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On interval-valued invex mappings and optimality conditions for interval-valued optimization problems

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Abstract

In this paper, we first introduce the concept of interval-valued invex mappings by using gH-differentiability and compare it with interval-valued weakly invex mappings. We can observe that interval-valued invex mappings are more general than interval-valued weakly invex mappings. In addition, the sufficient optimality condition for interval-valued objective functions is derived under invexity.

Keywords: interval-valued optimization; interval-valued invex mappings; sufficiency

1 Introduction

Convexity plays a vital role in many aspects of mathematical programming including, for example, sufficient optimality conditions and duality theorems. In inequality constrained optimization, the Kuhn-Tucker conditions are sufficient for optimality if the functions involved are convex. However, application of the Kuhn-Tucker conditions as sufficient conditions for optimality is not restricted to convex problems, and various generalizations of convexity have been made in order to explore the extent of this applicability.

An invex function, introduced by Hanson [1], is one of the generalized convex functions. He considered a differentiable function $f : R^n \rightarrow R$ for which there exists a vector-valued function $\eta : R^n \times R^n \rightarrow R^n$ such that, for all $x, y \in R^n$, the inequality

$$f(x) - f(y) \geq \nabla f(y)^t \eta(x, y) \quad (1)$$

holds. Hanson [1] proved that if, instead of the usual convexity conditions, the objective function and each of the constraints of a nonlinear constrained optimization problem are all invex for the same η , then both the sufficiency of Kuhn-Tucker conditions and weak and strong Wolfe duality still hold. Later, Craven [2] named functions satisfying (1) invex (with respect to η).

Ben-Israel and Mond [3] considered the preinvex function f with respect to η (not necessarily differentiable) for which there exists a vector-valued function $\eta : R^n \times R^n \rightarrow R^n$ such that, for all $x, y \in R^n$, the inequality

$$f(y + \lambda \eta(x, y)) \leq \lambda f(x) + (1 - \lambda) f(y) \quad (2)$$

holds. Moreover, they found that differentiable functions satisfying (2) satisfy (1). Further properties and applications of preinvexity and its some generalizations for some more general problems were studied by Antczak [4, 5], Bector *et al.* [6], Mohan and Neogy [7], Suneja *et al.* [8], and others.

However, the majority of real world optimization problems often involve data uncertainty or imprecision owing to measurement errors or some unexpected things. Interval-valued optimization [9] is an important model to deal with the problems with data uncertainty. Many approaches to interval-valued optimization problems have been explored in considerable details (see, for example, [10–12]). Recently, Wu has extended the concept of convexity for a real-valued function to LU-convexity for an interval-valued function, then he has established the Kuhn-Tucker conditions [13, 14] for an optimization problem with an interval-valued objective function under the assumption of LU-convexity. In [15], Wu studied the Kuhn-Tucker optimality conditions in multiobjective programming problems with an interval-valued objective function. Similar to the concept of non-dominated solution in vector optimization problems, Wu has proposed a solution concept in optimization problems with an interval-valued objective function based on a partial ordering on the set of all closed intervals. Then, the interval-valued Wolfe duality theory [16] and Lagrangian duality theory [17] for interval-valued optimization problems have been proposed. Wu [18] studied the duality theory for interval-valued linear programming problems. Chalco-Cano *et al.* [19] gave Kuhn-Tucker type optimality conditions, which are obtained using gH-derivative of interval-valued functions. Also, they discussed the relationship between the approach presented with other well-known approaches given by Wu [13]. However, these methods given by Chalco-Cano *et al.* [19] cannot solve a kind of optimization problems with interval-valued objective functions, which are not LU-convex but invex. For example, the interval-valued functions such as $f(x) = [x_1 - 2 \sin x_2, x_1 - \sin x_2 + 1]$ are not LU-convex but invex with respect to

$$\eta(x, y) = \left(\frac{\sin x_1 - \sin y_1}{\cos y_1}, \frac{\sin x_2 - \sin y_2}{\cos y_2} \right)^t$$

(the concept of invex can be seen in Definition 11). Zhang *et al.* [20] proposed the Kuhn-Tucker optimality conditions for an optimization problem with an interval-valued objective function under the assumptions of preinvexity and weak invexity. The definition of interval-valued invexity in this paper is more general than that of weak invexity given in [20] (see Theorem 4 and Example 4).

This paper aims at extending the Kuhn-Tucker optimality conditions to nonconvex optimization problem. First, we extend the concept of invexity using gH-derivative of interval-valued functions. The concept of invexity by using gH-differentiability of interval-valued functions is more general than the concept of invexity by using weak differentiability (see Theorem 4 and Example 4). Second, we present several properties of invex interval-valued functions. Finally, the Kuhn-Tucker optimality conditions are proposed for an interval-valued objective function under the assumptions of invexity.

2 Preliminaries

Let us denote by \mathcal{I} the class of all closed intervals in R . $A = [a^L, a^U] \in \mathcal{I}$ denotes a closed interval, where a^L and a^U mean the lower and upper bounds of A respectively. For every $a \in R$, we denote $a = [a, a]$.

Definition 1 Let $A = [a^L, a^U]$ and $B = [b^L, b^U]$ be in \mathcal{I} . We define

- (i) $A + B = \{a + b : a \in A \text{ and } b \in B\} = [a^L + b^L, a^U + b^U]$;
- (ii) $-A = \{-a : a \in A\} = [-a^U, -a^L]$;
- (iii) $A \times B = \{ab : a \in A \text{ and } b \in B\} = [\min_{ab}, \max_{ab}]$, where $\min_{ab} = \min\{a^L b^L, a^L b^U, a^U b^L, a^U b^U\}$ and $\max_{ab} = \max\{a^L b^L, a^L b^U, a^U b^L, a^U b^U\}$.

Then it is easy to conclude that

$$\begin{aligned}
 A - B &= A + (-B) = [a^L - b^U, a^U - b^L], \\
 kA &= \{ka : a \in A\} = \begin{cases} [ka^L, ka^U] & \text{if } k \geq 0, \\ |k|[-a^U, -a^L] & \text{if } k < 0, \end{cases} \tag{3}
 \end{aligned}$$

where k is a real number.

Hausdorff metric between two closed intervals A and B defined as

$$d_H(A, B) = \max\{|a^L - b^L|, |a^U - b^U|\}.$$

Definition 2 Let $A = [a^L, a^U]$ and $B = [b^L, b^U]$ in \mathcal{I} . We write $A \leq B$ if $a^L \leq b^L$ and $a^U \leq b^U$, $A < B$ if $A \leq B$ and $A \neq B$, i.e., the following (a1) or (a2), or (a3) is satisfied.

- (a1) $a^L < b^L$ and $a^U \leq b^U$;
- (a2) $a^L \leq b^L$ and $a^U < b^U$;
- (a3) $a^L < b^L$ and $a^U < b^U$.

Let $A, B \in \mathcal{I}$, if there exists $C \in \mathcal{I}$ such that $A = B + C$, then C is called the Hukuhara difference of A and B and written as $C = A \ominus B$; when we say that the H-difference C exists, it means that $a^L - b^L \leq a^U - b^U$ and $C = [a^L - b^L, a^U - b^U]$.

