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New bounds for the spectral radius for nonnegative tensors

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Abstract
A lower bound and an upper bound for the spectral radius for nonnegative tensors are obtained. A numerical example is given to show that the new bounds are sharper than the corresponding bounds obtained by Yang and Yang (SIAM J. Matrix Anal. Appl. 31:2517-2530, 2010), and that the upper bound is sharper than that obtained by Li *et al.* (Numer. Linear Algebra Appl. 21:39-50, 2014).

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1 Introduction

A real order m dimension n tensor $\mathcal{A} = (a_{i_1\cdots i_m})$, denoted by $\mathcal{A} \in R^{[m,n]}$, consists of n^m real entries:

$$a_{i_1\cdots i_m} \in R,$$

where $i_j = 1,\ldots,n$ for $j = 1,\ldots,m$. A tensor $\mathcal{A}$ is called nonnegative (positive), denoted by $\mathcal{A} \geq 0$ ($\mathcal{A} > 0$), if every entry $a_{i_1\cdots i_m} \geq 0$ ($a_{i_1\cdots i_m} > 0$, respectively). Given a tensor $\mathcal{A} = (a_{i_1\cdots i_m}) \in R^{[m,n]}$, if there are a complex number λ and a nonzero complex vector $x = (x_1, x_2, \ldots, x_n)^T$ that are solutions of the following homogeneous polynomial equations:

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

then λ is called an eigenvalue of $\mathcal{A}$ and x an eigenvector of $\mathcal{A}$ associated with λ [1–6], where $\mathcal{A}x^{m-1}$ and $x^{[m-1]}$ are vectors, whose ith entries are

$$\left(\mathcal{A}x^{m-1}\right)_i = \sum_{i_2,\ldots,i_m \in N} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m} \quad \left(N = \{1,2,\ldots,n\}\right)$$

and $(x^{[m-1]})_i = x_i^{m-1}$, respectively. Moreover, the spectral radius $\rho(\mathcal{A})$ [7] of the tensor $\mathcal{A}$ is defined as

$$\rho(\mathcal{A}) = \max\left\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathcal{A}\right\}.$$

Eigenvalues of tensors have become an important topic of study in numerical multilinear algebra, and they have a wide range of practical applications; see [4, 5, 8–21]. Recently, for the largest eigenvalue of a nonnegative tensor, Chang *et al.* [2] generalized the well-known Perron-Frobenius theorem for irreducible nonnegative matrices to irreducible nonnegative tensors. Here a tensor $\mathcal{A} = (a_{i_1\cdots i_m}) \in R^{m,n}$ is called reducible, if there exists a nonempty proper index subset $I \subset N$ such that

$$a_{i_1 i_2\cdots i_m} = 0 \quad \text{for all } i_1 \in I, \text{for all } i_2, \ldots, i_m \notin I.$$

If $\mathcal{A}$ is not reducible, then we call $\mathcal{A}$ irreducible.

Theorem 1 (Theorem 1.4 in [2]) *If $\mathcal{A} \in R^{[m,n]}$ is irreducible nonnegative, then $\rho(\mathcal{A})$ is a positive eigenvalue with an entrywise positive eigenvector x, i.e., $x > 0$, corresponding to it.*

Subsequently, Yang and Yang [21] extended this theorem to nonnegative tensors.

Theorem 2 (Theorem 2.3 in [21]) *If $\mathcal{A} \in R^{[m,n]}$ is nonnegative, then $\rho(\mathcal{A})$ is an eigenvalue with an entrywise nonnegative eigenvector x, i.e., $x \geq 0$, $x \neq 0$, corresponding to it.*

For the spectral radius of a nonnegative tensor, Yang and Yang [21] provided a lower bound and an upper bound for the spectral radius of a nonnegative tensor.

Theorem 3 (Lemma 5.2 in [21]) *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in R^{[m,n]}$ be nonnegative. Then*

$$R_{\min} \leq \rho(\mathcal{A}) \leq R_{\max},$$

where $R_{\min} = \min_{i\in N} R_i(\mathcal{A})$, $R_{\max} = \max_{i\in N} R_i(\mathcal{A})$, and $R_i(\mathcal{A}) = \sum_{i_2,\ldots,i_m\in N} a_{ii_2\cdots i_m}$.

