodies in $\mathcal{K}_o^n$ such that $K_j \to K_0 \in \mathcal{K}_o^n$, then, for $i = 0,1,\dots,n-1$, $S_{p,i}(K_j,\cdot) \to S_{p,i}(K_0,\cdot)$ weakly.*

Proof Suppose $f \in C(S^{n-1})$. Since $K_j \to K_0$, by the definition of support function, $h_{K_j} \to h_{K_0}$ uniformly on $\mathbb{S}^{n-1}$. Since the continuous function h_{K_0} is positive, h_{K_j} are uniformly bounded away from 0. It follows that $h_{K_j}^{1-p} \to h_{K_0}^{1-p}$ uniformly on $\mathbb{S}^{n-1}$, and thus that

$$fh_{K_j}^{1-p} \to fh_{K_0}^{1-p} \quad \text{uniformly on } \mathbb{S}^{n-1}.$$

But $K_j \to K_0$ also implies that

$$S_i(K_j,\cdot) \to S_i(K_0,\cdot) \quad \text{weakly on } \mathbb{S}^{n-1}$$

follows from the weak continuity of surface area measures (see, for example, Schneider [17, 39]). Hence,

$$\int_{\mathbb{S}^{n-1}} f(u)h(K_j,u)^{1-p}\,dS_i(K_j,u) \to \int_{\mathbb{S}^{n-1}} f(u)h(K_0,u)^{1-p}\,dS_i(K_0,u),$$

or equivalently,

$$\int_{\mathbb{S}^{n-1}} f(u)\,dS_{p,i}(K_j,u) \to \int_{\mathbb{S}^{n-1}} f(u)\,dS_{p,i}(K_0,u).$$

□

An immediate consequence of Lemma 3.5 and definition (2.14) is the following.

Proposition 3.6 *If $p \geq 1$, $i \in \{0,1,\dots,n-1\}$, and $L \in \mathcal{S}_o^n$, then $W_{p,i}(\cdot,L^*) : \mathcal{K}_o^n \to (0,\infty)$ is continuous.*

Lemma 3.7 *Suppose $K_j \to K_0 \in \mathcal{K}_o^n$ and $L_j \to L_0 \in \mathcal{K}_o^n$. If $p \geq 1$ and $i \in \{0,1,\dots,n-1\}$, then $W_{p,i}(K_j,L_j) \to W_{p,i}(K_0,L_0)$.*

Proof Since $h_{L_j} \to h_{L_0}$ uniformly on $\mathbb{S}^{n-1}$ and h_L is continuous, then h_{L_i} are uniformly bounded on $\mathbb{S}^{n-1}$. Hence,

$$h_{L_j}^p \to h_{L_0}^p \quad \text{uniformly on } \mathbb{S}^{n-1}.$$

By Lemma 3.5 $K_j \to K_0$ implies that

$$S_{p,i}(K_j,\cdot) \to S_{p,i}(K_0,\cdot) \quad \text{weakly on } \mathbb{S}^{n-1}.$$

Hence,

$$\int_{\mathbb{S}^{n-1}} h_{L_j}^p \, dS_{p,i}(K_j,u) \to \int_{\mathbb{S}^{n-1}} h_{L_0}^p \, dS_{p,i}(K_0,u). \qquad \square$$

By the definition of dual p-mixed quermassintegrals and the continuity of the radial function, we have the following.

Lemma 3.8 *Suppose $K_j \to K_0 \in \mathcal{S}_o^n$ and $L_j \to L_0 \in \mathcal{S}_o^n$. If $p \geq 1$, $i \in \mathbb{R}$, and $i \neq n$, $i \neq n+p$, then $\widetilde{W}_{-p,i}(K_j,L_j) \to \widetilde{W}_{-p,i}(K_0,L_0)$.*

An immediate consequence of the definition of $\Omega_p^{(i)}$, definition (2.14), and Proposition 3.6 is the following.

Proposition 3.9 *For $p \geq 1$ and $i \in \{0,1,\dots n-1\}$, the function $\Omega_p^{(i)} : \mathcal{K}_o^n \to [0,\infty)$ is upper semicontinuous.*

Proof of Theorem 1.4 From the definition of $\Omega_p^{(i)}(K)$, it can be seen that in order to prove the theorem, it need be shown that

$$\inf\bigl\{ W_{p,i}\bigl(K,Q^*\bigr)\widetilde{W}_i(Q)^{\frac{p}{n-i}} : Q \in \mathcal{S}_o^n \bigr\} = \left[\frac{1}{n}\int_{\mathbb{S}^{n-1}} f_{p,i}(K,u)^{\frac{n-i}{n+p-i}}\, dS(u)\right]^{\frac{n+p-i}{n-i}}. \tag{3.1}$$

Recall that $\Lambda_{p,i}K \in \mathcal{S}_o^n$ is defined by $f_{p,i}(K,\cdot) = \omega_n \rho(\Lambda_{p,i}K,\cdot)^{n+p-i}/\widetilde{W}_i(\Lambda_{p,i}K)$. From this and the formula for the ith dual quermassintegrals, it follows that the quantity on the right in (3.1) is just $\omega_n \widetilde{W}_i(\Lambda_{p,i}K)^{\frac{p}{n-i}}$. By Proposition 2.2,

$$W_{p,i}\bigl(K,Q^*\bigr)\widetilde{W}_i(Q)^{\frac{p}{n-i}} = \omega_n \widetilde{W}_{-p,i}(\Lambda_{p,i}K,Q)\widetilde{W}_i(Q)^{\frac{p}{n-i}}/\widetilde{W}_i(\Lambda_{p,i}K).$$

Hence to prove (3.1) it need only be shown that

$$\inf\bigl\{\widetilde{W}_{-p,i}(\Lambda_{p,i}K,Q)\widetilde{W}_i(Q)^{\frac{p}{n-i}} : Q \in \mathcal{S}_o^n\bigr\} = \widetilde{W}_i(\Lambda_{p,i}K)^{\frac{n+p-i}{n-i}}. \tag{3.2}$$

The fact that the quantity on the left in (3.2) is no less than the quantity on the right is a simple consequence of the dual p-mixed quermassintegrals inequality (2.11). To see that the quantity on the right in (3.2) is no less than the quantity on the left, take $Q = \Lambda_{p,i}K$ and note that

$$\widetilde{W}_{-p,i}(\Lambda_{p,i}K,\Lambda_{p,i}K)\widetilde{W}_i(\Lambda_{p,i}K)^{p/(n-i)} = \widetilde{W}_i(\Lambda_{p,i}K)^{(n+p-i)/(n-i)}.$$

Thus the result of Theorem 1.4 is obtained. $\square$

An immediate consequence of the definition of the ith p-curvature image and the integral representations of $\Omega_p^{(i)}$ as well as $\widetilde{W}_i$ is as follows.

Proposition 3.10 *If $p \geq 1$, $i \in \{0,1,\dots,n-1\}$, and $K \in \mathcal{F}_{i,o}^n$, then*

$$\Omega_p^{(i)}(K) = n\omega_n^{\frac{n-i}{n+p-i}}\widetilde{W}_i(\Lambda_{p,i}K)^{\frac{p}{n+p-i}}. \tag{3.3}$$

The following lemmas will be needed.

Lemma 3.11 (see [40]) *Suppose $K \in \mathcal{K}_o^n$ and $0 < i < n$, $i \in \mathbb{R}$, then*

$$\widetilde{W}_i(K) \le V(K)^{\frac{n-i}{n}} \omega_n^{\frac{i}{n}} \tag{3.4}$$

with equality if and only if K is an n-ball (*centered at the origin*).