Proposition 1 Let $A = [a^L, a^U]$ and $B = [b^L, b^U]$ be two closed intervals in \mathcal{I} . If $a^L - b^L \leq a^U - b^U$, then the H-difference C exists and $C = [a^L - b^L, a^U - b^U]$.

It follows from Proposition 1 that the H-difference is unique, but it does not always exist. To address this issue, a generalization of the Hukuhara difference is proposed in [21].

Definition 3 ([21]) Let $A = [a^L, a^U]$ and $B = [b^L, b^U]$ be two closed intervals, the gH-difference of A and B is defined by

$$[a^L, a^U] \ominus_g [b^L, b^U] = [\min(a^L - b^L, a^U - b^U), \max(a^L - b^L, a^U - b^U)].$$

For example, $[1, 3] \ominus_g [0, 3] = [0, 1]$, $[0, 3] \ominus_g [1, 3] = [-1, 0]$. And $a - b = [a, a] \ominus_g [b, b] = [a - b, a - b] = a - b$.

Proposition 2 ([21])

- (i) For every pair $A, B \in \mathcal{I}$, $A \ominus_g B$ always exists and $A \ominus_g B \in \mathcal{I}$.
- (ii) $A \ominus_g B \leq 0$ if and only if $A \leq B$.

The function $f : R^n \rightarrow \mathcal{I}$ defined on the Euclidean space R^n is called an *interval-valued function*, i.e., $f(x) = f(x_1, \dots, x_n)$ is a closed interval in R for each $x \in R^n$. f can be also

written as $f(x) = [f^L(x), f^U(x)]$, where f^L and f^U are two real-valued functions defined on R^n and satisfy $f^L(x) \leq f^U(x)$ for every $x \in R^n$. Based on the above concept, Wu [13] has introduced the concepts of limit, continuity and two kinds of differentiation of interval-valued functions.

Let f be an interval-valued function defined on R^n and $A = [a^L, a^U]$ be an interval in R , $c \in R^n$. If for every $\epsilon > 0$ there exists $\delta > 0$ such that, for $0 < \|x - c\| < \delta$, we have $d_H(f(x), A) < \epsilon$, then

$$\lim_{x \rightarrow c} f(x) = A.$$

Proposition 3 ([13]) *Let f be an interval-valued function defined on R^n and $A = [a^L, a^U]$ be an interval in R . Then $\lim_{x \rightarrow c} f(x) = A$ if and only if $\lim_{x \rightarrow c} f^L(x) = a^L$ and $\lim_{x \rightarrow c} f^U(x) = a^U$.*

Proposition 4 ([13]) *Let f be an interval-valued function defined on R^n . Then f is continuous at $c \in R^n$ if and only if both f^L and f^U are continuous at c .*

Proposition 5 ([13]) *Let X be an open set in R . An interval-valued function $f : X \rightarrow \mathcal{I}$ with $f(x) = [f^L(x), f^U(x)]$ is called weakly differentiable at x_0 if the real-valued functions f^L and f^U are differentiable at x_0 (in the usual sense).*

Definition 4 ([13]) Let X be an open set in R . An interval-valued function $f : X \rightarrow \mathcal{I}$ is called H-differentiable at x_0 if there exists a closed interval $A(x_0) \in \mathcal{I}$ such that the limits

$$\lim_{h \rightarrow 0^+} \frac{f(x_0 + h) \ominus f(x_0)}{h}$$

and

$$\lim_{h \rightarrow 0^+} \frac{f(x_0) \ominus f(x_0 - h)}{h}$$

both exist and equal $A(x_0)$. In this case, $A(x_0)$ is called the H-derivative of f at x_0 .

The following concept is particularization of the fuzzy concepts presented in [22] to the interval case. These are defined by using the usual Hukuhara difference \ominus .

Definition 5 ([22]) Let $T = (a, b)$ and let $t_0 \in T$. Given $f : T \rightarrow \mathcal{I}$, we say that f is strongly generalized differentiable (G-differentiable) at t_0 if there exists an element $f'(t_0) \in \mathcal{I}$ such that for all $h > 0$ sufficiently small,

- (i) $\exists f(x_0 + h) \ominus f(x_0), f(x_0) \ominus f(x_0 - h)$ and

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \rightarrow 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0),$$

or

- (ii) $\exists f(x_0) \ominus f(x_0 + h), f(x_0 - h) \ominus f(x_0)$ and

$$\lim_{h \rightarrow 0} \frac{f(x_0) \ominus f(x_0 + h)}{-h} = \lim_{h \rightarrow 0} \frac{f(x_0 - h) \ominus f(x_0)}{-h} = f'(x_0),$$

or

(iii) $\exists f(x_0 + h) \ominus f(x_0), f(x_0 - h) \ominus f(x_0)$ and

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \rightarrow 0} \frac{f(x_0 - h) \ominus f(x_0)}{-h} = f'(x_0),$$

or

(iv) $\exists f(x_0) \ominus f(x_0 + h), f(x_0) \ominus f(x_0 - h)$ and

$$\lim_{h \rightarrow 0} \frac{f(x_0) \ominus f(x_0 + h)}{-h} = \lim_{h \rightarrow 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0).$$

Based on the gH-difference, Stefanini [21] proposed the following differentiation.

Definition 6 ([21]) Let $x_0 \in (a, b)$ and h be such that $x_0 + h \in (a, b)$, then the gH-derivative of a function $f : (a, b) \rightarrow \mathcal{I}$ at x_0 is defined as

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) \ominus_g f(x_0)}{h}. \tag{4}$$

If $f'(x_0) \in \mathcal{I}$ satisfying (4) exists, we say that f is generalized Hukuhara differentiable (gH-differentiable for short) at x_0 .

The next two results express the gH-derivative in terms of the endpoints of the interval-valued function.

Theorem 1 ([23]) Let $f : (a, b) \rightarrow \mathcal{I}$ be such that $f(x) = [f^L(x), f^U(x)]$. If $f^L(x)$ and $f^U(x)$ are differentiable functions at $t \in (a, b)$, then $f(x)$ is gH-differentiable at t , and

$$f'(x) = [\min\{(f^L)'(x), (f^U)'(x)\}, \max\{(f^L)'(x), (f^U)'(x)\}].$$

Theorem 2 ([23]) Let $f : (a, b) \rightarrow \mathcal{I}$ be such that $f(x) = [f^L(x), f^U(x)]$. Then $f(x)$ is gH-differentiable at $t \in (a, b)$ if and only if one of the following cases holds:

- (a) $f^L(x)$ and $f^U(x)$ are differentiable at t ;
- (b) the lateral derivatives $(f^L)'_-(t), (f^L)'_+(t)$ and $(f^U)'_-(t), (f^U)'_+(t)$ exist and satisfy $(f^L)'_-(t) = (f^U)'_+(t)$ and $(f^L)'_+(t) = (f^U)'_-(t)$.

Let f be an interval-valued function defined on $X \subseteq R^n$, comparing above definitions, the following statements hold.