In order to obtain much sharper bounds of the spectral radius of a nonnegative tensor, Li *et al.* [22] have given an upper bound which estimates the spectral radius more precisely than that in Theorem 3.

Theorem 4 (Theorems 3.3 and 3.5 in [22]) *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in R^{[m,n]}$ be nonnegative with $n \geq 2$. Then*

$$\rho(\mathcal{A}) \leq \Omega_{\max},$$

where

$$\Omega_{\max} = \max_{\substack{i,j\in N,\\ j\neq i}} \frac{1}{2}\Big(a_{i\cdots i} + a_{j\cdots j} + r_i^j(\mathcal{A}) + \sqrt{\big(a_{i\cdots i} - a_{j\cdots j} + r_i^j(\mathcal{A})\big)^2 + 4a_{ij\cdots j}r_j(\mathcal{A})}\Big).$$

Furthermore, $\Omega_{\max} \leq R_{\max}$.

In this paper, we continue this research, and we give a lower bound and an upper bound for $\rho(\mathcal{A})$ of a nonnegative tensor $\mathcal{A}$, which all depend only on the entries of $\mathcal{A}$. It is proved that these bounds are shaper than the corresponding bounds in [21] and [22]. A numerical example is also given to verify the obtained results.

2 New bounds for the spectral radius of nonnegative tensors

In this section, bounds for the spectral radius of a nonnegative tensors are obtained. We first give some notation. Given a nonnegative tensor $\mathcal{A} = (a_{i_1\cdots i_m}) \in R^{[m,n]}$, we denote

$$\Theta_i = \bigl\{(i_2, i_3, \ldots, i_m) : i_j = i \text{ for some } j \in \{2, \ldots, m\}, \text{where } i, i_2, \ldots, i_m \in N\bigr\},$$

$$\overline{\Theta}_i = \bigl\{(i_2, i_3, \ldots, i_m) : i_j \neq i \text{ for any } j \in \{2, \ldots, m\}, \text{where } i, i_2, \ldots, i_m \in N\bigr\},$$

$$r_i(\mathcal{A}) = \sum_{\substack{i_2,\ldots,i_m \in N,\\ \delta_{ii_2\cdots i_m}=0}} a_{ii_2\cdots i_m} = \sum_{i_2,\ldots,i_m \in N} a_{ii_2\cdots i_m} - a_{i\cdots i} = R_i(\mathcal{A}) - a_{i\cdots i},$$

$$r_i^j(\mathcal{A}) = \sum_{\substack{\delta_{ii_2\ldots i_m}=0,\\ \delta_{ji_2\cdots i_m}=0}} a_{ii_2\cdots i_m} = \sum_{\substack{i_2,\ldots,i_m \in N,\\ \delta_{ii_2\ldots i_m}=0}} a_{ii_2\cdots i_m} - a_{ij\cdots j} = r_i(\mathcal{A}) - a_{ij\cdots j},$$

$$r_i^{\Theta_i}(\mathcal{A}) = \sum_{\substack{(i_2,\ldots,i_m)\in\Theta_i,\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|, \qquad r_i^{\overline{\Theta}_i}(\mathcal{A}) = \sum_{(i_2,\ldots,i_m)\in\overline{\Theta}_i} |a_{ii_2\cdots i_m}|,$$

where

$$\delta_{i_1\cdots i_m} = \begin{cases} 1, & \text{if } i_1 = \cdots = i_m, \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, $r_i(\mathcal{A}) = r_i^{\Theta_i}(\mathcal{A}) + r_i^{\overline{\Theta}_i}(\mathcal{A})$, and $r_i^j(\mathcal{A}) = r_i^{\Theta_i}(\mathcal{A}) + r_i^{\overline{\Theta}_i}(\mathcal{A}) - |a_{ij\cdots j}|$.