Lemma 3.12 (see [40]) *If $K \in \mathcal{K}_o^n$ and $i \in \{1, \dots, n-1\}$, then*

$$\widetilde{W}_i(K) \le W_i(K) \tag{3.5}$$

with equality if and only if K is an n-ball (*centered at the origin*).

Proof of Theorem 1.5 From the definition of $\Omega_p^{(i)}(K)$ it follows that, for $K \in \mathcal{K}_c^n$ and all $Q \in \mathcal{S}_o^n$,

$$\Omega_p^{(i)}(K)^{n+p-i} \le n^{n+p-i} W_{p,i}\big(K, Q^*\big)^{n-i} \widetilde{W}_i(Q)^p.$$

Taking $Q = K^*$, we have

$$\Omega_p^{(i)}(K)^{n+p-i} \le n^{n+p-i} W_i(K)^{n-i} \widetilde{W}_i\big(K^*\big)^p. \tag{3.6}$$

Hence, together with Lemma 3.11 and the Blaschke-Santaló inequality (2.2), it follows that

$$\begin{aligned}\Omega_p^{(i)}(K)^{n+p-i} \widetilde{W}_i(K)^p &\le n^{n+p-i} W_i(K)^{n-i} \big(\widetilde{W}_i\big(K^*\big) \widetilde{W}_i(K)\big)^p \\ &\le n^{n+p-i} W_i(K)^{n-i} \omega_n^{\frac{2ip}{n}} \big(V(K) V\big(K^*\big)\big)^{\frac{(n-i)p}{n}} \\ &\le n^{n+p-i} \omega_n^{2p} W_i(K)^{n-i}.\end{aligned}$$

Therefore

$$\Omega_p^{(i)}(K)^{n+p-i} \le n^{n+p-i} \omega_n^{2p} W_i(K)^{n-i} \widetilde{W}_i(K)^{-p}.$$

By the equality condition of the Blaschke-Santaló inequality (2.2) and Lemma 3.11, equality holds in the inequality of Theorem 1.5 for $i = 0$ if and only if K is an ellipsoid, for $0 < i \le n-1$ if and only if K is an n-ball centered at the origin. □

Proof of Theorem 1.6 Together with inequality (3.6), Lemma 3.12, equation (2.5), and the Blaschke-Santaló inequality (2.2), we have

$$\begin{aligned}\Omega_p^{(i)}(K)^{n+p-i} &\le n^{n+p-i} W_i(K)^{n-i} W_i\big(K^*\big)^p \\ &= n^{n+p-i} W_i(K)^{n-p-i} \big[W_i(K) W_i\big(K^*\big)\big]^p \\ &= n^{n+p-i} W_i(K)^{n-p-i} \omega_i^{2p} \binom{n}{i}^{-2p} \big[V_{n-i}(K) V_{n-i}\big(K^*\big)\big]^p \\ &\le n^{n+p-i} (\omega_i \omega_{n-i})^{2p} \binom{n}{i}^{-2p} W_i(K)^{n-p-i}.\end{aligned}$$

In the proof process, we can easily know that for $i = 0$ equality of inequality (1.10) holds if and only if K is an ellipsoid, and for $0 < i \leq n-1$ if and only if all $(n-i)$-dimensional sub-convex bodies contained in K are $(n-i)$-ball centered at the origin. □

Remark 3.13 More recently, the author in [26] defined the notion of ith p-geominimal surface area: For $K \in \mathcal{K}_o^n$, $p \geq 1$, $i \in \{0,1,\dots,n-1\}$, then

$$\omega_n^{\frac{p}{n-i}} G_{p,i}(K) = \inf\bigl\{n W_{p,i}(K,Q)\widetilde{W}_i\bigl(Q^*\bigr)^{\frac{p}{n-i}} : Q \in \mathcal{K}_o^n\bigr\}. \tag{3.7}$$

Comparing to (1.7) and (3.7), we easily obtain that if $K \in \mathcal{K}_o^n$, $p \geq 1$, $i \in \{0,1,\dots,n-1\}$, then

$$\Omega_p^{(i)}(K)^{n+p-i} \leq (n\omega_n)^p G_{p,i}(K)^{n-i}. \tag{3.8}$$

The inequality above is (1.8) of the article [26], and from the proof in [26] we know that equality holds in (3.8) if and only if $K \in \mathcal{W}_{p,i}^n$, where symbol $\mathcal{W}_{p,i}^n$ is defined in (3.17).

Some results of this paper can immediately be given by (3.8). For example, Theorem 4.2 of [26] implies our Theorem 1.5; Theorem 1.4 of [26] implies our Theorem 1.6. But we need to note in particular that the condition of equality holds in inequalities (1.9) and (1.10) cannot be determined.

Proof of Theorem 1.7 By using the definition of $\Omega_p^{(i)}(K)$, it follows that

$$n^{-\frac{p}{n-i}}\Omega_p^{(i)}(K)^{\frac{n+p-i}{n-i}} \leq nW_{p,i}\bigl(K,Q^*\bigr)\widetilde{W}_i(Q)^{\frac{p}{n-i}}$$

for any $Q \in \mathcal{S}_o^n$. Taking K^* for Q and noticing that $K^{**} = K$, we get

$$n^{-\frac{p}{n-i}}\Omega_p^{(i)}(K)^{\frac{n+p-i}{n-i}} \leq nW_i(K)\widetilde{W}_i\bigl(K^*\bigr)^{\frac{p}{n-i}}. \tag{3.9}$$

Taking K^* for K, we get

$$n^{-\frac{p}{n-i}}\Omega_p^{(i)}\bigl(K^*\bigr)^{\frac{n+p-i}{n-i}} \leq nW_i\bigl(K^*\bigr)\widetilde{W}_i(K)^{\frac{p}{n-i}}. \tag{3.10}$$

Combining with (3.9), (3.10), and Lemma 3.12, it follows that

$$\begin{aligned}\bigl(\Omega_p^{(i)}(K)\Omega_p^{(i)}\bigl(K^*\bigr)\bigr)^{\frac{n+p-i}{n-i}} &\leq n^{\frac{2(n+p-i)}{n-i}} W_i(K)W_i\bigl(K^*\bigr)\bigl(\widetilde{W}_i(K)\widetilde{W}_i\bigl(K^*\bigr)\bigr)^{\frac{p}{n-i}} \\ &\leq n^{\frac{2(n+p-i)}{n-i}}\bigl(W_i(K)W_i\bigl(K^*\bigr)\bigr)^{\frac{n+p-i}{n-i}}.\end{aligned}$$

Therefore,

$$\Omega_p^{(i)}(K)\Omega_p^{(i)}\bigl(K^*\bigr) \leq n^2 W_i(K)W_i\bigl(K^*\bigr). \tag{3.11}$$

According to the conditions of equality holds in the inequality of Lemma 3.12, we see that equality holds in inequality (1.11) for $i = 0$ if and only if K is an ellipsoid, and for $0 < i \leq n-1$ if and only if K is a ball centered at the origin. □

Proof of Corollary 1.8 Considering inequality (3.11), together with equation (2.5) and the Blaschke-Santaló inequality (2.2), we now obtain the desired results. □

From inequality (3.9), this is summarized as follows.