Proposition 6

- (i) If f is H-differentiable at $x_0 \in X$, then it is G-differentiable at x_0 , the converse is not true.
- (ii) If f is G-differentiable at $x_0 \in X$, then it is gH-differentiable at x_0 , the converse is not true.
- (iii) If f is weakly differentiable at x_0 , then it is gH-differentiable at x_0 , the converse is not true.

Definition 7 ([24]) Let $f(x)$ be an interval-valued function defined on Ω , where Ω is an open subset of R^n . Let D_{x_i} ($i = 1, 2, \dots, n$) stand for the partial differentiation with respect

to the i th variable x_i . Assume that $f^L(x)$ and $f^U(x)$ have continuous partial derivatives so that $D_{x_i}f^L(x)$ and $D_{x_i}f^U(x)$ are continuous. For $i = 1, 2, \dots, n$, define

$$D_{x_i}f(x) = [\min(D_{x_i}f^L(x), D_{x_i}f^U(x)), \max(D_{x_i}f^L(x), D_{x_i}f^U(x))],$$

we will say that $f(x)$ is differentiable at x , and we write

$$\nabla f(x) = (D_{x_1}f(x), D_{x_2}f(x), \dots, D_{x_n}f(x))^t.$$

We call $\nabla f(x)$ the gradient of the interval-valued function at x .

Example 1 Let $f : R^2 \rightarrow \mathcal{I}$ be defined by $f(x) = [x_1^2 + x_2^2, 2x_1^2 + 2x_2^2 + 3]$. So $f^L(x) = x_1^2 + x_2^2$ and $f^U(x) = 2x_1^2 + 2x_2^2 + 3$. $D_{x_1}f^L(x) = 2x_1$, $D_{x_2}f^L(x) = 2x_2$, $D_{x_1}f^U(x) = 4x_1$, $D_{x_2}f^U(x) = 4x_2$. Thus,

$$D_{x_1}f(x) = \begin{cases} [2x_1, 4x_1] & \text{if } x_1 \geq 0, \\ [4x_1, 2x_1] & \text{if } x_1 < 0, \end{cases} \tag{5}$$

$$D_{x_2}f(x) = \begin{cases} [2x_2, 4x_2] & \text{if } x_2 \geq 0, \\ [4x_2, 2x_2] & \text{if } x_2 < 0. \end{cases} \tag{6}$$

Thus,

$$\nabla f(x) = \begin{cases} ([2x_1, 4x_1], [2x_2, 4x_2])^t & \text{if } x_1 \geq 0, x_2 \geq 0, \\ ([2x_1, 4x_1], [4x_2, 2x_2])^t & \text{if } x_1 \geq 0, x_2 < 0, \\ ([4x_1, 2x_1], [2x_2, 4x_2])^t & \text{if } x_1 < 0, x_2 \geq 0, \\ ([4x_1, 2x_1], [4x_2, 4x_2])^t & \text{if } x_1 < 0, x_2 < 0. \end{cases} \tag{7}$$

Further,

$$\nabla^L f(x) = \begin{cases} (2x_1, 2x_2)^t & \text{if } x_1 \geq 0, x_2 \geq 0, \\ (2x_1, 4x_2)^t & \text{if } x_1 \geq 0, x_2 < 0, \\ (4x_1, 2x_2)^t & \text{if } x_1 < 0, x_2 \geq 0, \\ (4x_1, 4x_2)^t & \text{if } x_1 < 0, x_2 < 0, \end{cases} \tag{8}$$

$$\nabla^U f(x) = \begin{cases} (4x_1, 4x_2)^t & \text{if } x_1 \geq 0, x_2 \geq 0, \\ (4x_1, 2x_2)^t & \text{if } x_1 \geq 0, x_2 < 0, \\ (2x_1, 4x_2)^t & \text{if } x_1 < 0, x_2 \geq 0, \\ (2x_1, 4x_2)^t & \text{if } x_1 < 0, x_2 < 0. \end{cases} \tag{9}$$

3 Preinvexity and invexity of interval-valued functions

The concept of convexity plays an important role in the optimization theory. In recent years, the concept of convexity has been generalized in several directions. An important generalization of convex functions is a preinvex function, which was introduced by Weir and Mond [25]. The concepts of preinvexity and invexity have been extended to interval-valued functions by Zhang *et al.* [20], and the Kuhn-Tucker optimality conditions have been derived for preinvex and invex optimization problems with an interval-valued objective function under the conditions of weakly continuous differentiability and Hukuhara differentiability.

In what follows, we show the connection between preinvex and invex interval-valued mappings. Here, we recall the definition of preinvex interval-valued mappings.

Definition 8 Let $y \in X \subseteq R^n$. Then we say that X is invex at y with respect to $\eta : X \times X \rightarrow R^n$ if for each $x \in X, \lambda \in [0, 1], y + \lambda\eta(x, y) \in X$. X is said to be an invex set with respect to η if X is invex at each $y \in X$.

Definition 9 ([20]) Let $K \subseteq R^n$ be an invex set with respect to $\eta : K \times K \rightarrow R^n, f(x) = [f^L(x), f^U(x)]$ be an interval-valued function defined on K . We say that f is *preinvex* at x^* with respect to η if

$$f(x + \lambda\eta(x^*, x)) \preceq \lambda f(x^*) + (1 - \lambda)f(x)$$

for each $\lambda \in [0, 1]$ and each $x \in K$.

Theorem 3 Let K be an invex subset of R^n with respect to $\eta : K \times K \rightarrow R^n$ and f be an interval-valued function defined on K . Then f is preinvex at x^* if and only if f^L and f^U are preinvex at x^* with respect to the same $\eta : K \times K \rightarrow R^n$, i.e.,

$$f^L(x + \lambda\eta(x^*, x)) \leq \lambda f^L(x^*) + (1 - \lambda)f^L(x), \tag{10}$$

$$f^U(x + \lambda\eta(x^*, x)) \leq \lambda f^U(x^*) + (1 - \lambda)f^U(x) \tag{11}$$

for each $\lambda \in [0, 1]$ and each $x \in K$.

Definition 10 ([20]) Let $K \subseteq R^n$ be an invex set with respect to $\eta : K \times K \rightarrow R^n, f(x) = [f^L(x), f^U(x)]$ be an interval-valued function defined on K . We say that f is *invex* at x^* if the real-valued functions f^L and f^U are invex at x^* , i.e.,

$$f^L(x) - f^L(x^*) \geq \eta(x, x^*)^t \nabla f^L(x^*), \tag{12}$$

$$f^U(x) - f^U(x^*) \geq \eta(x, x^*)^t \nabla f^U(x^*) \tag{13}$$

for each $x \in K$.

Remark 1 Since the definition of interval-valued invex functions defined in [20] considered the end-point functions, we call them weakly invex functions in this paper.

Based on the gH-differentiability, we give the definition of interval-valued invex functions as follows.

Definition 11 A gH-differentiable interval-valued mapping $f : X \rightarrow \mathcal{I}$ is said to be invex on the invex set $X \subseteq R^n$ with respect to η if for any $x, y \in X, \lambda \in [0, 1]$,

$$f(x) \ominus_g f(x^*) \geq \eta(x, x^*)^t \nabla f(x^*) \tag{14}$$

for each $x \in K$.