For an irreducible nonnegative tensor, we give the following bounds for the spectral radius.

Lemma 1 *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in R^{[m,n]}$ be an irreducible nonnegative tensor with $n \geq 2$. Then*

$$\Delta_{\min} \leq \rho(\mathcal{A}) \leq \Delta_{\max},$$

where

$$\Delta_{\min} = \min_{\substack{i,j\in N,\\ j\neq i}} \Delta_{i,j}(\mathcal{A}), \qquad \Delta_{\max} = \max_{\substack{i,j\in N,\\ j\neq i}} \Delta_{i,j}(\mathcal{A})$$

and

$$\Delta_{i,j}(\mathcal{A}) = \frac{1}{2}\Bigl(a_{i\cdots i} + a_{j\cdots j} + r_i^{\Theta_i}(\mathcal{A}) + \sqrt{\bigl(a_{i\cdots i} - a_{j\cdots j} + r_i^{\Theta_i}(\mathcal{A})\bigr)^2 + 4r_i^{\overline{\Theta}_i}(\mathcal{A})r_j(\mathcal{A})}\Bigr).$$

Proof Let $x = (x_1, x_2, \ldots, x_n)^T$ be an entrywise positive eigenvector of $\mathcal{A}$ corresponding to $\rho(\mathcal{A})$, that is,

$$\mathcal{A}x^{m-1} = \rho(\mathcal{A})x^{[m-1]}. \tag{1}$$

Without loss of generality, suppose that

$$x_{t_n} \geq x_{t_{n-1}} \geq \cdots \geq x_{t_2} \geq x_{t_1} > 0.$$

(i) We first prove

$$\Delta_{\min} = \min_{\substack{i,j\in N,\\ j\neq i}} \Delta_{i,j}(\mathcal{A}) \leq \rho(\mathcal{A}).$$

From (1), we have

$$\sum_{i_2,\ldots,i_m\in N} a_{t_1 i_2\cdots i_m} x_{i_2}\cdots x_{i_m} = \rho(\mathcal{A}) x_{t_1}^{m-1},$$

equivalently,

$$\bigl(\rho(\mathcal{A}) - a_{t_1\cdots t_1}\bigr) x_{t_1}^{m-1} = \sum_{\substack{(i_2,\ldots,i_m)\in\Theta_{t_1},\\ \delta_{t_1 i_2\ldots i_m}=0}} a_{t_1 i_2\cdots i_m} x_{i_2}\cdots x_{i_m} + \sum_{(i_2,\ldots,i_m)\in\overline{\Theta}_{t_1}} a_{t_1 i_2\cdots i_m} x_{i_2}\cdots x_{i_m}.$$

Hence,

$$\begin{aligned}\bigl(\rho(\mathcal{A}) - a_{t_1\cdots t_1}\bigr) x_{t_1}^{m-1} &\geq \sum_{\substack{(i_2,\ldots,i_m)\in\Theta_{t_1},\\ \delta_{t_1 i_2\ldots i_m}=0}} a_{t_1 i_2\cdots i_m} x_{t_1}^{m-1} + \sum_{(i_2,\ldots,i_m)\in\overline{\Theta}_{t_1}} a_{t_1 i_2\cdots i_m} x_{t_2}^{m-1}\\ &= r_{t_1}^{\Theta_{t_1}}(\mathcal{A}) x_{t_1}^{m-1} + r_{t_1}^{\overline{\Theta}_{t_1}}(\mathcal{A}) x_{t_2}^{m-1},\end{aligned}$$

i.e.,

$$\bigl(\rho(\mathcal{A}) - a_{t_1\cdots t_1} - r_{t_1}^{\Theta_{t_1}}(\mathcal{A})\bigr) x_{t_1}^{m-1} \geq r_{t_1}^{\overline{\Theta}_{t_1}}(\mathcal{A}) x_{t_2}^{m-1} \geq 0. \tag{2}$$

