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# Sharp power-type Heronian mean bounds for the Sándor and Yang means

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**Abstract**

We prove that the double inequalities $H_\alpha(a,b) < X(a,b) < H_\beta(a,b)$ and $H_\lambda(a,b) < U(a,b) < H_\mu(a,b)$ hold for all $a,b>0$ with $a\neq b$ if and only if $\alpha \leq 1/2$, $\beta \geq \log 3/(1+\log 2) = 0.6488\cdots$, $\lambda \leq 2\log 3/(2\log\pi - \log 2) = 1.3764\cdots$, and $\mu \geq 2$, where $H_p(a,b)$, $X(a,b)$, and $U(a,b)$ are, respectively, the $p$th power-type Heronian mean, Sándor mean, and Yang mean of $a$ and $b$.

**MSC:** 26E60

**Keywords:** power-type Heronian mean; Sándor mean; Yang mean

## 1 Introduction

For $p \in \mathbb{R}$, the $p$th power-type Heronian mean $H_p(a,b)$ of two positive real numbers $a$ and $b$ is defined by

$$H_p(a,b) = \left[\frac{a^p + (ab)^{p/2} + b^p}{3}\right]^{1/p} \quad (p \neq 0), \qquad H_0(a,b) = \sqrt{ab}. \tag{1.1}$$

It is well known that $H_p(a,b)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a,b>0$ with $a\neq b$.

Let $G(a,b) = \sqrt{ab}$, $L(a,b) = (a-b)/(\log a - \log b)$, $P(a,b) = (a-b)/[2\arcsin((a-b)/(a+b))]$, $I(a,b) = (a^a/b^b)^{1/(a-b)}/e$, $A(a,b) = (a+b)/2$, $T(a,b) = (a-b)/[2\arctan((a-b)/(a+b))]$, $Q(a,b) = \sqrt{(a^2+b^2)/2}$ and $M_r(a,b) = [(a^r+b^r)/2]^{1/r}$ $(r\neq 0)$, and $M_0(a,b) = \sqrt{ab}$ be, respectively, the geometric, logarithmic, first Seiffert, identric, arithmetic, second Seiffert, quadratic, and $r$th power means of two distinct positive real numbers $a$ and $b$. Then it is well known that the inequalities

$$\begin{aligned} G(a,b) = M_0(a,b) < L(a,b) < P(a,b) < I(a,b) \\ < A(a,b) = M_1(a,b) < T(a,b) < Q(a,b) = M_2(a,b) \end{aligned}$$

hold for all $a,b>0$ with $a\neq b$.

Let $a,b>0$. Then the Sándor mean $X(a,b)$ [1] and Yang mean $U(a,b)$ [2] are given by

$$X(a,b) = A(a,b)e^{\frac{G(a,b)}{P(a,b)}-1} \tag{1.2}$$

and

$$U(a,b) = \frac{a-b}{\sqrt{2}\arctan(\frac{a-b}{\sqrt{2ab}})} \quad (a \neq b), \qquad U(a,a) = a, \tag{1.3}$$

respectively.

The Yang mean $U(a,b)$ is the special case of the Seiffert type mean $T_{M,p}(a,b) = (a-b)/[p\arctan((a-b)/(pM(a,b)))]$ defined by Toader in [3], where $M(a,b)$ is a bivariate mean and $p$ is a positive real number. Indeed, $U(a,b) = T_{G,\sqrt{2}}(a,b)$. Recently, the power-type Heronian, Sándor, and Yang means have been the subject of intensive research.

For all $a,b>0$ with $a\neq b$, Yang [4] and Sándor [5] proved that the double inequality

$$M_{1/2}(a,b) < H_1(a,b) < I(a,b)$$

holds, and the inequality $H_1(a,b) < M_{2/3}(a,b)$ can be found in the literature [6].

Jia and Cao [7] proved that the inequalities

$$\begin{aligned} &L(a,b) < H_p(a,b) < M_q(a,b), \\ &A(a,b) = M_1(a,b) < H_{\log 3/\log 2}(a,b) \end{aligned} \tag{1.4}$$

hold for all $a,b>0$ with $a\neq b$ if $p \geq 1/2$ and $q \geq 2p/3$. Inequality (1.4) can also be found in the literature [8], p.64 and [9].

