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# Discrete Grüss type inequality on fractional calculus

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**Abstract**
We give a discrete Grüss type inequality on fractional calculus.

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**Keywords:** discrete fractional calculus; discrete Grüss inequality

## 1 Introduction

Motivated by Grüss [1], our purpose is to prove more general versions of Grüss type inequalities for delta discrete fractional calculus. It is well known that Grüss type inequalities in continuous and discrete cases play a crucial role in studying the qualitative behavior of differential and difference equations, respectively, as well as many other areas of mathematics [2–9]. For the background and a summary on these particular subjects, we refer the interested reader to the excellent references [2, 10–18].

The study of discrete fractional calculus was pioneered by Diaz and Osler [19]. In the mentioned work, the authors used an infinite sum to give a definition of discrete fractional sum, whereas Gray and Zhang used a finite sum in [20]. In the last decade, new results in this area have been established [21–24], as well as importance has been gained by inequalities on discrete fractional calculus in [10, 24–27]

## 2 Preliminaries

We begin with basic definitions and results from [10].

**Definition 1** The $\nu$th fractional sum of $f$ is defined by

$$\Delta^{-\nu}f(t,a)=\frac{1}{\Gamma(\nu)}\sum_{s=a}^{t-\nu}(t-s-1)^{\underline{\nu-1}}f(s),$$

where $f$ and $\Delta^{-\nu}f$ are defined for $s=a \bmod(1)$ and $t=(a+\nu)\bmod(1)$, respectively. In particular, $\Delta^{-\nu}$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+\nu}$, where $\mathbb{N}_t=\{t,t+1,t+2,\dots\}$.

Here,

$$t^{\underline{\nu}}:=\frac{\Gamma(t+1)}{\Gamma(t-\nu+1)}.$$

From now on in this context for convenience we set $\Delta^{-\nu}f(t,a)=\Delta^{-\nu}f(t)$.

**Theorem 1** [28] *Let $f$ be a real valued function defined on $\mathbb{N}_a$ and let $\mu, \nu > 0$. Then*

$$\Delta^{-\nu}\left(\Delta^{-\mu}f(t)\right) = \Delta^{-(\mu+\nu)}f(t) = \Delta^{-\mu}\left(\Delta^{-\nu}f(t)\right) \quad \text{for all } t \in \mathbb{N}_{a+\mu+\nu}.$$

**Theorem 2** [21] *For $\nu > 0$ and $p$ a positive integer we have*

$$\Delta^{-\nu}\Delta^{p}f(t) = \Delta^{p}\Delta^{-\nu}f(t) - \sum_{k=0}^{\nu-1}\frac{(t-a)^{\underline{\nu-p+k}}}{\Gamma(\nu+k-p+1)}\Delta^{k}f(a),$$

*where $f$ is defined on $\mathbb{N}_a$.*

**Remark 1** Let $\mu > 0$ and $m-1 < \mu < m$, $m = \lceil\mu\rceil$, where $m$ is a positive integer, and set $\nu = m - \mu > 0$. Then by Theorem 2 we have

$$\Delta^{-\nu}\Delta^{m}f(t) = \Delta^{m}\Delta^{-\nu}f(t) - \sum_{k=0}^{m-1}\frac{(t-a)^{\underline{\nu-m+k}}}{\Gamma(\nu+k-m+1)}\Delta^{k}f(a),$$

where $f$ is defined on $\mathbb{N}_a$ and hence

$$\Delta^{m}\Delta^{-\nu}f(t) = \Delta_{*}^{\mu}f(t) + \sum_{k=0}^{m-1}\frac{(t-a)^{\underline{\nu-m+k}}}{\Gamma(\nu+k-m+1)}\Delta^{k}f(a). \tag{2.1}$$

**Definition 2** [21] The $\mu$th fractional Riemann-Liouville type difference is defined by

$$\Delta^{\mu}f(t) := \Delta^{m-\nu}f(t) := \Delta^{m}\left(\Delta^{-\nu}f(t)\right),$$

where $\mu > 0$, $m-1 < \mu < m$, and $\nu = m - \mu > 0$.

So from (2.1) we get

$$\Delta^{\mu}f(t) = \Delta_{*}^{\mu}f(t) + \sum_{k=0}^{m-1}\frac{(t-a)^{\underline{\nu-m+k}}}{\Gamma(\nu+k-m+1)}\Delta^{k}f(a), \tag{2.2}$$

where $f$ is defined on $\mathbb{N}_a$.