Similarly, we have, from (1),

$$\sum_{i_2,\ldots,i_m\in N} a_{t_2 i_2\cdots i_m} x_{i_2}\cdots x_{i_m} = \rho(\mathcal{A}) x_{t_2}^{m-1}$$

and

$$\bigl(\rho(\mathcal{A}) - a_{t_2\cdots t_2}\bigr) x_{t_2}^{m-1} \geq r_{t_2}(\mathcal{A}) x_{t_1}^{m-1} \geq 0. \tag{3}$$

Multiplying inequality (3) with inequality (2) gives

$$\bigl(\rho(\mathcal{A}) - a_{t_1\cdots t_1} - r_{t_1}^{\Theta_{t_1}}(\mathcal{A})\bigr)\bigl(\rho(\mathcal{A}) - a_{t_2\cdots t_2}\bigr) x_{t_1}^{m-1} x_{t_2}^{m-1} \geq r_{t_2}(\mathcal{A}) r_{t_1}^{\overline{\Theta}_{t_1}}(\mathcal{A}) x_{t_1}^{m-1} x_{t_2}^{m-1}.$$

Note that $x_{t_2} \geq x_{t_1} > 0$, hence

$$\bigl(\rho(\mathcal{A}) - a_{t_1\cdots t_1} - r_{t_1}^{\Theta_{t_1}}(\mathcal{A})\bigr)\bigl(\rho(\mathcal{A}) - a_{t_2\cdots t_2}\bigr) \geq r_{t_2}(\mathcal{A}) r_{t_1}^{\overline{\Theta}_{t_1}}(\mathcal{A}),$$

that is,

$$\rho(\mathcal{A})^2 - \bigl(a_{t_1\cdots t_1} + a_{t_2\cdots t_2} + r_{t_1}^{\Theta_{t_1}}(\mathcal{A})\bigr)\rho(\mathcal{A}) + a_{t_2\cdots t_2}\bigl(a_{t_1\cdots t_1} + r_{t_1}^{\Theta_{t_1}}(\mathcal{A})\bigr) \geq r_{t_2}(\mathcal{A}) r_{t_1}^{\overline{\Theta}_{t_1}}(\mathcal{A}).$$

Furthermore, since

$$\left(a_{t_1\cdots t_1}+a_{t_2\cdots t_2}+r_{t_1}^{\Theta_{t_1}}(\mathcal{A})\right)^2-4a_{t_2\cdots t_2}\left(a_{t_1\cdots t_1}+r_{t_1}^{\Theta_{t_1}}(\mathcal{A})\right)=\left(a_{t_1\cdots t_1}-a_{t_2\cdots t_2}+r_{t_1}^{\Theta_{t_1}}(\mathcal{A})\right)^2,$$

then solving for $\rho(\mathcal{A})$ gives

$$\rho(\mathcal{A})\geq\Delta_{t_1,t_2}(\mathcal{A})\geq\min_{\substack{i,j\in N,\\ j\neq i}}\Delta_{i,j}(\mathcal{A})=\Delta_{\min}.$$

(ii) We now prove

$$\rho(\mathcal{A})\leq\max_{\substack{i,j\in N,\\ j\neq i}}\Delta_{i,j}(\mathcal{A})=\Delta_{\max}.$$

From (1), we have

$$\sum_{i_2,\ldots,i_m\in N}a_{t_n i_2\cdots i_m}x_{i_2}\cdots x_{i_m}=\rho(\mathcal{A})x_{t_n}^{m-1}$$

and

$$\sum_{i_2,\ldots,i_m\in N}a_{t_{n-1}i_2\cdots i_m}x_{i_2}\cdots x_{i_m}=\rho(\mathcal{A})x_{t_{n-1}}^{m-1}.$$

Similar to the proof in (i), we obtain easily

$$\rho(\mathcal{A})\leq\Delta_{t_n,t_{n-1}}(\mathcal{A})\leq\max_{\substack{i,j\in N,\\ j\neq i}}\Delta_{i,j}(\mathcal{A})=\Delta_{\max}.$$

The conclusion follows from (i) and (ii). □

Now we establish upper and lower bounds for $\rho(\mathcal{A})$ of a nonnegative tensor $\mathcal{A}$.