Proposition 3.14 *If $p \geq 1$, $i \in \{0,1,\ldots,n-1\}$ and $K \in \mathcal{K}_o^n$, then*

$$\left(\frac{\Omega_p^{(i)}(K)^{n+p-i}}{n^{n+p-i}W_i(K)^{n-p-i}}\right)^{\frac{1}{p}} \leq W_i(K)\widetilde{W}_i(K^*). \tag{3.12}$$

Lemma 3.15 (see [28]) *Suppose $K, L \in \mathcal{K}_o^n$, and $\mathcal{B} \subset \mathcal{K}_o^n$ is a class of bodies such that $K, L \in \mathcal{B}$. If $0 \leq i < n$ and $n-i \neq p > 1$, and if*

$$W_{p,i}(K,Q) = W_{p,i}(L,Q) \quad \textit{for all } Q \in \mathcal{B}, \tag{3.13}$$

then $K = L$. If $0 \leq i < n-1$, $p = n-i$ and satisfies (3.13), *then K and L are dilates.*

The next proposition shows that for $p \neq n-i$, the functional $\Lambda_{p,i} : \mathcal{F}_{i,o}^n \to \mathcal{S}_o^n$ is injective.

Proposition 3.16 *Suppose that $K, L \in \mathcal{F}_{i,o}^n$ are such that $\Lambda_{p,i}K = \Lambda_{p,i}L$. If $p = n-i$, then K and L are dilates, and if $n-i \neq p > 1$, then $K = L$.*

Proof From Proposition 2.2, it follows that $\Lambda_{p,i}K = \Lambda_{p,i}L$ implies that

$$W_{p,i}(K,Q) = W_{p,i}(L,Q) \quad \text{for all } Q \in \mathcal{K}_o^n.$$

The desired result is now a consequence of (3.13). □

Lemma 3.17 (see [38]) *Suppose $K, L \in \mathcal{S}_o^n$, $p \geq 1$, $i \in \mathbb{R}$ and $i \neq n$, $i \neq n+p$. Then, for all $Q \in \mathcal{S}_o^n$, either*

$$\widetilde{W}_{-p,i}(K,Q) = \widetilde{W}_{-p,i}(L,Q) \quad \textit{or} \quad \widetilde{W}_{-p,i}(Q,K) = \widetilde{W}_{-p,i}(Q,L)$$

is true if and only if $K = L$.

From (2.20), (2.8), and (2.10), and noting that

$$\widetilde{W}_{-p,i}(\lambda K, \lambda L) = \lambda^{n-i}\widetilde{W}_{-p,i}(K,L),$$

it is easy to know that for $K \in \mathcal{F}_{i,o}^n$ and real $\lambda > 0$, then

$$\Lambda_{p,i}\lambda K = \lambda^{\frac{n-p-i}{p}}\Lambda_{p,i}K. \tag{3.14}$$

More generally, the next proposition shows that $\Lambda_{p,i}^*$ commutes with members of $O(n)$.

Proposition 3.18 *If $K \in \mathcal{F}_{i,o}^n$ and $i \in \{0,1,\ldots,n-1\}$, then, for any $\phi \in O(n)$,*

$$\Lambda_{p,i}\phi K = \phi^{-t}\Lambda_{p,i}K.$$

Proof Since $\phi \in O(n)$, then from Proposition 2.3, Lemmas 3.1 and 3.2, we have

$$\begin{aligned}\frac{\widetilde{W}_{-p,i}(\Lambda_{p,i}\phi K, Q)}{\widetilde{W}_i(\Lambda_{p,i}\phi K)} &= \frac{W_{p,i}(\phi K, Q^*)}{\omega_n} = \frac{W_{p,i}(\phi K, \phi\phi^{-1}Q^*)}{\omega_n} \\ &= \frac{W_{p,i}(K, \phi^{-1}Q^*)}{\omega_n} = \frac{W_{p,i}(K, (\phi^t Q)^*)}{\omega_n} = \frac{\widetilde{W}_{-p,i}(\Lambda_{p,i}K, \phi^t Q)}{\widetilde{W}_i(\Lambda_{p,i}K)} \\ &= \frac{\widetilde{W}_{-p,i}(\phi^{-t}\Lambda_{p,i}K, \phi^{-t}\phi^t Q)}{\widetilde{W}_i(\phi^{-t}\Lambda_{p,i}K)} = \frac{\widetilde{W}_{-p,i}(\phi^{-t}\Lambda_{p,i}K, Q)}{\widetilde{W}_i(\phi^{-t}\Lambda_{p,i}K)}.\end{aligned}$$

Take $Q = \Lambda_{p,i}\phi K$ and note that $\widetilde{W}_{-p,i}(\phi^{-t}\Lambda_{p,i}K, \phi^{-t}\Lambda_{p,i}K) = \widetilde{W}_i(\phi^{-t}\Lambda_{p,i}K)$, it follows that

$$\widetilde{W}_{-p,i}\big(\phi^{-t}\Lambda_{p,i}K, \phi^{-t}\Lambda_{p,i}K\big) = \widetilde{W}_{-p,i}\big(\phi^{-t}\Lambda_{p,i}K, \Lambda_{p,i}\phi K\big). \tag{3.15}$$

Together with (3.15) and Lemma 3.17, we immediately get the result. □

Recall that $\Lambda_{p,i}$ maps B_n, the centered unit ball, into B_n; *i.e.*, $\Lambda_{p,i}B_n = B_n$. Since $(\phi Q)^* = \phi^{-t}Q^*$, for $\phi \in O(n)$ and $Q \in \mathcal{K}_o^n$, Proposition 3.18 shows that if E is a centered ellipsoid and $E = \phi B_n$, then

$$\Lambda_{p,i}E = \Lambda_{p,i}\phi B_n = \phi^{-t}\Lambda_{p,i}B_n = \phi^{-t}B_n = \phi^{-t}B_n^* = (\phi B_n)^* = E^*,$$

namely,

$$\Lambda_{p,i}E = E^*. \tag{3.16}$$

It follows from Proposition 3.16 that for $K \in \mathcal{F}_{i,o}^n$ and $p > 1$, the body $\Lambda_{p,i}K$ is a centered ellipsoid if and only if K is a centered ellipsoid. Define

$$\mathcal{W}_{p,i}^n = \big\{K \in \mathcal{F}_{i,o}^n : \text{ there exists } Q \in \mathcal{K}_o^n \text{ with } f_{p,i}(K,\cdot) = h(Q,\cdot)^{-(n+p-i)}\big\}. \tag{3.17}$$

An immediate consequence of the definition of $\mathcal{W}_{p,i}^n$ and the definition of $\Lambda_{p,i}$ is the following.

Proposition 3.19 *If $p \geq 1$, $i \in \{0,1,\ldots,n-1\}$ and $K \in \mathcal{F}_{i,o}^n$, then*

$$K \in \mathcal{W}_{p,i}^n \quad \textit{if and only if} \quad \Lambda_{p,i}K \in \mathcal{K}_o^n.$$

It follows from Propositions 3.18 and 3.19 that $\mathcal{W}_{p,i}^n$ is an orthogonal transformation invariant class.