**Theorem 3** [10] *For $\mu > 0$, $\mu$ noninteger, $m = \lceil\mu\rceil$, $\nu = m - \mu$, the following holds*:

$$f(t) = \sum_{k=0}^{m-1}\frac{(t-a)^{\underline{k}}}{k!}\Delta^{k}f(a) + \frac{1}{\Gamma(\mu)}\sum_{s=a+\nu}^{t-\mu}(t-s-1)^{\underline{\mu-1}}\Delta_{*}^{\mu}f(s) \tag{2.3}$$

*for all $t \in \mathbb{N}_{a+m}$, where $f$ is defined on $\mathbb{N}_a$ with $a \in \mathbb{Z}^{+} := \{0, 1, 2, \ldots\}$.*

**Remark 2** Here $[a, b]$ denotes the discrete interval $[a, b] = [a, a+1, a+2, \ldots, b]$, where $a < b$ and $a, b \in \{0, 1, \ldots\}$. Let $\mu > 0$ be noninteger such that $m-1 < \mu < m$, *i.e.* $m = \lceil\mu\rceil$. Consider a function $f$ defined on $[a, b]$. Then clearly the fractional discrete Taylor formula (2.3) is valid only for $t \in [a+m, b]$, $a+m < b$.

We now give a discrete Caputo type fractional extended Taylor formula.

**Theorem 4** [10] *Let $\mu > p$, $p \in \mathbb{N}$, $\mu$ not integer, $m = \lceil \mu \rceil$, $\nu = m - \mu$. Then*

$$\Delta^p f(t) = \sum_{k=p}^{m-1} \frac{(t-a)^{\underline{k-p}}}{(k-p)!} \Delta^k f(a) + \frac{1}{\Gamma(\mu - p)} \sum_{s=a+\nu}^{t-\mu+p} (t-s-1)^{\underline{\mu-p-1}} \Delta_*^\mu f(s) \tag{2.4}$$

*for all $t \in \mathbb{N}_{a+m-p}$, where $f$ is defined on $\mathbb{N}_a$, $a \in \mathbb{Z}^+$.*

**Remark 3** We assume that $f$ is defined on $[a,b]$. Then (2.4) is valid only for $[a+m-p, b]$ with $a+m-p < b$. Notice $p = 0$ applied to (2.4) yields (2.3).

**Remark 4** For $\mu > 0$, $\mu$ not an integer, $m = \lceil \mu \rceil$, $\nu = m - \mu$, $f$ defined on $\mathbb{N}_a$, $a \in \mathbb{Z}^+$ and $\Delta^k f(a)$ for $k = 0, \ldots, m-1$, we get

$$f(t) = \frac{1}{\Gamma(\mu)} \sum_{s=a+\nu}^{t-\mu} (t-s-1)^{\underline{\mu-1}} \Delta_*^\mu f(s) \quad \text{for all } t \in \mathbb{N}_{a+m}. \tag{2.5}$$

**Remark 5** For $\mu > p$, $p \in \mathbb{N}$, $\mu$ noninteger, $m = \lceil \mu \rceil$, $\nu = m - \mu$; $f$ defined on $\mathbb{N}_a$, $a \in \mathbb{Z}^+$, if we assume that $\Delta^k f(a) = 0$, $k = p, \ldots, m-1$, then we obtain

$$\Delta^p f(t) = \frac{1}{\Gamma(\mu - p)} \sum_{s=a+\nu}^{t-\mu+p} (t-s-1)^{\underline{\mu-p-1}} \Delta_*^\mu f(s) \quad \text{for all } t \in \mathbb{N}_{a+m-p}. \tag{2.6}$$

## 3 Main results

We present the following discrete delta Grüss type inequality.

**Theorem 5** *Let $\mu > p$, $p \in \mathbb{Z}^+$, $\mu$ not an integer, $m = \lceil \mu \rceil$, $\nu = m - \mu$, $f, g$ be defined on $\mathbb{N}_a$, $a \in \mathbb{Z}^+$ and $a + m - p < b$, $b \in \mathbb{N}$. Assume that*

$$\Delta^k f(a) = \Delta^k g(a) = 0 \quad \textit{for } k = p+1, \ldots, m-1, p < m-2,$$