In [10], the authors proved that the double inequality

$$H_p(a,b) < T(a,b) < H_q(a,b)$$

holds for all $a,b>0$ with $a\neq b$ if and only if $p \leq \log 3/(\log\pi - \log 2)$ and $q \geq 5/2$.

Sándor [11] presented the inequalities

$$\begin{aligned} &X(a,b) < \frac{P^2(a,b)}{A(a,b)}, \qquad \frac{A(a,b)G(a,b)}{P(a,b)} < X(a,b) < \frac{A(a,b)P(a,b)}{2P(a,b)-G(a,b)}, \\ &X(a,b) > \frac{A(a,b)L(a,b)}{P(a,b)} e^{\frac{G(a,b)}{L(a,b)}-1}, \qquad X(a,b) > \frac{A(a,b)[P(a,b)+G(a,b)]}{3P(a,b)-G(a,b)}, \\ &\frac{A^2(a,b)G(a,b)}{P(a,b)L(a,b)} e^{\frac{L(a,b)}{A(a,b)}-1} < X(a,b) < A(a,b)\left[\frac{1}{e} + \left(1-\frac{1}{e}\right)\frac{G(a,b)}{P(a,b)}\right], \\ &A(a,b)+G(a,b)-P(a,b) < X(a,b) < A^{-1/3}(a,b)\left[\frac{A(a,b)+G(a,b)}{2}\right]^{4/3}, \\ &P^{1/(\log\pi-\log 2)}(a,b)A^{1-1/(\log\pi-\log 2)}(a,b) < X(a,b) < P^{-1}(a,b)\left[\frac{A(a,b)+G(a,b)}{2}\right]^2 \end{aligned}$$

for all $a,b>0$ with $a\neq b$.

Yang *et al.* [12] proved that the double inequality

$$M_p(a,b) < X(a,b) < M_q(a,b)$$

holds for all $a,b>0$ with $a\neq b$ if and only if $p \leq 1/3$ and $q \geq \log 2/(1+\log 2)$.

In [2], Yang established the inequalities

$$P(a,b) < U(a,b) < T(a,b), \qquad \frac{G(a,b)T(a,b)}{A(a,b)} < U(a,b) < \frac{P(a,b)Q(a,b)}{A(a,b)},$$
$$Q^{1/2}(a,b)\left[\frac{2G(a,b)+Q(a,b)}{3}\right]^{1/2} < U(a,b) < Q^{2/3}(a,b)\left[\frac{G(a,b)+Q(a,b)}{2}\right]^{1/3},$$
$$\frac{G(a,b)+Q(a,b)}{2} < U(a,b) < \left[\frac{2}{3}\left(\frac{G(a,b)+Q(a,b)}{2}\right)^{1/2} + \frac{1}{3}Q^{1/2}(a,b)\right]^2$$

for all $a,b > 0$ with $a \neq b$.

In [13, 14], the authors proved that the double inequalities

$$\begin{aligned}&\left[\frac{2}{3}\left(\frac{G(a,b)+Q(a,b)}{2}\right)^{p} + \frac{1}{3}Q^{p}(a,b)\right]^{1/p}\\ &\quad< U(a,b) < \left[\frac{2}{3}\left(\frac{G(a,b)+Q(a,b)}{2}\right)^{q} + \frac{1}{3}Q^{q}(a,b)\right]^{1/q},\\ &\frac{2^{1-\lambda}(G(a,b)+Q(a,b))^{\lambda}Q(a,b)+G(a,b)Q^{\lambda}(a,b)}{2^{1-\lambda}(G(a,b)+Q(a,b))^{\lambda}+Q^{\lambda}(a,b)}\\ &\quad< U(a,b) < \frac{2^{1-\mu}(G(a,b)+Q(a,b))^{\mu}Q(a,b)+G(a,b)Q^{\mu}(a,b)}{2^{1-\mu}(G(a,b)+Q(a,b))^{\mu}+Q^{\mu}(a,b)},\\ &M_{\alpha}(a,b) < U(a,b) < M_{\beta}(a,b)\end{aligned}$$