Proposition 3.20 *Suppose $K \in \mathcal{F}_{i,o}^n$ and $i \in \{0,1,\ldots,n-1\}$. If $p \geq 1$ and $\phi \in O(n)$, then $K \in \mathcal{W}_{p,i}^n$ if and only if $\phi K \in \mathcal{W}_{p,i}^n$.*

Define

$$\mathcal{E}_i^n = \big\{K \in \mathcal{F}_{i,o}^n : K^* \text{ and } \Lambda_i K \text{ are dilates}\big\}. \tag{3.18}$$

For the case $i = 0$, the subscript i in $\mathcal{E}_i^n$ will often be suppressed. Namely, $\mathcal{E}_0^n = \mathcal{E}^n$. Obviously, $\mathcal{E}_i^n \subset \mathcal{W}_{p,i}^n$ for all $p \geq 1$ and $i \in \{0,1,\ldots,n-1\}$. From Proposition 3.18 and (3.16) it follows that all centered ellipsoids belong to $\mathcal{E}_i^n$. If $K \in \mathcal{E}_i^n$, then from definition (2.21) of the ith p-curvature image, (2.19) and (2.22), and noting that $\Lambda_i K = \lambda K^*$ with arbitrary $\lambda > 0$, we have

$$\begin{aligned}
\rho(\Lambda_{p,i}K,\cdot)^{n+p-i} &= \frac{\widetilde{W}_i(\Lambda_{p,i}K)}{\omega_n} f_{p,i}(K,\cdot) \\
&= \frac{\widetilde{W}_i(\Lambda_{p,i}K)}{\omega_n} h(K,\cdot)^{1-p} f_i(K,\cdot) \\
&= \frac{\widetilde{W}_i(\Lambda_{p,i}K)}{\widetilde{W}_i(\Lambda_i K)} \rho\bigl(K^*,\cdot\bigr)^{p-1} \rho(\Lambda_i K,\cdot)^{n+1-i} \\
&= \frac{\widetilde{W}_i(\Lambda_{p,i}K)}{\widetilde{W}_i(\lambda K^*)} \rho\bigl(K^*,\cdot\bigr)^{p-1} \rho\bigl(\lambda K^*,\cdot\bigr)^{n+1-i} \\
&= \frac{\lambda \widetilde{W}_i(\Lambda_{p,i}K)}{\widetilde{W}_i(K^*)} \rho\bigl(K^*,\cdot\bigr)^{n+p-i}.
\end{aligned}$$

Taking

$$\lambda = \frac{\widetilde{W}_i(K^*)}{\widetilde{W}_i(\Lambda_{p,i}K)} \times \left(\frac{W_i(K)}{\omega_n}\right)^{\frac{n+p-i}{p}},$$

and then

$$\rho(\Lambda_{p,i}K,\cdot)^{n+p-i} = \left(\frac{W_i(K)}{\omega_n}\right)^{\frac{n+p-i}{p}} \rho\bigl(K^*,\cdot\bigr)^{n+p-i},$$

it follows immediately that

$$\Lambda_{p,i}K = \bigl[W_i(K)/\omega_n\bigr]^{1/p} K^*$$

for all $p \geq 1$ and $i \in \{0,1,\ldots,n-1\}$.

On the other hand, if $p \geq 1$, $i \in \{0,1,\ldots,n-1\}$ and the body $K \in \mathcal{F}_{i,o}^n$ is such that $\Lambda_{p,i}K$ and K^* are dilates, then let $\Lambda_{p,i}K = \lambda K^*$ with $\lambda > 0$. From definition (2.21) of the ith p-curvature image, (2.19), (2.20), and (2.1) as well as (3.14), it follows that

$$\begin{aligned}
(2.21) \quad &\Longrightarrow \quad \rho\bigl(\lambda K^*,\cdot\bigr)^{n+p-i} = \frac{\widetilde{W}_i(\lambda K^*)}{\omega_n} f_{p,i}(K,\cdot) \\
&\Longrightarrow \quad \lambda^p \rho\bigl(K^*,\cdot\bigr)^{n+p-i} = \frac{\widetilde{W}_i(K^*)}{\omega_n} h(K,\cdot)^{1-p} f_i(K,\cdot) \\
&\Longrightarrow \quad \lambda^p \rho\bigl(K^*,\cdot\bigr)^{n+1-i} = \frac{\widetilde{W}_i(K^*)}{\omega_n} f_i(K,\cdot) \\
&\Longrightarrow \quad \rho\bigl(\lambda K^*,\cdot\bigr)^{n+1-i} = \frac{\widetilde{W}_i(\lambda K^*)}{\omega_n} f_i\bigl(\lambda^{\frac{1-p}{n-i-1}} K,\cdot\bigr).
\end{aligned} \tag{3.19}$$

Comparing to (2.22) and (3.19) and using Proposition 3.19, we let $Q = \lambda^{\frac{1-p}{n-i-1}}K$ and $\lambda K^* = \Lambda_i Q$. Then from (3.14) we get

$$\lambda K^* = \Lambda_i Q = \Lambda_i\bigl(\lambda^{\frac{1-p}{n-i-1}}K\bigr) = \lambda^{1-p}\Lambda_i K,$$

that is,

$$\Lambda_i K = \lambda^p K^*.$$

Accordingly, $K \in \mathcal{E}_i^n$. Thus, the sets defined for $p \geq 1$ and $i \in \{0,1,\dots,n-1\}$ by $\mathcal{E}_{p,i}^n = \{K \in \mathcal{F}_{i,o}^n : K^*$ and $\Lambda_{p,i}K$ are dilates$\}$ are one and the same. Namely, $\mathcal{E}_{p,i}^n = \mathcal{E}_i^n$ for all $p \geq 1$ and $i \in \{0,1,\dots,n-1\}$.

It is known that if ∂K is a regular C^2 hypersurface and $K \in \mathcal{E}^n$, then K must be an ellipsoid. It is known that if $K \in \mathcal{E}^n$ and K is a body of revolution, then K must be an ellipsoid. It is also known that $\mathcal{E}^2$ consists only of centered ellipses. For all these facts, see Petty [12]. It has been conjectured that $\mathcal{E}^n$ is exactly the class of centered ellipsoids (see [17]). Therefore, we conjecture that $\mathcal{E}_i^n$, $i = 0,1,\dots,n-1$, are exactly the class of centered ellipsoids. None of the facts stated in this paragraph will be used in this article.

For $K \in \mathcal{K}_o^n$, define the ith p-curvature ratio of K as

$$\left(\frac{\omega_n^{n-i}\widetilde{W}_i(\Lambda_{p,i}K)^p}{W_i(K)^{n-p-i}}\right)^{\frac{1}{p}}.$$

Since $K \in \mathcal{E}_i^n$ implies that $\Lambda_{p,i}K = [W_i(K)/\omega_n]^{1/p}K^*$, it follows immediately that the ith p-curvature ratio of $K \in \mathcal{E}_i^n$ equals $W_i(K)\widetilde{W}_i(K^*)$ of K. Namely, if $K \in \mathcal{E}_i^n$, then

$$\left(\frac{\omega_n^{n-i}\widetilde{W}_i(\Lambda_{p,i}K)^p}{W_i(K)^{n-p-i}}\right)^{\frac{1}{p}} = W_i(K)\widetilde{W}_i\bigl(K^*\bigr)$$

for all $p \geq 1$ and $i \in \{0,1,\dots,n-1\}$. The next proposition shows that this characterizes bodies in $\mathcal{E}_i^n$.

Proposition 3.21 *If $p \geq 1$ and $K \in \mathcal{F}_{i,o}^n$ with $i \in \{0,1,\dots,n-1\}$, then*

$$\left(\frac{\omega_n^{n-i}\widetilde{W}_i(\Lambda_{p,i}K)^p}{W_i(K)^{n-p-i}}\right)^{\frac{1}{p}} \leq W_i(K)\widetilde{W}_i\bigl(K^*\bigr) \tag{3.20}$$

with equality if and only if $K \in \mathcal{E}_i^n$.

Proof Taking $Q = K^*$ in Proposition 2.2, we get

$$W_i(K) = \omega_n \widetilde{W}_{-p,i}\bigl(\Lambda_{p,i}K, K^*\bigr)/\widetilde{W}_i(\Lambda_{p,i}K).$$

The dual p-mixed quermassintegrals inequality (2.11) gives

$$W_i(K)^{n-i} \geq \omega_n^{n-i}\widetilde{W}_i(\Lambda_{p,i}K)^p\widetilde{W}_i\bigl(K^*\bigr)^{-p}$$

with equality if and only if $\Lambda_{p,i}K$ and K^* are dilates. □

For bodies with ith continuous curvature functions, the equality conditions for the inequality of Proposition 3.14 are easily obtained by combining Propositions 3.10 and 3.21.