*and*

$$m_1 \le \Delta_*^\mu f(s) \le M_1, \qquad m_2 \le \Delta_*^\mu g(s) \le M_2$$

*for $s = a+1, \ldots, b$, where $m_1$, $m_2$, $M_1$, and $M_2$ are positive constants. Then*

$$\begin{aligned} &\frac{1}{b-a-m+p} \sum_{j=a+m-p+1}^{b} \bigl[\bigl(\Delta^p f(j)\bigr)\bigl(\Delta^p g(j)\bigr)\bigr] \\ &\quad - \frac{1}{(b-a-m+p)^2} \Biggl[\sum_{j=a+m-p+1}^{b} \Delta^p f(j)\Biggr]\Biggl[\sum_{j=a+m-p+1}^{b} \Delta^p g(j)\Biggr] \\ &\quad \le \frac{M_1 M_2 C_1 - m_1 m_2 C_2}{[\Gamma(\mu - p + 1)]^2}, \end{aligned}$$

*where*

$$C_1 := (b-a-m+p)\sum_{j=a+m-p+1}^{b}\bigl[(j-a-\nu)^{\underline{\mu-p}}\bigr]^2$$

*and*

$$C_2 := \frac{[(b-a-\nu+1)^{\underline{\mu-p+1}}-(m-p-\nu+1)^{\underline{\mu-p+1}}]^2}{(\mu-p+1)^2}.$$

*Proof* By (2.4), we have

$$\Delta^p f(j) = \sum_{k=p}^{m-1}\frac{(j-a)^{\underline{k-p}}}{(k-p)!}\Delta^k f(a) + \frac{1}{\Gamma(\mu-p)}\sum_{s=a+\nu}^{j-\mu+p}(j-s-1)^{\underline{\mu-p-1}}\Delta_*^\mu f(s).$$

By hypothesis $\Delta^k f(a) = 0$, $k = p+1,\ldots,m-1$, $p < m-2$. So we have

$$\Delta^p f(j) = \Delta^p f(a) + \frac{1}{\Gamma(\mu-p)}\sum_{s=a+\nu}^{j-\mu+p}(j-s-1)^{\underline{\mu-p-1}}\bigl(\Delta_*^\mu f(s)\bigr) \tag{3.1}$$

and

$$\Delta^p g(j) = \Delta^p g(a) + \frac{1}{\Gamma(\mu-p)}\sum_{s=a+\nu}^{j-\mu+p}(j-s-1)^{\underline{\mu-p-1}}\bigl(\Delta_*^\mu g(s)\bigr) \tag{3.2}$$

for all $j \in [a+m-p+1,\ldots,b]$. Multiplying (3.1) and (3.2) gives us

$$\begin{aligned}\bigl(\Delta^p f(j)\bigr)\bigl(\Delta^p g(j)\bigr) &= \bigl(\Delta^p f(a)\bigr)\bigl(\Delta^p g(a)\bigr) + \frac{1}{[\Gamma(\mu-p)]^2}\\ &\quad\times\left[\sum_{s=a+\nu}^{j-\mu+p}(j-s-1)^{\underline{\mu-p-1}}\bigl(\Delta_*^\mu f(s)\bigr)\right]\left[\sum_{s=a+\nu}^{j-\mu+p}(j-s-1)^{\underline{\mu-p-1}}\bigl(\Delta_*^\mu g(s)\bigr)\right]\\ &\quad+\frac{\Delta^p f(a)}{\Gamma(\mu-p)}\left[\sum_{s=a+\nu}^{j-\mu+p}(j-s-1)^{\underline{\mu-p-1}}\bigl(\Delta_*^\mu g(s)\bigr)\right]\\ &\quad+\frac{\Delta^p g(a)}{\Gamma(\mu-p)}\left[\sum_{s=a+\nu}^{j-\mu+p}(j-s-1)^{\underline{\mu-p-1}}\bigl(\Delta_*^\mu f(s)\bigr)\right].\end{aligned}$$

Summing from $a+m-p+1$ to $b$ yields

$$\begin{aligned}&\sum_{j=a+m-p+1}^{b}\bigl(\Delta^p f(j)\bigr)\bigl(\Delta^p g(j)\bigr)\\ &\quad=\sum_{j=a+m-p+1}^{b}\bigl(\Delta^p f(a)\bigr)\bigl(\Delta^p g(a)\bigr) + \frac{1}{[\Gamma(\mu-p)]^2}\\ &\qquad\times\sum_{j=a+m-p+1}^{b}\left\{\left[\sum_{s=a+\nu}^{j-\mu+p}(j-s-1)^{\underline{\mu-p-1}}\bigl(\Delta_*^\mu f(s)\bigr)\right]\left[\sum_{s=a+\nu}^{j-\mu+p}(j-s-1)^{\underline{\mu-p-1}}\bigl(\Delta_*^\mu g(s)\bigr)\right]\right\}\end{aligned}$$