hold for all $a,b > 0$ with $a \neq b$ if and only if $p \leq p_0$, $q \geq 1/5$, $\lambda \geq 1/5$, $\mu \leq p_1$, $\alpha \leq 2\log 2/(2\log\pi - \log 2)$, and $\beta \geq 4/3$, where $p_0 = 0.1941\cdots$ is the unique solution of the equation $p\log(2/\pi) - \log(1+2^{1-p}) + \log 3 = 0$ on the interval $(1/10,\infty)$, and $p_1 = \log(\pi - 2)/\log 2 = 0.1910\cdots$.

The main purpose of this paper is to present the best possible parameter $\alpha$, $\beta$, $\lambda$, and $\mu$ such that the double inequalities $H_{\alpha}(a,b) < X(a,b) < H_{\beta}(a,b)$ and $H_{\lambda}(a,b) < U(a,b) < H_{\mu}(a,b)$ hold for all $a,b > 0$ with $a \neq b$.

## 2 Lemmas

In order to prove our main results we need two lemmas, which we present in this section.

**Lemma 2.1** *Let $p \in (0,1)$ and*

$$\begin{aligned}f(x) &= (p-1)x^{3p+2} + (4p-3)x^{2p+2} + (p-3)x^{p+2} + 6x^{4p} + 6x^{3p} - 6x^{p}\\ &\quad+ 2x^{4p-2} + (3-p)x^{3p-2} + (3-4p)x^{2p-2} + (1-p)x^{p-2} - 2x^2 - 6.\end{aligned}\tag{2.1}$$

*Then the following statements are true*:

(1) *if $p = 1/2$, then $f(x) < 0$ for all $x > 1$*;

(2) *if $p = \log 3/(1+\log 2) = 0.6488\cdots$, then there exists $\lambda \in (1,\infty)$ such that $f(x) > 0$ for $x \in (1,\lambda)$ and $f(x) < 0$ for $x \in (\lambda,\infty)$.*

*Proof* For part (1), if $p = 1/2$, then (2.1) becomes

$$f(x) = -\frac{(x-1)(\sqrt{x}-1)^2}{2x^{3/2}}\left(x^3 + 4x^{5/2} + 13x^2 + 16x^{3/2} + 13x + 4\sqrt{x} + 1\right).\tag{2.2}$$

Therefore, part (1) follows from (2.2).

For part (2), let $p = \log 3/(1+\log 2)$, $f_1(x) = f'(x)/x$, $f_2(x) = x^{5-p}f_1'(x)$, $f_3(x) = f_2'(x)/(2px)$, and $f_4(x) = f_3'(x)/(2x)$. Then elaborated computations lead to

$$f(1) = 0, \qquad \lim_{x\to+\infty} f(x) = -\infty, \tag{2.3}$$

$$\begin{aligned} f_1(x) &= (p-1)(3p+2)x^{3p} + 2(p+1)(4p-3)x^{2p} + (p-3)(p+2)x^{p} \\ &\quad + 24px^{4p-2} + 18px^{3p-2} - 6px^{p-2} + 4(2p-1)x^{4p-4} + (3-p)(3p-2)x^{3p-4} \\ &\quad + 2(p-1)(3-4p)x^{2p-4} + (1-p)(p-2)x^{p-4} - 4, \end{aligned}$$

$$f_1(1) = 36(2p-1) > 0, \qquad \lim_{x\to+\infty} f_1(x) = -\infty, \tag{2.4}$$

$$\begin{aligned} f_2(x) &= 3p(p-1)(3p+2)x^{2p+4} + 4p(p+1)(4p-3)x^{p+4} + 48p(2p-1)x^{3p+2} \\ &\quad + 18p(3p-2)x^{2p+2} + 16(2p-1)(p-1)x^{3p} + (3-p)(3p-2)(3p-4)x^{2p} \\ &\quad + 4(3-4p)(p-1)(p-2)x^{p} + p(p-3)(p+2)x^{4} \\ &\quad - 6p(p-2)x^{2} + (1-p)(p-2)(p-4), \end{aligned}$$