Example 2 Consider the interval-valued mapping $f(x) = [1, 2]x^2$, $x \in R$. Then $f(x)$ is gH-differentiable on R by Theorem 1, and

$$\nabla f(x) = \begin{cases} [2x, 4x], & x \geq 0, \\ [4x, 2x], & x < 0. \end{cases}$$

Let $\eta(x, y) = x - y$, thus

$$\eta(x, y)^t \nabla f(y) = \begin{cases} [2y(x - y), 4y(x - y)], & y \geq 0, x - y \geq 0, \\ [2y(x - y), 4y(x - y)], & y < 0, x - y < 0, \\ [4y(x - y), 2y(x - y)], & y \geq 0, x - y < 0, \\ [4y(x - y), 2y(x - y)], & y < 0, x - y \geq 0, \end{cases} \tag{15}$$

$$f(x) \ominus_g f(y) = \begin{cases} [x^2 - y^2, 2x^2 - 2y^2], & x^2 \geq y^2, \\ [2x^2 - 2y^2, x^2 - y^2], & x^2 < y^2. \end{cases} \tag{16}$$

We can observe that $f(x)$ is invex with respect to $\eta(x, y) = x - y$ by Definition 11.

The following theorem shows the relationship between interval-valued invex functions and weakly invex functions.

Theorem 4 Let K be an invex subset of R^n with respect to $\eta : K \times K \rightarrow R^n$ and $f(x) = [f^L(x), f^U(x)]$ be an interval-valued function defined on K . If f is weakly invex, then it is invex, but the converse is not true in general.

Proof Since f is weakly invex at x^* , we have that real-valued functions f^L and f^U are invex at x^* , i.e.,

$$f^L(x) - f^L(x^*) \geq \eta(x, x^*)^t \nabla f^L(x^*), \tag{17}$$

$$f^U(x) - f^U(x^*) \geq \eta(x, x^*)^t \nabla f^U(x^*) \tag{18}$$

for each $\lambda \in [0, 1]$ and each $x \in K$.

(1) On the condition of $\eta(x, x^*)^t \nabla f^L(x^*) \leq \eta(x, x^*)^t \nabla f^U(x^*)$, we have

$$\eta(x, x^*)^t \nabla f(x^*) = [\eta(x, x^*)^t \nabla f^L(x^*), \eta(x, x^*)^t \nabla f^U(x^*)].$$

If $f(x) \ominus_g f(x^*) = [f^L(x) - f^L(x^*), f^U(x) - f^U(x^*)]$, then from (17) and (18) we have

$$f(x) \ominus_g f(x^*) \geq \eta(x, x^*)^t \nabla f(x^*).$$

If $f(x) \ominus_g f(x^*) = [f^U(x) - f^U(x^*), f^L(x) - f^L(x^*)]$, then

$$f^L(x) - f^L(x^*) \geq f^U(x) - f^U(x^*) \geq \eta(x, x^*)^t \nabla f^U(x^*) \geq \eta(x, x^*)^t \nabla f^L(x^*).$$

We have

$$f(x) \ominus_g f(x^*) \geq \nabla f(x^*)^t \eta(x, x^*).$$

(2) On the condition of $\eta(x, x^*)^t \nabla f^L(x^*) > \eta(x, x^*)^t \nabla f^U(x^*)$, we have

$$\eta(x, x^*)^t \nabla f(x^*) = [\eta(x, x^*)^t \nabla f^U(x^*), \eta(x, x^*)^t \nabla f^L(x^*)].$$

If $f(x) \ominus_g f(x^*) = [f^U(x) - f^U(x^*), f^L(x) - f^L(x^*)]$, then from (17) and (18) we have

$$f(x) \ominus_g f(x^*) \geq \eta(x, x^*)^t \nabla f(x^*).$$

If $f(x) \ominus_g f(x^*) = [f^L(x) - f^L(x^*), f^U(x) - f^U(x^*)]$, then

$$f^U(x) - f^U(x^*) \geq f^L(x) - f^L(x^*) \geq \eta(x, x^*)^t \nabla f^L(x^*) \geq \eta(x, x^*)^t \nabla f^U(x^*).$$

Thus

$$f(x) \ominus_g f(x^*) \geq \eta(x, x^*)^t \nabla f(x^*). \quad \square$$

Example 3 Considering the interval-valued function $f(x) = [x_1 - 2 \sin x_2, x_1 - \sin x_2 + 1]$, $x \in R^2$, we can prove that both $f^L(x)$ and $f^U(x)$ are weakly invex with respect to

$$\eta(x, y) = \left(\frac{\sin x_1 - \sin y_1}{\cos y_1}, \frac{\sin x_2 - \sin y_2}{\cos y_2} \right)^t.$$

Then $f(x) = [x_1 - 2 \sin x_2, x_1 - \sin x_2 + 1]$, $x \in R^2$ is invex with respect to the same $\eta(x, y)$ by Theorem 4.

Example 4 Considering the interval-valued function $f(x) = [-|x|, |x|]$, $x \in R$,

$$\eta(x, y) = \begin{cases} x - y, & xy \geq 0, \\ x + y, & xy < 0. \end{cases}$$

From Theorem 1, it follows that $f(x)$ is gH-differentiable on R , and $\nabla f(y) = [-1, 1]$, thus

$$\eta(x, y)^t \nabla f(y) = \begin{cases} [-(x - y), x - y], & x \geq y \geq 0 \text{ or } y \leq x \leq 0, \\ [x - y, -(x - y)], & y \geq x \geq 0 \text{ or } x \leq y \leq 0, \\ [-(x + y), x + y], & xy < 0 \text{ and } x + y > 0, \\ [x + y, -(x + y)], & xy < 0 \text{ and } x + y < 0, \end{cases}$$

$$f(x) \ominus_g f(y) = \begin{cases} [-(x - y), x - y], & x \geq y \geq 0 \text{ or } y \leq x \leq 0, \\ [x - y, -(x - y)], & y \geq x \geq 0 \text{ or } x \leq y \leq 0, \\ [-(x + y), x + y], & x > 0 > y \text{ and } x + y > 0, \\ [x + y, -(x + y)], & x > 0 > y \text{ and } x + y < 0, \\ [-(x + y), x + y], & x < 0 < y. \end{cases}$$

Then

$$f(x) \ominus_g f(y) \geq \eta(x, y) \nabla f(y)^t.$$

But $f(x)$ is not weakly invex since $f^L(x)$ is not weakly differentiable at $x = 0$.

Let $x = 2, y = -1, \eta(x, y) = 1,$

$$f^L(y + \lambda\eta(x, y)) = 1 - \lambda > -2 + \lambda = \lambda f^L(y) + (1 - \lambda)f^L(x).$$

Thus, $f(x)$ is not preinvex since $f^L(x)$ is not preinvex with respect to η .

The following theorem given in [20] illustrates the relations between weakly invex interval-valued and preinvex interval-valued functions.

Theorem 5 *Let $K \subseteq R^n$ be an invex set with respect to $\eta : K \times K \rightarrow R^n$; if $f(x) = [f^L(x), f^U(x)]$ is a weakly continuously differentiable and preinvex interval-valued function defined on K , then f is also a weakly invex interval-valued function with respect to the same η defined on K .*

We can prove the following result.