$$+\frac{\Delta^p f(a)}{\Gamma(\mu-p)}\sum_{j=a+m-p+1}^{b}\left[\sum_{s=a+\nu}^{j-\mu+p}(j-s-1)^{\underline{\mu-p-1}}\left(\Delta_*^{\mu}g(s)\right)\right]$$
$$+\frac{\Delta^p g(a)}{\Gamma(\mu-p)}\sum_{j=a+m-p+1}^{b}\left[\sum_{s=a+\nu}^{j-\mu+p}(j-s-1)^{\underline{\mu-p-1}}\left(\Delta_*^{\mu}f(s)\right)\right].$$

Then

$$\frac{1}{b-a-m+p}\sum_{j=a+m-p+1}^{b}\left(\Delta^p f(j)\right)\left(\Delta^p g(j)\right)$$
$$=\frac{1}{b-a-m+p}\sum_{j=a+m-p+1}^{b}\left[\left(\Delta^p f(a)\right)\left(\Delta^p g(a)\right)\right]+\frac{1}{(b-a-m+p)[\Gamma(\mu-p)]^2}$$
$$\times\sum_{j=a+m-p+1}^{b}\left\{\left[\sum_{s=a+\nu}^{j-\mu+p}(j-s-1)^{\underline{\mu-p-1}}\left(\Delta_*^{\mu}f(s)\right)\right]\left[\sum_{s=a+\nu}^{j-\mu+p}(j-s-1)^{\underline{\mu-p-1}}\left(\Delta_*^{\mu}g(s)\right)\right]\right\}$$
$$+\frac{\Delta^p f(a)}{(b-a-m+p)\Gamma(\mu-p)}\sum_{j=a+m-p+1}^{b}\left[\sum_{s=a+\nu}^{j-\mu+p}(j-s-1)^{\underline{\mu-p-1}}\left(\Delta_*^{\mu}g(s)\right)\right]$$
$$+\frac{\Delta^p g(a)}{(b-a-m+p)\Gamma(\mu-p)}\sum_{j=a+m-p+1}^{b}\left[\sum_{s=a+\nu}^{j-\mu+p}(j-s-1)^{\underline{\mu-p-1}}\left(\Delta_*^{\mu}f(s)\right)\right]. \tag{3.3}$$

On the other hand,

$$\frac{1}{b-a-m+p}\sum_{j=a+m-p+1}^{b}\left(\Delta^p f(j)\right)$$
$$=\Delta^p f(a)+\frac{1}{(b-a-m+p)\Gamma(\mu-p)}\sum_{j=a+m-p+1}^{b}\left[\sum_{s=a+\nu}^{j-\mu+p}(j-s-1)^{\underline{\mu-p-1}}\left(\Delta_*^{\mu}f(s)\right)\right]$$

and

$$\frac{1}{(b-a-m+p)}\sum_{j=a+m-p+1}^{b}\left(\Delta^p g(j)\right)$$
$$=\Delta^p g(a)+\frac{1}{(b-a-m+p)\Gamma(\mu-p)}\sum_{j=a+m-p+1}^{b}\left[\sum_{s=a+\nu}^{j-\mu+p}(j-s-1)^{\underline{\mu-p-1}}\left(\Delta_*^{\mu}g(s)\right)\right].$$

Multiplying the above two terms yields

$$\frac{1}{(b-a-m+p)^2}\left[\sum_{j=a+m-p+1}^{b}\left(\Delta^p f(j)\right)\right]\left[\sum_{j=a+m-p+1}^{b}\left(\Delta^p g(j)\right)\right]$$
$$=\left(\Delta^p f(a)\right)\left(\Delta^p g(a)\right)$$
$$+\frac{\Delta^p f(a)}{(b-a-m+p)\Gamma(\mu-p)}\sum_{j=a+m-p+1}^{b}\left[\sum_{s=a+\nu}^{j-\mu+p}(j-s-1)^{\underline{\mu-p-1}}\left(\Delta_*^{\mu}g(s)\right)\right]$$

$$+\frac{\Delta^p g(a)}{(b-a-m+p)\Gamma(\mu-p)}\sum_{j=a+m-p+1}^{b}\left[\sum_{s=a+\nu}^{j-\mu+p}(j-s-1)^{\underline{\mu-p-1}}\left(\Delta_*^{\mu}f(s)\right)\right]$$
$$+\frac{1}{(b-a-m+p)^2\Gamma(\mu-p)^2}\sum_{j=a+m-p+1}^{b}\left[\sum_{s=a+\nu}^{j-\mu+p}(j-s-1)^{\underline{\mu-p-1}}\left(\Delta_*^{\mu}f(s)\right)\right]$$
$$\times\sum_{j=a+m-p+1}^{b}\left[\sum_{s=a+\nu}^{j-\mu+p}(j-s-1)^{\underline{\mu-p-1}}\left(\Delta_*^{\mu}g(s)\right)\right]. \tag{3.4}$$