$$f_2(1) = 72(2p-1)^2 > 0, \qquad \lim_{x\to+\infty} f_2(x) = -\infty, \tag{2.5}$$

$$\begin{aligned} f_3(x) &= 3(p-1)(p+2)(3p+2)x^{2p+2} + 2(p+1)(p+4)(4p-3)x^{p+2} \\ &\quad + 24(2p-1)(3p+2)x^{3p} + 18(p+1)(3p-2)x^{2p} + 24(2p-1)(p-1)x^{3p-2} \\ &\quad + (3-p)(3p-2)(3p-4)x^{2p-2} + 2(p-1)(p-2)(3-4p)x^{p-2} \\ &\quad + 2(p-3)(p+2)x^{2} - 6(p-2), \end{aligned}$$

$$f_3(1) = 12\bigl(31p^2 - 12p - 5\bigr) > 0, \qquad \lim_{x\to+\infty} f_3(x) = -\infty, \tag{2.6}$$

$$\begin{aligned} f_4(x) &= 3(p-1)(p+1)(p+2)(3p+2)x^{2p} + (p+1)(p+2)(p+4)(4p-3)x^{p} \\ &\quad + 36p(2p-1)(3p+2)x^{3p-2} + 18p(p+1)(3p-2)x^{2p-2} \\ &\quad + 12(p-1)(2p-1)(3p-2)x^{3p-4} \\ &\quad + (p-1)(3-p)(3p-2)(3p-4)x^{2p-4} + (p-1)(p-2)^2(3-4p)x^{p-4} \\ &\quad + 2(p-3)(p+2). \end{aligned} \tag{2.7}$$

It follows from (2.7) and $p = \log 3/(1+\log 2) = 0.6488\cdots$ together with $13p^4 + 337p^3 - 80p^2 - 72 = -11.3153\cdots < 0$ that

$$\begin{aligned} f_4(x) &< 3(p-1)(p+1)(p+2)(3p+2) + (p+1)(p+2)(p+4)(4p-3) \\ &\quad + 36p(2p-1)(3p+2) + 18p(p+1)(3p-2)x^{2p-2} + 12(p-1)(2p-1)(3p-2) \\ &\quad + (p-1)(3-p)(3p-2)(3p-4)x^{2p-4} + (p-1)(p-2)^2(3-4p)x^{p-4} \\ &\quad + 2(p-3)(p+2) \\ &= 18p(p+1)(3p-2)x^{2p-2} + (p-1)(3-p)(3p-2)(3p-4)x^{2p-4} \\ &\quad + (p-1)(p-2)^2(3-4p)x^{p-4} + \bigl(13p^4 + 337p^3 - 80p^2 - 72\bigr) < 0 \end{aligned} \tag{2.8}$$

for $x \in (1, \infty)$.

Inequality (2.8) implies that $f_3(x)$ is strictly decreasing on $[1,\infty)$. Then (2.6) leads to the conclusion that there exists $\lambda_1 \in (1,\infty)$ such that $f_2(x)$ is strictly increasing on $[1,\lambda_1]$ and strictly decreasing on $[\lambda_1,\infty)$.

It follows from the piecewise monotonicity of $f_2$ and (2.5) that there exists $\lambda_2 \in (\lambda_1,\infty)$ such that $f_1(x)$ is strictly increasing on $[1,\lambda_2]$ and strictly decreasing on $[\lambda_2,\infty)$.

From (2.4) and the piecewise monotonicity of $f_1$ we clearly see that there exists $\lambda_3 \in (\lambda_2,\infty)$ such that $f(x)$ is strictly increasing on $[1,\lambda_3]$ and strictly decreasing on $[\lambda_3,\infty)$.