Lemma 2 (Lemma 3.3 in [21]) *Suppose* $0\leq\mathcal{A}<\mathcal{C}$. *Then* $\rho(\mathcal{A})\leq\rho(\mathcal{C})$.

Theorem 5 *Let* $\mathcal{A}=(a_{i_1\cdots i_m})\in R^{[m,n]}$ *be a nonnegative tensor with* $n\geq 2$. *Then*

$$\Delta_{\min}\leq\rho(\mathcal{A})\leq\Delta_{\max}.$$

Proof Let $\mathcal{A}_k=\mathcal{A}+\frac{1}{k}\mathcal{E}$, where $k=1,2,\ldots,$ and $\mathcal{E}$ denote the tensor with every entry being 1. Then $\mathcal{A}_k$ is a sequence of positive tensors satisfying

$$0\leq\mathcal{A}<\cdots\mathcal{A}_{k+1}<\mathcal{A}_k<\cdots<\mathcal{A}_1.$$

By Lemma 2, $\{\rho(\mathcal{A}_k)\}_{k=1}^{+\infty}$ is a monotone decreasing sequence with lower bound $\rho(\mathcal{A})$. From the proof of Theorem 2.3 in [21], we have

$$\lim_{k\to+\infty}\rho(\mathcal{A}_k)=\rho(\mathcal{A}).$$

Note that for any $i, j \in N$, $j \neq i$,

$$\Delta_{i,j}(\mathcal{A}) < \cdots < \Delta_{i,j}(\mathcal{A}_{k+1}) < \Delta_{i,j}(\mathcal{A}_k) < \cdots < \Delta_{i,j}(\mathcal{A}_1),$$

we obtain easily

$$\lim_{k \to +\infty} \Delta_{i,j}(\mathcal{A}_k) = \Delta_{i,j}(\mathcal{A}).$$

Furthermore, since $\mathcal{A}_k$ is positive and also irreducible nonnegative for $k = 1, 2, \ldots$, we have, from Lemma 1,

$$\min_{\substack{i,j \in N, \\ j \neq i}} \Delta_{i,j}(\mathcal{A}_k) \leq \rho(\mathcal{A}_k) \leq \max_{\substack{i,j \in N, \\ j \neq i}} \Delta_{i,j}(\mathcal{A}_k).$$

Letting $k \to +\infty$, then

$$\Delta_{\min} = \min_{\substack{i,j \in N, \\ j \neq i}} \Delta_{i,j}(\mathcal{A}) \leq \rho(\mathcal{A}) \leq \max_{\substack{i,j \in N, \\ j \neq i}} \Delta_{i,j}(\mathcal{A}) = \Delta_{\max}.$$

The proof is completed. □

We next compare the bounds in Theorem 5 with those in Theorem 3.

Theorem 6 *Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in R^{[m,n]}$ be a nonnegative tensor with $n \geq 2$. Then*

$$R_{\min} \leq \Delta_{\min} \leq \Delta_{\max} \leq R_{\max}. \tag{4}$$

Proof We first prove $R_{\min} \leq \Delta_{\min}$. For any $i, j \in N$, $j \neq i$, if $R_i(\mathcal{A}) \leq R_j(\mathcal{A})$, then

$$a_{ii\cdots i} - a_{jj\cdots j} + r_i^{\Theta_i}(\mathcal{A}) + r_i^{\overline{\Theta}_i}(\mathcal{A}) \leq r_j(\mathcal{A}).$$