Theorem 6 *Let $K \subseteq R^n$ be an invex set with respect to $\eta : K \times K \rightarrow R^n$. If $f(x) = [f^L(x), f^U(x)]$ is a weakly differentiable and preinvex interval-valued function defined on K , then f is also an interval-valued invex function with respect to the same η defined on K .*

Proof Since f is interval-valued preinvex and weakly differentiable, we have

$$\begin{aligned} f^L(y + \lambda\eta(x, y)) - f^L(y) &\leq \lambda[f^L(x) - f^L(y)], \\ f^U(y + \lambda\eta(x, y)) - f^U(y) &\leq \lambda[f^U(x) - f^U(y)], \end{aligned}$$

which for $\lambda \in (0, 1]$ implies

$$\begin{aligned} \frac{f^L(y + \lambda\eta(x, y)) - f^L(y)}{\lambda} &\leq f^L(x) - f^L(y), \\ \frac{f^U(y + \lambda\eta(x, y)) - f^U(y)}{\lambda} &\leq f^U(x) - f^U(y). \end{aligned}$$

By taking limits for $\lambda \rightarrow 0^+$, since f is weakly differentiable, we get

$$\eta(x, y)^t \nabla f^L(y) \leq f^L(x) - f^L(y), \tag{19}$$

$$\eta(x, y)^t \nabla f^U(y) \leq f^U(x) - f^U(y). \tag{20}$$

On the other hand, from Definition 3 and Theorem 1, we have

$$f(x) \ominus_g f(y) = \begin{cases} [f^L(x) - f^L(y), f^U(x) - f^U(y)], & f^L(x) - f^L(y) \leq f^U(x) - f^U(y), \\ [f^U(x) - f^U(y), f^L(x) - f^L(y)], & f^L(x) - f^L(y) > f^U(x) - f^U(y) \end{cases} \tag{21}$$

and

$$\begin{aligned} \eta(x, y)^t \nabla f(y) &= \eta(x, y)^t [\min\{\nabla f^L(y), \nabla f^U(y)\}, \max\{\nabla f^L(y), \nabla f^U(y)\}] \end{aligned} \tag{22}$$

$$= \begin{cases} [\eta(x, y)^t \nabla f^L(y), \eta(x, y)^t \nabla f^U(y)], & \eta(x, y)^t \nabla f^L(y) \leq \eta(x, y)^t \nabla f^U(y), \\ [\eta(x, y)^t \nabla f^U(y), \eta(x, y)^t \nabla f^L(y)], & \nabla \eta(x, y)^t f^L(y) > \eta(x, y)^t \nabla f^U(y). \end{cases} \tag{23}$$

From (19)-(23) it follows that

$$f(x) \ominus_g f(y) \geq \eta(x, y)^t \nabla f(y).$$

Thus, f is an invex interval-valued function. □

The following example given in [20] shows that a weakly invex interval-valued function may not be a preinvex interval-valued function, from Theorem 4 we can conclude that the converse of Theorem 6 is not true.

Example 5 The interval-valued function $f(x) = [1, 2] \cdot e^x$, $x \in R$ is invex with respect to $\eta = -1$, but not preinvex with respect to the same function η .

However, Mohan and Neogy [7] have proved that a differentiable invex real-valued function is also preinvex under the following condition.

Condition C We say that the function $\eta : R^n \rightarrow R^n$ satisfies Condition C if for any $x, y \in X$,

$$\eta(x, y + \lambda\eta(x, y)) = -\lambda\eta(x, y), \quad \eta(y, y + \lambda\eta(x, y)) = (1 - \lambda)\eta(x, y).$$

We can conclude that a continuously weakly differentiable invex interval-valued function $f : K \rightarrow \mathcal{I}$ is also a preinvex interval-valued function on K if the function η satisfies Condition C.

Theorem 7 Suppose that K is an invex set of R^n with respect to $\eta : K \times K \rightarrow R^n$ and $f : K \rightarrow \mathcal{I}$ is a continuously weakly differentiable interval-valued function on an open set containing K . If f is invex on K with respect to η and η satisfies Condition C, then f is preinvex with respect to η on K .

Proof Suppose that $x_1, x_2 \in X$. Let $0 < \lambda < 1$ be given and look at $\bar{x} = x_2 + \lambda\eta(x_1, x_2)$.

Note that $\bar{x} \in X$, by invexity of \tilde{f} , we have

$$f(x_1) \ominus_g f(\bar{x}) \geq \eta(x_1, \bar{x})^t \nabla f(\bar{x})$$

and

$$f(x_2) \ominus_g f(\bar{x}) \geq \eta(x_2, \bar{x})^t \nabla f(\bar{x}),$$

i.e.,

$$\begin{aligned} & [\min\{f^L(x_1) - f^L(\bar{x}), f^U(x_1) - f^U(\bar{x})\}, \max\{f^L(x_1) - f^L(\bar{x}), f^U(x_1) - f^U(\bar{x})\}] \\ & \geq \eta(x_1, \bar{x})^t [\min\{\nabla f^L(\bar{x}), \nabla f^U(\bar{x})\}, \max\{\nabla f^L(\bar{x}), \nabla f^U(\bar{x})\}] \end{aligned}$$

and

$$\begin{aligned} & [\min\{f^L(x_2) - f^L(\bar{x}), f^U(x_2) - f^U(\bar{x})\}, \max\{f^L(x_2) - f^L(\bar{x}), f^U(x_2) - f^U(\bar{x})\}] \\ & \geq \eta(x_2, \bar{x})^t [\min\{\nabla f^L(\bar{x}), \nabla f^U(\bar{x})\}, \max\{\nabla f^L(\bar{x}), \nabla f^U(\bar{x})\}]. \end{aligned}$$

(1) On the condition of $f^L(x_1) - f^L(\bar{x}) \leq f^U(x_1) - f^U(\bar{x})$, $\eta(x_1, \bar{x})^t \nabla f^L(\bar{x}) \leq \eta(x_1, \bar{x})^t \nabla f^U(\bar{x})$, $f^L(x_2) - f^L(\bar{x}) \leq f^U(x_2) - f^U(\bar{x})$, and $\eta(x_2, \bar{x})^t \nabla f^L(\bar{x}) \leq \eta(x_2, \bar{x})^t \nabla f^U(\bar{x})$, we have

$$\begin{aligned} f(x_1) \ominus_g f(\bar{x}) &= [f^L(x_1) - f^L(\bar{x}), f^U(x_1) - f^U(\bar{x})], \\ f(x_2) \ominus_g f(\bar{x}) &= [f^L(x_2) - f^L(\bar{x}), f^U(x_2) - f^U(\bar{x})] \end{aligned}$$

and

$$\begin{aligned} \eta(x_1, \bar{x})^t \nabla f(\bar{x}) &= [\eta(x_1, \bar{x})^t \nabla f^L(\bar{x}), \eta(x_1, \bar{x})^t \nabla f^U(\bar{x})], \\ \eta(x_2, \bar{x})^t \nabla f(\bar{x}) &= [\eta(x_2, \bar{x})^t \nabla f^L(\bar{x}), \eta(x_2, \bar{x})^t \nabla f^U(\bar{x})]. \end{aligned}$$