Therefore, part (2) follows from (2.3) and the piecewise monotonicity of $f$. □

**Lemma 2.2** *Let* $p \in (1,2]$ *and*

$$\begin{aligned} g(x) = {} & 2x^{4p+4} - 2x^{4p+2} + 10x^{4p} + 6x^{4p-2} + (p-1)x^{3p+6} - 2x^{3p+4} - 2x^{3p+2} \\ & + 14x^{3p} + (7-p)x^{3p-2} + 4(p-1)x^{2p+6} - 12x^{2p+4} + 12x^{2p} + 4(1-p)x^{2p-2} \\ & + (p-7)x^{p+6} - 14x^{p+4} + 2x^{p+2} + 2x^{p} + (1-p)x^{p-2} - 2\bigl(3x^6 + 5x^4 - x^2 + 1\bigr). \end{aligned} \tag{2.9}$$

*Then the following statements are true*:

(1) *if* $p = 2$, *then* $g(x) > 0$ *for all* $x > 1$;

(2) *if* $p = 2\log 3/(2\log\pi - \log 2) = 1.3764\cdots$, *then there exists* $\mu \in (1,\infty)$ *such that* $g(x) < 0$ *for* $x \in (1,\mu)$ *and* $g(x) > 0$ *for* $x \in (\mu,\infty)$.

*Proof* For part (1), if $p = 2$, then (2.9) becomes

$$g(x) = 3\bigl(x^4 - 1\bigr)^3. \tag{2.10}$$

Therefore, part (1) follows from (2.10).

For part (2), let $p = 2\log 3/(2\log\pi - \log 2)$, $g_1(x) = g'(x)/x$, $g_2(x) = g_1'(x)/x$, $g_3(x) = g_2'(x)/x$, and $g_4(x) = x^{9-p}g_3'(x)$. Then elaborated computations lead to

$$g(1) = 0, \qquad \lim_{x\to\infty} g(x) = +\infty, \tag{2.11}$$

$$\begin{aligned} g_1(x) = {} & 8(p+1)x^{4p+2} - 4(2p+1)x^{4p} + 40px^{4p-2} + 12(2p-1)x^{4p-4} + 3(p-1)(p+2)x^{3p+4} \\ & - 2(3p+4)x^{3p+2} - 2(3p+2)x^{3p} + 42px^{3p-2} + (7-p)(3p-2)x^{3p-4} \\ & + 8(p-1)(p+3)x^{2p+4} \\ & - 24(p+2)x^{2p+2} + 24px^{2p-2} - 8(1-p)^2x^{2p-4} + (p-7)(p+6)x^{p+4} \\ & - 14(p+4)x^{p+2} + 2(p+2)x^{p} + 2px^{p-2} + (1-p)(p-2)x^{p-4} - 36x^4 - 40x^2 + 4, \end{aligned}$$

$$g_1(1) = -144(2-p) < 0, \qquad \lim_{x\to\infty} g_1(x) = +\infty, \tag{2.12}$$

$$\begin{aligned} g_2(x) = {} & 16(p+1)(2p+1)x^{4p} - 16p(2p+1)x^{4p-2} + 80p(2p-1)x^{4p-4} \\ & + 48(p-1)(2p-1)x^{4p-6} \\ & + 3(p-1)(p+2)(3p+4)x^{3p+2} - 2(3p+2)(3p+4)x^{3p} - 6p(3p+2)x^{3p-2} \\ & + 42p(3p-2)x^{3p-4} + (7-p)(3p-2)(3p-4)x^{3p-6} \\ & + 16(p-1)(p+2)(p+3)x^{2p+2} - 48(p+1)(p+2)x^{2p} + 48p(p-1)x^{2p-4} \end{aligned}$$

$$
\begin{aligned}
&- 16(1-p)^2(p-2)x^{2p-6} + (p-7)(p+4)(p+6)x^{p+2} - 14(p+2)(p+4)x^p \\
&+ 2p(p+2)x^{p-2} + 2p(p-2)x^{p-4} + (1-p)(p-2)(p-4)x^{p-6} - 144x^2 - 80,
\end{aligned}
$$

$$
g_2(1) = -288\big(2+3p-2p^2\big) < 0, \qquad \lim_{x\to\infty} g_2(x) = +\infty, \tag{2.13}
$$