Hence,

$$\begin{aligned}
&\left(a_{i\cdots i} - a_{j\cdots j} + r_i^{\Theta_i}(\mathcal{A})\right)^2 + 4r_i^{\overline{\Theta}_i}(\mathcal{A}) r_j(\mathcal{A}) \\
&\quad \geq \left(a_{i\cdots i} - a_{j\cdots j} + r_i^{\Theta_i}(\mathcal{A})\right)^2 \\
&\qquad + 4r_i^{\overline{\Theta}_i}(\mathcal{A})\left(a_{ii\cdots i} - a_{jj\cdots j} + r_i^{\Theta_i}(\mathcal{A}) + r_i^{\overline{\Theta}_i}(\mathcal{A})\right) \\
&\quad = \left(a_{i\cdots i} - a_{j\cdots j} + r_i^{\Theta_i}(\mathcal{A})\right)^2 \\
&\qquad + 4r_i^{\overline{\Theta}_i}(\mathcal{A})\left(a_{ii\cdots i} - a_{jj\cdots j} + r_i^{\Theta_i}(\mathcal{A})\right) + 4\left(r_i^{\overline{\Theta}_i}(\mathcal{A})\right)^2 \\
&\quad = \left(a_{i\cdots i} - a_{j\cdots j} + r_i^{\Theta_i}(\mathcal{A}) + 2r_i^{\overline{\Theta}_i}(\mathcal{A})\right)^2.
\end{aligned}$$

When

$$a_{i\cdots i} - a_{j\cdots j} + r_i^{\Theta_i}(\mathcal{A}) + 2r_i^{\overline{\Theta}_i}(\mathcal{A}) > 0,$$

we have

$$\begin{aligned}
&a_{i\cdots i} + a_{j\cdots j} + r_i^{\Theta_i}(\mathcal{A}) + \sqrt{\left(a_{i\cdots i} - a_{j\cdots j} + r_i^{\Theta_i}(\mathcal{A})\right)^2 + 4r_i^{\overline{\Theta}_i}(\mathcal{A})r_j(\mathcal{A})} \\
&\quad \geq a_{i\cdots i} + a_{j\cdots j} + r_i^{\Theta_i}(\mathcal{A}) + \left(a_{i\cdots i} - a_{j\cdots j} + r_i^{\Theta_i}(\mathcal{A}) + 2r_i^{\overline{\Theta}_i}(\mathcal{A})\right) \\
&\quad = 2\left(a_{i\cdots i} + r_i^{\Theta_i}(\mathcal{A}) + r_i^{\overline{\Theta}_i}(\mathcal{A})\right) \\
&\quad = 2R_i(\mathcal{A}).
\end{aligned}$$

When

$$a_{i\cdots i} - a_{j\cdots j} + r_i^{\Theta_i}(\mathcal{A}) + 2r_i^{\overline{\Theta}_i}(\mathcal{A}) \leq 0,$$

that is,

$$a_{i\cdots i} + r_i^{\Theta_i}(\mathcal{A}) + 2r_i^{\overline{\Theta}_i}(\mathcal{A}) \leq a_{j\cdots j},$$

we have

$$\begin{aligned}
&a_{i\cdots i} + a_{j\cdots j} + r_i^{\Theta_i}(\mathcal{A}) + \sqrt{\left(a_{i\cdots i} - a_{j\cdots j} + r_i^{\Theta_i}(\mathcal{A})\right)^2 + 4r_i^{\overline{\Theta}_i}(\mathcal{A})r_j(\mathcal{A})} \\
&\quad \geq a_{i\cdots i} + a_{j\cdots j} + r_i^{\Theta_i}(\mathcal{A}) + \sqrt{\left(a_{i\cdots i} - a_{j\cdots j} + r_i^{\Theta_i}(\mathcal{A})\right)^2} \\
&\quad = a_{i\cdots i} + a_{j\cdots j} + r_i^{\Theta_i}(\mathcal{A}) - \left(a_{i\cdots i} - a_{j\cdots j} + r_i^{\Theta_i}(\mathcal{A})\right) \\
&\quad = 2a_{j\cdots j} \\
&\quad \geq 2\left(a_{i\cdots i} + r_i^{\Theta_i}(\mathcal{A}) + 2r_i^{\overline{\Theta}_i}(\mathcal{A})\right) \\
&\quad \geq 2\left(a_{i\cdots i} + r_i^{\Theta_i}(\mathcal{A}) + r_i^{\overline{\Theta}_i}(\mathcal{A})\right) \\
&\quad = 2R_i(\mathcal{A}).
\end{aligned}$$