Theorem 3.22 *If $p \geq 1$, $i \in \{0,1,\dots,n-1\}$, and $K \in \mathcal{F}^n_{i,o}$, then*

$$\left(\frac{\Omega_p^{(i)}(K)^{n+p-i}}{n^{n+p-i}W_i(K)^{n-p-i}}\right)^{\frac{1}{p}} \leq W_i(K)\widetilde{W}_i\bigl(K^*\bigr) \tag{3.21}$$

with equality if and only if $K \in \mathcal{E}^n_i$.

4 The cyclic inequalities and monotonicity of *i*th *p*-affine area ratio

Suppose that $1 \leq p < q < r$ and $K, L \in \mathcal{K}^n_o$. Since

$$h_L^q h_K^{1-q} = \bigl[h_L^p h_K^{1-p}\bigr]^{(r-q)/(r-p)}\bigl[h_L^r h_K^{1-r}\bigr]^{(q-p)/(r-p)},$$

the Hölder inequality, together with (2.6) and (2.7), yields the following.

Proposition 4.1 *If $K, L \in \mathcal{K}^n_o$, and $1 \leq p < q < r$, then, for $i \in \{0,1,\dots,n-1\}$,*

$$W_{q,i}(K,L)^{r-p} \leq W_{p,i}(K,L)^{r-q}W_{r,i}(K,L)^{q-p} \tag{4.1}$$

with equality if and only if there exists a constant $c > 0$ such that $h_L = ch_K$ almost everywhere with respect to $S_i(K,\cdot)$.

Suppose $1 \leq p < q$ and $K \in \mathcal{K}^n_o$ with $L \in \mathcal{S}^n_o$. Since

$$\rho_L^{-p}h_K^{1-p} = \bigl[\rho_L^{-q}h_K^{1-q}\bigr]^{p/q}h_K^{(q-p)/q},$$

the Hölder inequality yields the following.

Proposition 4.2 *Suppose $K \in \mathcal{K}^n_o$, $L \in \mathcal{S}^n_o$, and $1 \leq p < q$. Then, for any $i \in \{0,1,\dots,n-1\}$,*

$$\left(\frac{W_{p,i}(K,L^*)}{W_i(K)}\right)^{\frac{1}{p}} \leq \left(\frac{W_{q,i}(K,L^*)}{W_i(K)}\right)^{\frac{1}{q}} \tag{4.2}$$

with equality if and only if there exists a constant $c > 0$ such that $\rho_L = c/h_K$ almost everywhere with respect to $S_i(K,\cdot)$.

Suppose $1 \leq p < q$ and $K \in \mathcal{K}^n_o$ with $L \in \mathcal{S}^n_o$. From the integral representation of $W_{p,i}(K,L^*)$ the easy estimate follows

$$\bigl|W_{p,i}\bigl(K,L^*\bigr) - W_{q,i}\bigl(K,L^*\bigr)\bigr| \leq W_{p,i}\bigl(K,L^*\bigr)\max_{u\in S^{n-1}}\bigl|\bigl[\rho_L(u)h_K(u)\bigr]^{p-q} - 1\bigr|.$$

This gives the following proposition.

Proposition 4.3 *Suppose $K \in \mathcal{K}^n_o$, $L \in \mathcal{S}^n_o$, and $i \in \{0,1,\dots,n-1\}$. The function defined on $[1,\infty)$ by*

$$p \mapsto W_{p,i}\bigl(K,L^*\bigr)$$

is continuous.

From the equality conditions of Proposition 3.21 it follows that if $K \in \mathcal{E}_i^n$, then the ith p-curvature ratios are independent of p. The next proposition provides a strong converse by showing that unless $K \in \mathcal{E}_i^n$, the ith p-curvature ratios are (strictly) monotone increasing in p.

Proposition 4.4 *If $K \in \mathcal{F}_{i,o}^n$, $i \in \{0, 1, \ldots, n-1\}$, and $1 \le p < q$, then*

$$\left(\frac{\omega_n^{n-i}\widetilde{W}_i(\Lambda_{p,i}K)^p}{W_i(K)^{n-p-i}}\right)^{\frac{1}{p}} \le \left(\frac{\omega_n^{n-i}\widetilde{W}_i(\Lambda_{q,i}K)^q}{W_i(K)^{n-q-i}}\right)^{\frac{1}{q}} \tag{4.3}$$

with equality if and only if $K \in \mathcal{E}_i^n$.

Proof From Proposition 2.2, with $\Lambda_{q,i}K$ taken for Q, and Proposition 4.2 it follows that

$$\left(\frac{\omega_n \widetilde{W}_{-p,i}(\Lambda_{p,i}K, \Lambda_{q,i}K)}{W_i(K)\widetilde{W}_i(\Lambda_{p,i}K)}\right)^{\frac{1}{p}} \le \left(\frac{\omega_n \widetilde{W}_{-q,i}(\Lambda_{q,i}K, \Lambda_{q,i}K)}{W_i(K)\widetilde{W}_i(\Lambda_{q,i}K)}\right)^{\frac{1}{q}} = \left(\frac{\omega_n}{W_i(K)}\right)^{\frac{1}{q}}.$$

The dual ith p-mixed quermassintegrals inequality (2.11) now gives the desired inequality and shows that equality implies that $\Lambda_{p,i}K$ and $\Lambda_{q,i}K$ must be dilates. But definition (2.21) of ith p-curvature images and definition (2.22) of ith curvature images, together with (2.19), show that $\Lambda_{p,i}K$ and $\Lambda_{q,i}K$ can be dilates if and only if $K \in \mathcal{E}_i^n$. □

The following cyclic inequality will be needed.

Proposition 4.5 *If $K \in \mathcal{F}_{i,o}^n$, $i \in \{0, 1, \ldots, n-1\}$, and $1 \le p < q < r$, then*

$$\widetilde{W}_i(\Lambda_{q,i}K)^{q(r-p)} \le \widetilde{W}_i(\Lambda_{p,i}K)^{p(r-q)}\widetilde{W}_i(\Lambda_{r,i}K)^{r(q-p)} \tag{4.4}$$

with equality if and only if $K \in \mathcal{E}_i^n$.

Proof From (2.19) it follows that

$$f_{q,i}(K, \cdot)^{r-p} = f_{p,i}(K, \cdot)^{r-q} f_{r,i}(K, \cdot)^{q-p}.$$

Thus definition (2.21) of the ith p-curvature image shows that

$$\begin{aligned}&\widetilde{W}_i(\Lambda_{q,i}K)^{p-r}\rho(\Lambda_{q,i}K, \cdot)^{(n+q-i)(r-p)} \\ &\quad = \widetilde{W}_i(\Lambda_{p,i}K)^{q-r}\rho(\Lambda_{p,i}K, \cdot)^{(n+p-i)(r-q)}\widetilde{W}_i(\Lambda_{r,i}K)^{p-q}\rho(\Lambda_{r,i}K, \cdot)^{(n+r-i)(q-p)}.\end{aligned}$$

The Hölder inequality and formula (2.8) for ith dual quermassintegrals now yield the desired inequality and show that equality is possible if and only if $\Lambda_{p,i}K$ and $\Lambda_{r,i}K$ are dilates, or equivalently, if and only if $K \in \mathcal{E}_i^n$. □

In contrast to the inequality of Proposition 4.4, there is the following proposition.