Thus,

$$\begin{aligned} f^L(x_1) - f^L(\bar{x}) &\geq \eta(x_1, \bar{x})^t \nabla f^L(\bar{x}), \\ f^U(x_1) - f^U(\bar{x}) &\geq \eta(x_1, \bar{x})^t \nabla f^U(\bar{x}) \end{aligned}$$

and

$$\begin{aligned} f^L(x_2) - f^L(\bar{x}) &\geq \eta(x_2, \bar{x})^t \nabla f^L(\bar{x}), \\ f^U(x_2) - f^U(\bar{x}) &\geq \eta(x_2, \bar{x})^t \nabla f^U(\bar{x}). \end{aligned}$$

Therefore,

$$\lambda f^L(x_1) + (1 - \lambda)f^L(x_2) - f^L(\bar{x}) \geq (\lambda \eta(x_1, \bar{x}) + (1 - \lambda)\eta(x_2, \bar{x}))^t \nabla f^L(\bar{x}).$$

However, by Condition C, $\lambda \eta(x_1, \bar{x}) + (1 - \lambda)\eta(x_2, \bar{x}) = 0$. Hence,

$$f^L(x_2 + \lambda \eta(x_1, x_2)) \leq \lambda f^L(x_1) + (1 - \lambda)f^L(x_2).$$

By a similar way,

$$f^U(x_2 + \lambda \eta(x_1, x_2)) \leq \lambda f^U(x_1) + (1 - \lambda)f^U(x_2).$$

Thus f is preinvex with respect to η by Theorem 3.

(2) On the condition of $f^L(x_1) - f^L(\bar{x}) \leq f^U(x_1) - f^U(\bar{x})$, $\eta(x_1, \bar{x})^t \nabla f^L(\bar{x}) \leq \eta(x_1, \bar{x})^t \nabla f^U(\bar{x})$, $f^L(x_2) - f^L(\bar{x}) \leq f^U(x_2) - f^U(\bar{x})$, and $\eta(x_2, \bar{x})^t \nabla f^L(\bar{x}) > \eta(x_2, \bar{x})^t \nabla f^U(\bar{x})$, we have

$$\begin{aligned} f(x_1) \ominus_g f(\bar{x}) &= [f^L(x_1) - f^L(\bar{x}), f^U(x_1) - f^U(\bar{x})], \\ f(x_2) \ominus_g f(\bar{x}) &= [f^L(x_2) - f^L(\bar{x}), f^U(x_2) - f^U(\bar{x})] \end{aligned}$$

and

$$\begin{aligned} \eta(x_1, \bar{x})^t \nabla f(\bar{x}) &= [\eta(x_1, \bar{x})^t \nabla f^L(\bar{x}), \eta(x_1, \bar{x})^t \nabla f^U(\bar{x})], \\ \eta(x_2, \bar{x})^t \nabla f(\bar{x}) &= [\eta(x_2, \bar{x})^t \nabla f^U(\bar{x}), \eta(x_2, \bar{x})^t \nabla f^L(\bar{x})]. \end{aligned}$$

Thus,

$$f^L(x_1) - f^L(\bar{x}) \geq \eta(x_1, \bar{x})^t \nabla f^L(\bar{x}),$$

$$f^U(x_1) - f^U(\bar{x}) \geq \eta(x_1, \bar{x})^t \nabla f^U(\bar{x})$$

and

$$f^L(x_2) - f^L(\bar{x}) \geq \eta(x_2, \bar{x})^t \nabla f^U(\bar{x}),$$

$$f^U(x_2) - f^U(\bar{x}) \geq \eta(x_2, \bar{x})^t \nabla f^L(\bar{x}).$$

Suppose $\min\{\eta(x_1, \bar{x})^t \nabla f^L(\bar{x}), \eta(x_2, \bar{x})^t \nabla f^U(\bar{x})\} = \eta(x_1, \bar{x})^t \nabla f^L(\bar{x})$. Therefore,

$$\begin{aligned} \lambda f^L(x_1) + (1 - \lambda)f^L(x_2) - f^L(\bar{x}) &\geq \lambda \eta(x_1, \bar{x})^t \nabla f^L(\bar{x}) + (1 - \lambda)\eta(x_2, \bar{x})^t \nabla f^U(\bar{x}) \\ &\geq \lambda \eta(x_1, \bar{x})^t \nabla f^L(\bar{x}) + (1 - \lambda)\eta(x_2, \bar{x})^t \nabla f^L(\bar{x}) \\ &= (\lambda \eta(x_1, \bar{x}) + (1 - \lambda)\eta(x_2, \bar{x}))^t \nabla f^L(\bar{x}). \end{aligned}$$

However, by Condition C, $\lambda \eta(x_1, \bar{x}) + (1 - \lambda)\eta(x_2, \bar{x}) = 0$. Hence,

$$f^L(x_2 + \lambda \eta(x_1, x_2)) \leq \lambda f^L(x_1) + (1 - \lambda)f^L(x_2).$$

By a similar way,

$$f^U(x_2 + \lambda \eta(x_1, x_2)) \leq \lambda f^U(x_1) + (1 - \lambda)f^U(x_2).$$

Thus f is preinvex with respect to η by Theorem 3.

(3) On the condition of $f^L(x_1) - f^L(\bar{x}) \leq f^U(x_1) - f^U(\bar{x})$, $\eta(x_1, \bar{x})^t \nabla f^L(\bar{x}) \leq \eta(x_1, \bar{x})^t \nabla f^U(\bar{x})$, $f^L(x_2) - f^L(\bar{x}) > f^U(x_2) - f^U(\bar{x})$, and $\eta(x_2, \bar{x})^t \nabla f^L(\bar{x}) \leq \eta(x_2, \bar{x})^t \nabla f^U(\bar{x})$, we have

$$f(x_1) \ominus_g f(\bar{x}) = [f^L(x_1) - f^L(\bar{x}), f^U(x_1) - f^U(\bar{x})],$$

$$f(x_2) \ominus_g f(\bar{x}) = [f^U(x_2) - f^U(\bar{x}), f^L(x_2) - f^L(\bar{x})]$$

and

$$\eta(x_1, \bar{x})^t \nabla f(\bar{x}) = [\eta(x_1, \bar{x})^t \nabla f^L(\bar{x}), \eta(x_1, \bar{x})^t \nabla f^U(\bar{x})],$$

$$\eta(x_2, \bar{x})^t \nabla f(\bar{x}) = [\eta(x_2, \bar{x})^t \nabla f^L(\bar{x}), \eta(x_2, \bar{x})^t \nabla f^U(\bar{x})].$$