So, using (3.3) and (3.4), we get

$$\frac{1}{b-a-m+p}\sum_{j=a+m-p+1}^{b}\left[\left(\Delta^p f(j)\right)\left(\Delta^p g(j)\right)\right]$$
$$-\frac{1}{(b-a-m+p)^2}\left[\sum_{j=a+m-p+1}^{b}\left(\Delta^p f(j)\right)\right]\left[\sum_{j=a+m-p+1}^{b}\left(\Delta^p g(j)\right)\right]$$
$$\le\frac{M_1M_2}{(b-a-m+p)^2[\Gamma(\mu-p)]^2}\sum_{j=a+m-p+1}^{b}\left[\sum_{s=a+\nu}^{j-\mu+p}(j-s-1)^{\underline{\mu-p-1}}\right]^2$$
$$-\frac{m_1m_2}{(b-a-m+p)^2[\Gamma(\mu-p)]^2}\left[\sum_{j=a+m-p+1}^{b}\left[\sum_{s=a+\nu}^{j-\mu+p}(j-s-1)^{\underline{\mu-p-1}}\right]\right]^2.$$

Now, calculating the sums:

$$\sum_{s=a+\nu}^{j-\mu+p}(j-s-1)^{\underline{\mu-p-1}}=\int_{\tau=a+\nu}^{j-\mu+p+1}\left(j-\sigma(\tau)\right)^{\underline{\mu-p-1}}\Delta\tau=\frac{1}{\mu-p}(j-a-\nu)^{\underline{\mu-p}},$$

We get

$$\sum_{j=a+m-p+1}^{b}\left[\sum_{s=a+\nu}^{j-\mu+p}(j-s-1)^{\underline{\mu-p-1}}\right]^2\le\frac{1}{(\mu-p)^2}\sum_{j=a+m-p+1}^{b}\left[(j-a-\nu)^{\underline{\mu-p}}\right]^2,$$
$$\left[\sum_{j=a+m-p+1}^{b}\left[\sum_{s=a+\nu}^{j-\mu+p}(j-s-1)^{\underline{\mu-p-1}}\right]\right]=\frac{1}{(\mu-p)^2}\left[\sum_{j=a+m-p+1}^{b-a-\nu}(s)^{\underline{\mu-p}}\right]^2$$
$$=\frac{1}{(\mu-p)^2}\frac{1}{(\mu-p+1)^2}$$
$$\times\left[(b-a-\nu+1)^{\underline{\mu-p+1}}-(m-p-\nu+1)^{\underline{\mu-p+1}}\right]^2.$$

Consequently, we get

$$\frac{1}{b-a-m+p}\sum_{j=a+m-p+1}^{b}\left[\left(\Delta^p f(j)\right)\left(\Delta^p g(j)\right)\right]$$
$$-\frac{1}{(b-a-m+p)^2}\left[\sum_{j=a+m-p+1}^{b}\left(\Delta^p f(j)\right)\right]\left[\sum_{j=a+m-p+1}^{b}\left(\Delta^p g(j)\right)\right]$$

$$
\begin{aligned}
&\leq \frac{M_1M_2}{(b-a-m+p)[\Gamma(\mu-p)]^2}\frac{1}{(\mu-p)^2}\sum_{j=a+m-p+1}^{b}\left[(j-a-\nu)^{\underline{\mu-p}}\right]^2 \\
&\quad -\frac{m_1m_2}{(b-a-m+p)^2[\Gamma(\mu-p)]^2}\frac{1}{(\mu-p)^2(\mu-p+1)^2} \\
&\quad \times\left[(b-a-\nu+1)^{\underline{\mu-p+1}}-(m-p-\nu+1)^{\underline{\mu-p+1}}\right]^2 \\
&= \frac{M_1M_2C_1-m_1m_2C_2}{(b-a-m+p)^2[\Gamma(\mu-p+1)]^2}. \qquad \square
\end{aligned}
$$

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally in writing this article and collaborated in its design in coordination. All authors read and approved the final paper.

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