Proposition 4.6 *If $K \in \mathcal{F}_{i,o}^n$, $i \in \{0, 1, \ldots, n-1\}$, and $1 \le p < q$, then*

$$\left(\frac{\widetilde{W}_i(\Lambda_{q,i}K)}{\widetilde{W}_i(K^*)}\right)^q \le \left(\frac{\widetilde{W}_i(\Lambda_{p,i}K)}{\widetilde{W}_i(K^*)}\right)^p \tag{4.5}$$

with equality if and only if $K \in \mathcal{E}_i^n$.

Proof From (2.19) it follows that

$$f_{q,i}(K,\cdot) = f_{p,i}(K,\cdot)h(K,\cdot)^{-(q-p)}.$$

Definition (2.21) of the ith p-curvature image thus gives

$$\widetilde{W}_i(\Lambda_{q,i}K)^{-1}\rho(\Lambda_{q,i}K,\cdot)^{n+q-i} = \widetilde{W}_i(\Lambda_{p,i}K)^{-1}\rho(\Lambda_{p,i}K,\cdot)^{n+p-i}h(K,\cdot)^{-(q-p)}.$$

The Hölder inequality, together with formula (2.8) for ith dual quermassintegrals, now yields the desired inequality and shows that equality can occur if and only if $\Lambda_{p,i}K$ and K^* are dilates, or equivalently, if and only if $K \in \mathcal{E}_i^n$. □

It turns out that there is an inequality between the ith p-affine surface areas of a convex body that is similar to the classical cyclic inequality between the quermassintegrals of the convex body.

Theorem 4.7 *Suppose $K \in \mathcal{K}_o^n$, $i \in \{0,1,\dots,n-1\}$ and $1 \le p < q < r$. Then*

$$\Omega_q^{(i)}(K)^{(n+q-i)(r-p)} \le \Omega_p^{(i)}(K)^{(n+p-i)(r-q)}\Omega_r^{(i)}(K)^{(n+r-i)(q-p)}.$$

Obviously, the case $i = 0$ of Theorem 4.7 is just the cyclic inequality for p-affine surface areas of a convex body by Lutwak (see [17]).

Proof To show this, define $Q_3 \in \mathcal{S}_o^n$ by

$$\rho_{Q_3}^{q(r-p)} = \rho_{Q_1}^{p(r-q)}\rho_{Q_2}^{r(q-p)}.$$

Since

$$\rho_{Q_3}^{n-i} = \rho_{Q_1}^{\frac{p(r-q)(n-i)}{q(r-p)}}\rho_{Q_2}^{\frac{r(q-p)(n-i)}{q(r-p)}},$$

the Hölder inequality and the dual quermassintegrals formula give

$$\widetilde{W}_i(Q_3)^{q(r-p)} \le \widetilde{W}_i(Q_1)^{p(r-q)}\widetilde{W}_i(Q_2)^{r(q-p)}. \tag{4.6}$$

Since

$$\rho_{Q_3}^{-q}h_K^{1-q} = \left[\rho_{Q_1}^{-p}h_K^{1-p}\right]^{\frac{r-q}{r-p}}\left[\rho_{Q_2}^{-r}h_K^{1-r}\right]^{\frac{q-p}{r-p}},$$

the Hölder inequality, together with (2.6) and (2.7), yields

$$W_{q,i}\bigl(K,Q_3^*\bigr)^{r-p} \le W_{p,i}\bigl(K,Q_1^*\bigr)^{r-q}W_{r,i}\bigl(K,Q_2^*\bigr)^{q-p}. \tag{4.7}$$

Definition (2.21) of the ith p-affine surface area, together with (4.6) and (4.7), yields

$$\Omega_q^{(i)}(K)^{(n+q-i)(r-p)} \le \Omega_p^{(i)}(K)^{(n+p-i)(r-q)}\Omega_r^{(i)}(K)^{(n+r-i)(q-p)}.$$ □

Note that if K is a polytope, then there is equality in the inequality of Theorem 4.7. For bodies with ith continuous curvature functions, the equality conditions of inequality of Theorem 4.7 are easily obtained from Propositions 3.10 and 4.5.

Proposition 4.8 *Suppose $K \in \mathcal{F}^n_{i,o}$, $i \in \{0,1,\ldots,n-1\}$ and $1 \le p < q < r$. Then*

$$\Omega_q^{(i)}(K)^{(n+q-i)(r-p)} \le \Omega_p^{(i)}(K)^{(n+p-i)(r-q)}\Omega_r^{(i)}(K)^{(n+r-i)(q-p)} \tag{4.8}$$

with equality if and only if $K \in \mathcal{E}^n_i$.

For $K \in \mathcal{K}^n_o$, we define the ith p-affine area ratio of K by

$$\left(\frac{\Omega_p^{(i)}(K)^{n+p-i}}{n^{n+p-i}W_i(K)^{n-p-i}}\right)^{\frac{1}{p}}. \tag{4.9}$$

We can rewrite definition (1.7) for $\Omega_p^{(i)}(K)$ of $K \in \mathcal{K}^n_o$ by

$$W_i(K)\left(\frac{\Omega_p^{(i)}(K)}{nW_i(K)}\right)^{\frac{n+p-i}{p}} = \inf\bigl\{\bigl[W_{p,i}\bigl(K,Q^*\bigr)/W_i(K)\bigr]^{\frac{n-i}{p}}\widetilde{W}_i(Q) : Q \in \mathcal{S}^n_o\bigr\}. \tag{4.10}$$

Together with definition (4.10) and Proposition 4.2, the following shows that the ith p-affine area ratios are monotone nondecreasing in p.

Proposition 4.9 *If $K \in \mathcal{K}^n_o$ and $i \in \{0,1,\ldots,n-1\}$, then, for $1 \le p \le q$,*

$$\left(\frac{\Omega_p^{(i)}(K)^{n+p-i}}{n^{n+p-i}W_i(K)^{n-p-i}}\right)^{\frac{1}{p}} \le \left(\frac{\Omega_q^{(i)}(K)^{n+q-i}}{n^{n+q-i}W_i(K)^{n-q-i}}\right)^{\frac{1}{q}}. \tag{4.11}$$

Proof of Theorem 1.9 Note that if K is a polytope, then there is equality in inequality (4.11). For bodies with ith continuous curvature functions, the equality conditions of the inequality of Proposition 4.9 follow directly from Propositions 3.10 and 4.4. This completes the proof of Theorem 1.9. □

In contrast to the inequality of Proposition 4.9, we have the following proposition.

Proposition 4.10 *If $K \in \mathcal{K}^n_o$ and $i \in \{0,1,\ldots,n-1\}$, then, for $1 \le p \le q$,*

$$\left(\frac{\Omega_q^{(i)}(K)}{n\widetilde{W}_i(K^*)}\right)^{n+q-i} \le \left(\frac{\Omega_p^{(i)}(K)}{n\widetilde{W}_i(K^*)}\right)^{n+p-i}. \tag{4.12}$$

Proof The inequality of Proposition 4.10 follows immediately from the definition of ith p-affine surface area once the following fact is established: Given $Q \in \mathcal{S}^n_o$, there exists $\overline{Q} \in \mathcal{S}^n_o$ such that

$$W_{q,i}\bigl(K,\overline{Q}^*\bigr)^{n-i}\frac{\widetilde{W}_i(\overline{Q})^q}{\widetilde{W}_i(K^*)^q} \le W_{p,i}\bigl(K,Q^*\bigr)^{n-i}\frac{\widetilde{W}_i(Q)^p}{\widetilde{W}_i(K^*)^p}. \tag{4.13}$$

To show this, define $\overline{Q} \in \mathcal{S}^n_o$ by

$$\rho_{\overline{Q}} = \bigl[\widetilde{W}_i\bigl(K^*\bigr)^{p-q}\widetilde{W}_i(Q)^{-p}\bigr]^{\frac{1}{q(n-i)}}\rho_Q^{\frac{p}{q}}\rho_{K^*}^{\frac{q-p}{q}}. \tag{4.14}$$