Thus,

$$f^L(x_1) - f^L(\bar{x}) \geq \eta(x_1, \bar{x})^t \nabla f^L(\bar{x}),$$

$$f^U(x_1) - f^U(\bar{x}) \geq \eta(x_1, \bar{x})^t \nabla f^U(\bar{x})$$

and

$$f^L(x_2) - f^L(\bar{x}) \geq \eta(x_2, \bar{x})^t \nabla f^U(\bar{x}),$$

$$f^U(x_2) - f^U(\bar{x}) \geq \eta(x_2, \bar{x})^t \nabla f^L(\bar{x}).$$

Suppose $\min\{\eta(x_1, \bar{x})^t \nabla f^L(\bar{x}), \eta(x_2, \bar{x})^t \nabla f^U(\bar{x})\} = \eta(x_1, \bar{x})^t \nabla f^L(\bar{x})$. Thus,

$$\begin{aligned} \lambda f^L(x_1) + (1 - \lambda)f^L(x_2) - f^L(\bar{x}) &\geq \lambda \eta(x_1, \bar{x})^t \nabla f^U(\bar{x}) + (1 - \lambda)\eta(x_2, \bar{x})^t \nabla f^L(\bar{x}) \\ &\geq \lambda \eta(x_1, \bar{x})^t \nabla f^L(\bar{x}) + (1 - \lambda)\eta(x_2, \bar{x})^t \nabla f^L(\bar{x}) \\ &= (\lambda \eta(x_1, \bar{x}) + (1 - \lambda)\eta(x_2, \bar{x}))^t \nabla f^L(\bar{x}). \end{aligned}$$

However, by Condition C, $\lambda \eta(x_1, \bar{x}) + (1 - \lambda)\eta(x_2, \bar{x}) = 0$. Hence,

$$f^L(x_2 + \lambda \eta(x_1, x_2)) \leq \lambda f^L(x_1) + (1 - \lambda)f^L(x_2).$$

By a similar way,

$$f^U(x_2 + \lambda \eta(x_1, x_2)) \leq \lambda f^U(x_1) + (1 - \lambda)f^U(x_2).$$

Thus f is preinvex with respect to η by Theorem 3.

(4) On the condition of $f^L(x_1) - f^L(\bar{x}) \leq f^U(x_1) - f^U(\bar{x})$, $\eta(x_1, \bar{x})^t \nabla f^L(\bar{x}) \leq \eta(x_1, \bar{x})^t \nabla f^U(\bar{x})$, $f^L(x_2) - f^L(\bar{x}) > f^U(x_2) - f^U(\bar{x})$, and $\eta(x_2, \bar{x})^t \nabla f^L(\bar{x}) > \eta(x_2, \bar{x})^t \nabla f^U(\bar{x})$, we have

$$\begin{aligned} f(x_1) \ominus_g f(\bar{x}) &= [f^L(x_1) - f^L(\bar{x}), f^U(x_1) - f^U(\bar{x})], \\ f(x_2) \ominus_g f(\bar{x}) &= [f^U(x_2) - f^U(\bar{x}), f^L(x_2) - f^L(\bar{x})] \end{aligned}$$

and

$$\begin{aligned} \eta(x_1, \bar{x})^t \nabla f(\bar{x}) &= [\eta(x_1, \bar{x})^t \nabla f^L(\bar{x}), \eta(x_1, \bar{x})^t \nabla f^U(\bar{x})], \\ \eta(x_2, \bar{x})^t \nabla f(\bar{x}) &= [\eta(x_2, \bar{x})^t \nabla f^U(\bar{x}), \eta(x_2, \bar{x})^t \nabla f^L(\bar{x})]. \end{aligned}$$

Thus,

$$\begin{aligned} f^L(x_1) - f^L(\bar{x}) &\geq \eta(x_1, \bar{x})^t \nabla f^L(\bar{x}), \\ f^U(x_1) - f^U(\bar{x}) &\geq \eta(x_1, \bar{x})^t \nabla f^U(\bar{x}) \end{aligned}$$

and

$$\begin{aligned} f^L(x_2) - f^L(\bar{x}) &\geq \eta(x_2, \bar{x})^t \nabla f^L(\bar{x}), \\ f^U(x_2) - f^U(\bar{x}) &\geq \eta(x_2, \bar{x})^t \nabla f^U(\bar{x}). \end{aligned}$$

Thus,

$$\lambda f^L(x_1) + (1 - \lambda)f^L(x_2) - f^L(\bar{x}) \geq (\lambda \eta(x_1, \bar{x}) + (1 - \lambda)\eta(x_2, \bar{x}))^t \nabla f^L(\bar{x}).$$

However, by Condition C, $\lambda \eta(x_1, \bar{x}) + (1 - \lambda)\eta(x_2, \bar{x}) = 0$. Hence,

$$f^L(x_2 + \lambda \eta(x_1, x_2)) \leq \lambda f^L(x_1) + (1 - \lambda)f^L(x_2).$$

By a similar way,

$$f^U(x_2 + \lambda\eta(x_1, x_2)) \leq \lambda f^U(x_1) + (1 - \lambda)f^U(x_2).$$

As a consequence, f is preinvex with respect to η by Theorem 3.

We can prove the result on the other four conditions by a similar way. □

4 The Kuhn-Tucker optimality conditions with interval-valued objective functions

An interval-valued objective minimization problem is

$$\begin{aligned} \text{(IVOP)} \quad \min f(x) &= [f^L(x), f^U(x)] \\ \text{subject to } g_i(x) &\leq 0, \quad i = 1, 2, \dots, m. \end{aligned}$$

Let $P = \{x \in R^n : g_i(x) \leq 0, i = 1, 2, \dots, m\}$ be a feasible set of (IVOP).

Definition 12 Let x^* be a feasible solution of the primal problem (IVOP). We say that x^* is a non-dominated solution of problem (IVOP) if and only if there exists no $x \in P$ such that $f(x) < f(x^*)$. In this case, $f(x^*)$ is called the non-dominated objective value of f .

Theorem 8 Let $f(x)$ be gH-differentiable on $X \subseteq R^n$ and interval-valued invex with respect to $\eta : X \times X \rightarrow R^n$, and $g_i(x)$ ($i = 1, 2, \dots, m$) be invex with respect to the same η . If there exist $x^* \in P$ and v_i ($i = 1, 2, \dots, m$) such that

$$\{\nabla f(x^*)\}^L + \sum_{i=1}^n v_i \nabla g_i(x^*) = 0, \tag{24}$$

$$\sum_{i=1}^n v_i g_i(x^*) = 0, \tag{25}$$

$$v_i \geq 0, \tag{26}$$

then x^* is a non-dominated solution of problem (IVOP).

Proof For any $x^* \in P$ satisfying $g_i(x^*) \leq 0, i = 1, 2, \dots, m$, since $f(x)$ is gH-differentiable on $X \subseteq R^n$ and interval-valued invex with respect to η , then $[\min\{f^L(x) - f^L(x^*), f^U(x) - f^U(x^*)\}, \max\{f^L(x) - f^L(x^*), f^U(x) - f^U(x^*)\}] \geq \eta(x, x^*)^t \nabla f(x^*)$.