Therefore,

$$\frac{1}{2}\left(a_{i\cdots i} + a_{j\cdots j} + r_i^{\Theta_i}(\mathcal{A}) + \sqrt{\left(a_{i\cdots i} - a_{j\cdots j} + r_i^{\Theta_i}(\mathcal{A})\right)^2 + 4r_i^{\overline{\Theta}_i}(\mathcal{A})r_j(\mathcal{A})}\right) \geq R_i(\mathcal{A}),$$

which implies

$$\begin{aligned}
&\min_{\substack{i,j\in N,\\ j\neq i}} \frac{1}{2}\left(a_{i\cdots i} + a_{j\cdots j} + r_i^{\Theta_i}(\mathcal{A}) + \sqrt{\left(a_{i\cdots i} - a_{j\cdots j} + r_i^{\Theta_i}(\mathcal{A})\right)^2 + 4r_i^{\overline{\Theta}_i}(\mathcal{A})r_j(\mathcal{A})}\right) \\
&\quad \geq \min_{i\in N} R_i(\mathcal{A}),
\end{aligned}$$

i.e., $R_{\min} \leq \Delta_{\min}$.

On the other hand, if for any $i, j \in N$, $j \neq i$,

$$R_j(\mathcal{A}) \leq R_i(\mathcal{A}),$$

then

$$a_{jj\cdots j} - a_{ii\cdots i} - r_i^{\Theta_i}(\mathcal{A}) + r_j(\mathcal{A}) \le r_i^{\overline{\Theta}_i}(\mathcal{A}).$$

Similarly, we can also obtain

$$\frac{1}{2}\Bigl(a_{i\cdots i} + a_{j\cdots j} + r_i^{\Theta_i}(\mathcal{A}) + \sqrt{\bigl(a_{i\cdots i} - a_{j\cdots j} + r_i^{\Theta_i}(\mathcal{A})\bigr)^2 + 4r_i^{\overline{\Theta}_i}(\mathcal{A})r_j(\mathcal{A})}\Bigr) \ge R_j(\mathcal{A}),$$

and that $R_{\min} \le \Delta_{\min}$. Hence, the first inequality in (4) holds. In a similar way, we can prove that the last inequality in (4) also holds. The conclusion follows. □

Example 1 Consider the nonnegative tensor

$$\mathcal{A} = \bigl[A(:,:,1), A(:,:,2), A(:,:,3)\bigr] \in R^{[3,3]},$$

where

$$A(:,:,1) = \begin{pmatrix} 0.2192 & 0.4411 & 0.5232 \\ 0.7637 & 0.5239 & 0.8330 \\ 0.7993 & 0.3710 & 0.5328 \end{pmatrix},$$

$$A(:,:,2) = \begin{pmatrix} 0.4380 & 0.0482 & 0.1325 \\ 0.1803 & 0.6729 & 0.1809 \\ 0.3773 & 0.1079 & 0.8965 \end{pmatrix},$$

$$A(:,:,3) = \begin{pmatrix} 0.0779 & 0.1982 & 0.4691 \\ 0.5135 & 0.8284 & 0.7352 \\ 0.1135 & 0.1163 & 0.8645 \end{pmatrix}.$$

We now compute the bounds for $\rho(\mathcal{A})$. By Theorem 3, we have

$$2.5474 \le \rho(\mathcal{A}) \le 5.2318.$$

By Theorem 4, we have

$$\rho(\mathcal{A}) \le 5.0753.$$

By Theorem 5, we have

$$3.0097 \le \rho(\mathcal{A}) \le 4.7894.$$

It is easy to see that the bounds in Theorem 5 are sharper than those in Theorem 3 (Lemma 5.2 of [21]), and that the upper bound in Theorem 5 is sharper than that in Theorem 4 (Theorem 3.3 of [22]) in some cases.

3 Conclusions

In this paper, we obtain a lower and an upper bound for the spectral radius of a nonnegative tensor, which improved the known bounds obtained by Yang and Yang [21], and Li *et al.* [22].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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