From (4.14) we have

$$\rho_{\overline{Q}}^{-q} h_K^{1-q} = \widetilde{W}_i\big(K^*\big)^{\frac{q-p}{n-i}} \widetilde{W}_i(Q)^{\frac{p}{n-i}} \rho_Q^{-p} h_K^{1-p},$$

the integral representation of $W_{p,i}(K, Q^*)$ shows that

$$W_{q,i}\big(K, \overline{Q}^*\big) = \widetilde{W}_i\big(K^*\big)^{\frac{q-p}{n-i}} \widetilde{W}_i(Q)^{\frac{p}{n-i}} W_{p,i}\big(K, Q^*\big). \tag{4.15}$$

The definition of $\overline{Q}$, together with the Hölder inequality and the formula for ith dual quermassintegrals, shows that

$$\begin{aligned}\widetilde{W}_i(\overline{Q}) &= \widetilde{W}_i\big(K^*\big)^{\frac{p-q}{q}} \widetilde{W}_i(Q)^{-\frac{p}{q}} \left[\frac{1}{n}\int_{\mathbb{S}^{n-1}} \rho_Q(u)^{\frac{(n-i)p}{q}} \rho_{K^*}(u)^{\frac{(n-i)(q-p)}{q}}\, dS(u)\right] \\ &\le \widetilde{W}_i\big(K^*\big)^{\frac{p-q}{q}} \widetilde{W}_i(Q)^{-\frac{p}{q}} \left(\frac{1}{n}\int_{\mathbb{S}^{n-1}} \rho_Q^{n-i}(u)\, dS(u)\right)^{\frac{p}{q}} \left(\int_{\mathbb{S}^{n-1}} \rho_{K^*}^{n-i}(u)\, dS(u)\right)^{\frac{q-p}{q}} \\ &= 1.\end{aligned} \tag{4.16}$$

Together with (4.15) and (4.16), we show that (4.13), and this completes the argument. □

If K is a polytope there is equality in the inequality of Proposition 4.10. For bodies with ith continuous curvature functions, the equality conditions in inequality (4.12) follow immediately from Propositions 3.10 and 4.6.

Theorem 4.11 *If $K \in \mathcal{F}_{i,o}^n$ and $i \in \{0, 1, \ldots, n-1\}$, then, for $1 \le p < q$,*

$$\left(\frac{\Omega_q^{(i)}(K)}{n\widetilde{W}_i(K^*)}\right)^{n+q-i} \le \left(\frac{\Omega_p^{(i)}(K)}{n\widetilde{W}_i(K^*)}\right)^{n+p-i} \tag{4.17}$$

with equality if and only if $K \in \mathcal{E}_i^n$.

An immediate consequence of Propositions 4.9 and 4.10 is the following.

Proposition 4.12 *If $K \in \mathcal{K}_o^n$, $i \in \{0, 1, \ldots, n-1\}$, and $\Omega_p^{(i)} = 0$ for some $p \in [1, \infty)$, then $\Omega_p^{(i)} = 0$ for all p.*

The cyclic inequality of Theorem 4.7 shows that the function defined on $[1, \infty)$ by

$$p \mapsto (n + p - i) \log \Omega_p^{(i)}(K)$$

is convex. The continuity of this function on $[1, \infty)$ follows from this and Proposition 4.9. The continuity of this function immediately gives the following.

Proposition 4.13 *If $K \in \mathcal{K}_o^n$ and $i \in \{0, 1, \ldots, n-1\}$, then the function defined on $[1, \infty)$ by*

$$p \mapsto \Omega_p^{(i)}(K)$$

is continuous.

5 Extremal *i*th affine surface area

Define the generalized Santaló product of $K \in \mathcal{K}_o^n$ by $W_i(K)\widetilde{W}_i(K^*)$. Proposition 4.9 states that, for $K \in \mathcal{K}_o^n$, the ith p-affine area ratio

$$\left(\frac{\Omega_p^{(i)}(K)^{n+p-i}}{n^{n+p-i}W_i(K)^{n-p-i}}\right)^{\frac{1}{p}}$$

is monotone nondecreasing in p, and Theorem 3.22 states that this ratio is bounded by the generalized Santaló product of K.

In order to facilitate the formulation of the ith p-affine area ratio for the case $p = \infty$, it will be helpful to introduce a quermassintegrals-normalized version of ith p-mixed quermassintegrals. If K, L are convex bodies that contain the origin in their interiors, then for each real $p > 0$ define

$$\overline{W}_{p,i}(K,L) = \left(\frac{W_{p,i}(K,L)}{W_i(K)}\right)^{\frac{1}{p}} = \left[\frac{1}{nW_i(K)}\int_{S^{n-1}}\left(\frac{h_L(u)}{h_K(u)}\right)^p h_K(u)\,dS_i(K,u)\right]^{\frac{1}{p}},$$

and for $p = \infty$ define

$$\begin{aligned}\overline{W}_{\infty,i}(K,L) &= \lim_{p\to\infty}\left(\frac{W_{p,i}(K,L)}{W_i(K)}\right)^{\frac{1}{p}}\\ &= \max\left\{\frac{h_L(u)}{h_K(u)} : u \in \operatorname{supp} S_i(K,\cdot)\right\}.\end{aligned} \tag{5.1}$$

Note that $\frac{1}{n}h_K\,dS_i(K,\cdot)/W_i(K) = \frac{1}{n}h_K^{1-i}\,dS(K,\cdot)/W_i(K)$ is a probability measure on supp $S_i(K,\cdot)$ (or $S(K,\cdot)$).

According to (5.1), we can define ith ∞-mixed quermassintegrals, $W_{\infty,i}(K,L)$, of $K, L \in \mathcal{S}_o^n$ by

$$W_{\infty,i}(K,L) = W_i(K)\overline{W}_{\infty,i}(K,L). \tag{5.2}$$

From definition (1.7) of ith p-affine surface area $\Omega_p^{(i)}(K)$, definition (5.1) of $\overline{W}_{\infty,i}(K,L)$ and definition (5.2) of $W_{\infty,i}(K,L)$, we have

$$\begin{aligned}\lim_{p\to\infty}\left(\frac{\Omega_p^{(i)}(K)^{n+p-i}}{n^{n+p-i}W_i(K)^{n-p-i}}\right)^{\frac{1}{p}} &= \lim_{p\to\infty}\left(\frac{\inf_{Q\in\mathcal{S}_o^n}\{n^{n+p-i}W_{p,i}(K,Q^*)^{n-i}\widetilde{W}_i(Q)\}}{n^{n+p-i}W_i(K)^{n-p-i}}\right)^{\frac{1}{p}}\\ &= \inf_{Q\in\mathcal{S}_o^n}\left\{\lim_{p\to\infty}\left(\frac{W_{p,i}(K,Q^*)}{W_i(K)}\right)^{\frac{n-i}{p}}\lim_{p\to\infty}\frac{\widetilde{W}_i(Q)}{W_i(K)^{-1}}\right\}\\ &= \inf_{Q\in\mathcal{S}_o^n}\left\{\overline{W}_{\infty,i}\big(K,Q^*\big)^{n-i}\cdot\frac{\widetilde{W}_i(Q)}{W_i(K)^{-1}}\right\}\\ &= \frac{\inf_{Q\in\mathcal{S}_o^n}\{n^{n+1-i}W_{\infty,i}(K,Q^*)^{n-i}\widetilde{W}_i(Q)\}}{n^{n+1-i}W_i(K)^{n-1-i}}.\end{aligned}$$