(i) On the condition of $\min\{f^L(x) - f^L(x^*), f^U(x) - f^U(x^*)\} = f^L(x) - f^L(x^*), \eta(x, x^*)^t \times \nabla f(x^*) = [\eta(x, x^*)^t \{\nabla f(x^*)\}^L, \eta(x, x^*)^t \{\nabla f(x^*)\}^U]$,

$$\begin{aligned} f^L(x) - f^L(x^*) &\geq \eta(x, x^*)^t \{\nabla f(x^*)\}^L \\ &= -\eta(x, x^*)^t \sum_{i=1}^n v_i \nabla g_i(x^*) \\ &\geq -\sum_{i=1}^m v_i (g_i(x) - g_i(x^*)) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{i=1}^m (-v_i g_i(x) + v_i g_i(x^*)) \\
 &= \sum_{i=1}^m -v_i g_i(x) \\
 &\geq 0.
 \end{aligned}$$

Thus, $f(x) < f(x^*)$ does not hold.

(ii) On the condition of $\min\{f^L(x) - f^L(x^*), f^U(x) - f^U(x^*)\} = f^U(x) - f^U(x^*)$, $\eta(x, x^*)^t \times \nabla f(x^*) = [\eta(x, x^*)^t \{\nabla f(x^*)\}^L, \eta(x, x^*)^t \{\nabla f(x^*)\}^U]$,

$$\begin{aligned}
 f^U(x) - f^U(x^*) &\geq \eta(x, x^*)^t \{\nabla f(x^*)\}^L \\
 &= -\eta(x, x^*)^t \sum_{i=1}^n v_i \nabla g_i(x^*) \\
 &\geq -\sum_{i=1}^m v_i (g_i(x) - g_i(x^*)) \\
 &= \sum_{i=1}^m (-v_i g_i(x) + v_i g_i(x^*)) \\
 &= \sum_{i=1}^m -v_i g_i(x) \\
 &\geq 0.
 \end{aligned}$$

Thus, $f(x) < f(x^*)$ does not hold.

Similarly, we can prove the other two conditions. □

The proof of the following theorem is similar to the one of Theorem 8.

Theorem 9 *Let $f(x)$ be gH-differentiable on $X \subseteq R^n$ and interval-valued invex with respect to $\eta : X \times X \rightarrow R^n$, and $g_i(x)$ ($i = 1, 2, \dots, m$) be invex with respect to the same η . If there exist $x^* \in P$ and v_i ($i = 1, 2, \dots, m$) such that*

$$\{\nabla f(x^*)\}^U + \sum_{i=1}^n v_i \nabla g_i(x^*) = 0, \tag{27}$$

$$\sum_{i=1}^n v_i g_i(x^*) = 0, \tag{28}$$

$$v_i \geq 0, \tag{29}$$

then x^* is a non-dominated solution of problem (IVOP).

Example 6

$$\begin{aligned}
 &\text{minimize } f(x) = [x^2 + x + 1, x^2 + 3] \\
 &\text{subject to } g_1(x) = x - 2 \leq 0, \\
 &\qquad\qquad g_2(x) = -x - 6 \leq 0.
 \end{aligned}$$

From Theorem 1 we have $\nabla f(x) = [2x, 2x + 1]$.

(i) It is easy to see that the problem satisfies the assumptions of Theorem 8. Then

$$\begin{aligned} \{[2x^*, 2x^* + 1]\}^L + v_1 - v_2 &= 0, \\ v_1(x^* - 2) &= 0, \\ v_2(-x^* - 6) &= 0. \end{aligned}$$

We obtain $v_1 = v_2 = 0$, and $x^* = 0$ is a non-dominated solution.

(ii) It is easy to see that the problem satisfies the assumptions of Theorem 9. Then

$$\begin{aligned} \{[2x^*, 2x^* + 1]\}^U + v_1 - v_2 &= 0, \\ v_1(x^* - 2) &= 0, \\ v_2(-x^* - 6) &= 0. \end{aligned}$$

We obtain $v_1 = v_2 = 0$, and $x^* = -\frac{1}{2}$ is also a non-dominated solution.

Example 7 Consider the following interval-valued programming problem:

$$\begin{aligned} \text{minimize } f(x) &= [1, 2] \sin^2 x \\ \text{subject to } g(x) &= (\sin x - 1)^2 - \frac{1}{8} \leq 0, \\ x &\in \left(0, \frac{\pi}{2}\right). \end{aligned}$$

Note that functions f and g are invex with respect to

$$\eta(x, y) = \frac{\sin x - \sin y}{\cos y}.$$

And $\nabla f(x) = [2 \sin x \cos x, 4 \sin x \cos x]$, $\nabla g(x) = 2(\sin x - 1) \cos x$.

It is easy to see that the problem satisfies the assumptions of Theorem 9. Then

$$\begin{aligned} 2 \sin x^* \cos x^* + 2v(\sin x^* - 1) \cos x^* &= 0, \\ v\left((\sin x^* - 1)^2 - \frac{1}{8}\right) &= 0. \end{aligned}$$

After some algebraic calculations, we obtain $x^* = \sin^{-1}\left(1 - \frac{1}{2\sqrt{2}}\right)$, $v = 2\sqrt{2} - 1$. Therefore, $x^* = \sin^{-1}\left(1 - \frac{1}{2\sqrt{2}}\right)$ is a non-dominated solution.

The following example also shows the advantages of our method in respect to [19].

Example 8 Consider the following interval-valued programming problem:

$$\begin{aligned} \text{minimize } f(x) &= [x_1 - 2 \sin x_2, x_1 - \sin x_2 + 1] \\ \text{subject to } g_1(x) &= 2 \sin x_1 + 7 \sin x_2 + x_1 - 6 \leq 0, \end{aligned}$$

$$g_2(x) = 2x_1 + 2x_2 - 3 \leq 0,$$

$$g_3(x) = -\sin x_1 \leq 0,$$

$$g_4(x) = -\sin x_2 \leq 0.$$

Then f is gH-differentiable and weakly differentiable. Since f^L is not LU-convex and $f^L + f^U$ is not LU-convex, methods in [19] cannot be used.

Note that functions f and g_i ($i = 1, 2, 3, 4$) are invex with respect to

$$\eta(x, y) = \left(\frac{\sin x_1 - \sin y_1}{\cos y_1}, \frac{\sin x_2 - \sin y_2}{\cos y_2} \right)^t.$$

It is easy to see that the problem satisfies the assumptions of Theorem 8. Then

$$1 + v_1(2 \cos x_1 + 1) + 2v_2 - v_3 \cos x_1 = 0,$$

$$-2 \cos x_2 + 7v_1 \cos x_2 + 2v_2 - v_4 \cos x_2 = 0,$$

$$v_1(2 \sin x_1 + 7 \sin x_2 + x_1 - 6) = 0,$$

$$v_2(2x_1 + 2x_2 - 3) = 0,$$

$$v_3(-\sin x_1) = 0,$$

$$v_4(-\sin x_1) = 0.$$

After some algebraic calculations, we obtain $x^* = (0, \sin^{-1} \frac{6}{7})^t$, $v = (\frac{2}{7}, 0, \frac{13}{7}, 0)^t$. Therefore, x^* is a non-dominated solution.

5 Conclusion

The concept of convex interval-valued mappings has been studied in the literature by many researchers. The aim of this paper is to introduce the concept of invex interval-valued mappings with gH-differentiable functions. Then we discussed the relationships between interval-valued invex mappings and interval-valued weakly invex mappings. Finally, the sufficient optimality condition for interval-valued objective functions has been derived under invexity.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have equal contributions. All authors read and approved the final manuscript.

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