$$
\begin{aligned}
g_3(x) ={}& 64p(p+1)(2p+1)x^{4p-2} - 32p\big(4p^2-1\big)x^{4p-4} + 320p(p-1)(2p-1)x^{4p-6} \\
&+ 96(p-1)(2p-1)(2p-3)x^{4p-8} + 3(p-1)(p+2)(3p+2)(3p+4)x^{3p} \\
&- 6p(3p+2)(3p+4)x^{3p-2} - 6p\big(9p^2-4\big)x^{3p-4} + 42p(3p-2)(3p-4)x^{3p-6} \\
&+ 3(7-p)(p-2)(3p-2)(3p-4)x^{3p-8} \\
&+ 32\big(p^2-1\big)(p+2)(p+3)x^{2p} - 96p(p+1)(p+2)x^{2p-2} + 96p(p-1)(p-2)x^{2p-6} \\
&- 32(1-p)^2(p-2)(p-3)x^{2p-8} + (p-7)(p+2)(p+4)(p+6)x^p \\
&- 14p(p+2)(p+4)x^{p-2} + 2p\big(p^2-4\big)x^{p-4} + 2p(p-2)(p-4)x^{p-6} \\
&+ (1-p)(p-2)(p-4)(p-6)x^{p-8} - 288,
\end{aligned}
$$

$$
g_3(1) = 48\big(43p^3 - 100p^2 + 58p - 36\big) < 0, \qquad \lim_{x\to\infty} g_3(x) = +\infty, \tag{2.14}
$$

$$
\begin{aligned}
g_4(x) ={}& 128p(p+1)\big(4p^2-1\big)x^{3p+6} - 128p\big(4p^2-1\big)(p-1)x^{3p+4} \\
&+ 640p(p-1)(2p-1)(2p-3)x^{3p+2} + 384(p-1)(p-2)(2p-1)(2p-3)x^{3p} \\
&+ 9p(p-1)(p+2)(3p+2)(3p+4)x^{2p+8} - 6p\big(9p^2-4\big)(3p+4)x^{2p+6} \\
&- 6p\big(9p^2-4\big)(3p-4)x^{2p+4} + 126p(p-2)(3p-2)(3p-4)x^{2p+2} \\
&+ 3(7-p)(p-2)(3p-2)(3p-4)(3p-8)x^{2p} + 64p\big(p^2-1\big)(p+2)(p+3)x^{p+8} \\
&- 192p\big(p^2-1\big)(p+2)x^{p+6} + 192p(p-1)(p-2)(p-3)x^{p+2} \\
&- 64(1-p)^2(p-2)(p-3)(p-4)x^p \\
&+ p(p-7)(p+2)(p+4)(p+6)x^8 - 14p\big(p^2-4\big)(p+4)x^6 + 2p\big(p^2-4\big)(p-4)x^4 \\
&+ 2p(p-2)(p-4)(p-6)x^2 + (1-p)(p-2)(p-4)(p-6)(p-8) \\
=:{}& a_1x^{3p+6} + a_4x^{3p+4} + a_8x^{3p+2} + a_{11}x^{3p} + a_0x^{2p+8} + a_3x^{2p+6} + a_7x^{2p+4} + a_{10}x^{2p+2} \\
&+ a_{14}x^{2p} + a_2x^{p+8} + a_6x^{p+6} + a_{13}x^{p+2} + a_{16}x^p + a_5x^8 + a_9x^6 \\
&+ a_{12}x^4 + a_{15}x^2 + a_{17},
\end{aligned} \tag{2.15}
$$

$$
\sum_{n=0}^{8} a_n = 2p\big(73p^4 + 1306p^3 - 3344p^2 + 3272p - 1328\big) > 0, \tag{2.16}
$$

$$
a_9 + a_{10} = 16p\big(70p^3 - 287p^2 + 350p - 112\big) > 0, \tag{2.17}
$$

$$
\sum_{n=11}^{15} a_n = -81p^5 + 2839p^4 - 13904p^3 + 25652p^2 - 19600p + 4992 > 0, \tag{2.18}
$$

$$
a_{16} + a_{17} = 5\big(-13p^5 + 145p^4 - 608p^3 + 1196p^2 - 1104p + 384\big) > 0. \tag{2.19}
$$