Therefore, we can define ith ∞-affine surface area $\Omega_\infty^{(i)}(K)$ of $K \in \mathcal{K}_o^n$ by

$$n^{-\frac{1}{n-i}}\Omega_\infty^{(i)}(K)^{\frac{n+1-i}{n-i}} = \inf\left\{nW_{\infty,i}\big(K,Q^*\big)\widetilde{W}_i(Q)^{\frac{1}{n-i}} : Q \in \mathcal{S}_o^n\right\}.$$

Then

$$\frac{\Omega_{\infty}^{(i)}(K)^{n+1-i}}{n^{n+1-i}W_i(K)^{n-1-i}} = \lim_{p\to\infty}\left(\frac{\Omega_p^{(i)}(K)^{n+p-i}}{n^{n+p-i}W_i(K)^{n-p-i}}\right)^{\frac{1}{p}}. \tag{5.3}$$

An immediate consequence of Proposition 3.3 and the definition of $\Omega_{\infty}^{(i)}(K)$ is that $\Omega_{\infty}^{(i)}(K)$ is invariant under orthogonal transformations of K.

Proposition 5.1 *If $i \in \{0,1,\ldots n-1\}$ and $K \in \mathcal{K}_o^n$, then*

$$\Omega_{\infty}^{(i)}(\phi K) = \Omega_{\infty}^{(i)}(K)$$

for all $\phi \in O(n)$.

An immediate consequence of Proposition 3.4 and the definition of $\Omega_{\infty}^{(i)}$ is as follows.

Proposition 5.2 *If $p \geq 1$ and $P \in \mathcal{K}_o^n$ is a polytope, then $\Omega_{\infty}^{(i)}(P) = 0$ for any $i \in \{0,1,\ldots, n-1\}$.*

From Proposition 4.9 and the definition of $\Omega_{\infty}^{(i)}$ the proposition follows.

Proposition 5.3 *If $K \in \mathcal{K}_o^n$ and $p \geq 1$, $i \in \{0,1,\ldots n-1\}$, then*

$$\left(\frac{\Omega_p^{(i)}(K)^{n+p-i}}{n^{n+p-i}W_i(K)^{n-p-i}}\right)^{\frac{1}{p}} \leq \frac{\Omega_{\infty}^{(i)}(K)^{n+1-i}}{n^{n+1-i}W_i(K)^{n-1-i}}.$$

If K has the ith continuous curvature function, then the equality conditions in Proposition 5.3 are easily obtained. Note that from Theorem 1.9 it follows that if $K \in \mathcal{F}_{i,o}^n \setminus \mathcal{E}_i^n$, then the limit

$$\lim_{p\to\infty}\left(\frac{\Omega_p^{(i)}(K)^{n+p-i}}{n^{n+p-i}W_i(K)^{n-p-i}}\right)^{\frac{1}{p}} = \frac{\Omega_{\infty}^{(i)}(K)^{n+1-i}}{n^{n+1-i}W_i(K)^{n-1-i}}$$

is the limit of a strictly increasing function of p. Hence, from Theorem 1.9 and the definition of $\Omega_{\infty}^{(i)}$, the proposition follows.

Proposition 5.4 *If $p \geq 1$, $i \in \{0,1,\ldots n-1\}$ and $K \in \mathcal{F}_{i,o}^n$, then*

$$\left(\frac{\Omega_p^{(i)}(K)^{n+p-i}}{n^{n+p-i}W_i(K)^{n-p-i}}\right)^{\frac{1}{p}} \leq \frac{\Omega_{\infty}^{(i)}(K)^{n+1-i}}{n^{n+1-i}W_i(K)^{n-1-i}} \tag{5.4}$$

with equality if and only if $K \in \mathcal{E}_i^n$.

From Proposition 3.14 and the definition of $\Omega_{\infty}^{(i)}$ we have

Proposition 5.5 *If $K \in \mathcal{K}_o^n$ and $i \in \{0,1,\ldots,n-1\}$, then*

$$\frac{\Omega_{\infty}^{(i)}(K)^{n+1-i}}{n^{n+1-i}W_i(K)^{n-1-i}} \leq W_i(K)\widetilde{W}_i\big(K^*\big). \tag{5.5}$$

This immediately yields

Proposition 5.6 *Suppose $K \in \mathcal{K}_o^n$ and $i \in \{0,1,\ldots,n-1\}$, then*

$$\Omega_\infty^{(i)}(K)\Omega_\infty^{(i)}\big(K^*\big) \le n^2 W_i(K) W_i\big(K^*\big). \tag{5.6}$$

The inequality of Proposition 5.5 compares $W_i(K)\widetilde{W}_i(K^*)$ and $\Omega_\infty^{(i)}(K)$ for $K \in \mathcal{K}_o^n$. The next proposition shows that for an important class of bodies, these quantities are the same.

Proposition 5.7 *If $i \in \{0,1,\ldots,n-1\}$ and $K \in \mathcal{F}_{i,o}^n$, then*

$$\frac{\Omega_\infty^{(i)}(K)^{n+1-i}}{n^{n+1-i} W_i(K)^{n-1-i}} = W_i(K)\widetilde{W}_i\big(K^*\big). \tag{5.7}$$

Proof Since $f_{p,i}(K,\cdot) = h_K^{1-p} f_i(K,\cdot)$, and h_K and $f_i(K,\cdot)$ are positive continuous functions, it is easily seen that

$$\lim_{p\to\infty} f_{p,i}(K,\cdot)^{\frac{n-i}{n+p-i}} = h_K^{-(n-i)} \quad \text{uniformly on } \mathbb{S}^{n-1}.$$

The formula for the ith dual quermassintegrals, together with the integral representation of Theorem 1.4, now yields the desired result. □

Proposition 5.7 shows that when restricted to $\mathcal{F}_{i,o}^n$, the function $\Omega_\infty^{(i)} : \mathcal{F}_{i,o}^n \to (0,\infty)$ is continuous.

When Theorem 1.9 and Proposition 5.4 are combined with Proposition 5.7, result is that for $K \in \mathcal{F}_{i,o}^n$ and $1 \le p \le q$,

$$\begin{aligned}\frac{\Omega^{(i)}(K)^{n+1-i}}{n^{n+1-i} W_i(K)^{n-1-i}} &\le \left(\frac{\Omega_p^{(i)}(K)^{n+p-i}}{n^{n+p-i} W_i(K)^{n-p-i}}\right)^{\frac{1}{p}} \le \left(\frac{\Omega_q^{(i)}(K)^{n+q-i}}{n^{n+q-i} W_i(K)^{n-q-i}}\right)^{\frac{1}{q}} \\ &\le \frac{\Omega_\infty^{(i)}(K)^{n+1-i}}{n^{n+1-i} W_i(K)^{n-1-i}} = W_i(K)\widetilde{W}_i\big(K^*\big).\end{aligned} \tag{5.8}$$

Finally, we propose the following open question.

Conjecture 5.8 *Suppose $K \in \mathcal{F}_{i,o}^n$, $i \in \{0,1,\ldots,n-1\}$ and $p \ge 1$. Does it follow that*

$$\Omega_p^{(i)}(K)^{n+p-i} \le n^{n+p-i}\omega_n^{2p} W_i(K)^{n-p-i}? \tag{5.9}$$

with equality in inequality for $i = 0$ if and only if K is an ellipsoid, and for $0 < i \le n-1$ if and only if K is a ball.

Obviously, the case $i = 0$ of Conjecture 5.8 is just the p-affine isoperimetric inequality by Lutwak (see [17]).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors completed the paper and read and approved the final manuscript.

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