Note that

$$\begin{aligned} &2p+8>3p+6>p+8>2p+6>3p+4>8>p+6>2p+4 \\ &\qquad >3p+2>6>2p+2>3p>4>p+2>2p>2>p>1. \end{aligned} \tag{2.20}$$

It follows from (2.15)-(2.20) that

$$g_4(x) > \left(\sum_{n=0}^{8} a_n\right)x^{p+8} + (a_9+a_{10})x^{2p+2} + \left(\sum_{n=11}^{15} a_n\right)x^{2p} + (a_{16}+a_{17})x^p > 0 \tag{2.21}$$

for $x \in (1,\infty)$.

Therefore, part (2) follows easily from (2.11)-(2.14) and (2.21). □

## 3 Main results

**Theorem 3.1** *The double inequality*

$$H_\alpha(a,b) < X(a,b) < H_\beta(a,b)$$

*holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha\leq 1/2$ and $\beta\geq \log 3/(1+\log 2) = 0.6488\cdots$.*

*Proof* Since $X(a,b)$ and $H_p(a,b)$ are symmetric and homogeneous of degree one, we assume that $a>b$. Let $x=\sqrt{a/b}\in(1,\infty)$ and $p\in\mathbb{R}$. Then (1.1) and (1.2) lead to

$$\begin{aligned} &\log\bigl[X(a,b)\bigr] - \log\bigl[H_p(a,b)\bigr] \\ &\quad = \log\left(\frac{x^2+1}{2}\right) + \frac{2x}{x^2-1}\arcsin\left(\frac{x^2-1}{x^2+1}\right) - \frac{1}{p}\log\left(\frac{x^{2p}+x^p+1}{3}\right) - 1 := F(x). \end{aligned} \tag{3.1}$$

Simple computations lead to

$$F(1) = 0, \tag{3.2}$$

$$\lim_{x\to+\infty} F(x) = \frac{1}{p}\log 3 - \log 2 - 1, \tag{3.3}$$

$$F'(x) = \frac{2(1+x^2)}{(x^2-1)^2}F_1(x), \tag{3.4}$$

where

$$F_1(x) = \frac{(x^2-1)(2x^{2p}+x^{p+2}+x^p+2x^2)}{2x(1+x^2)(x^{2p}+x^p+1)} - \arcsin\left(\frac{1-x^2}{1+x^2}\right),$$

$$F_1(1) = 0, \qquad \lim_{x\to+\infty} F_1(x) = +\infty, \tag{3.5}$$

$$F_1'(x) = -\frac{x^2-1}{2(x^2+1)^2(x^{2p}+x^p+1)^2}f(x), \tag{3.6}$$

where $f(x)$ is defined as in Lemma 2.1.

If $p=1/2$, then from Lemma 2.1(1), (3.1), (3.2), and (3.4)-(3.6) we clearly see that

$$X(a,b) > H_{1/2}(a,b)$$

for all $a,b>0$ with $a\neq b$.

If $p = \log 3/(1 + \log 2)$, then (3.3) becomes

$$\lim_{x\to+\infty} F(x) = 0. \tag{3.7}$$

It follows from Lemma 2.2(2) and (3.6) that there exists $\lambda \in (1,\infty)$ such that $F_1(x)$ is strictly decreasing on $[1,\lambda]$ and strictly increasing on $[\lambda,\infty)$.

Equations (3.4) and (3.5) together with the piecewise monotonicity of $F_1$ lead to the conclusion that there exists $\lambda^* \in (1,\infty)$ such that $F(x)$ is strictly decreasing on $[1,\lambda^*]$ and strictly increasing on $[\lambda^*,\infty)$.

Therefore,

$$X(a,b) < H_{\log 3/(1+\log 2)}(a,b)$$

for all $a, b > 0$ with $a \neq b$ follows from (3.1), (3.2), (3.7), and the piecewise monotonicity of $F$.

Next, we prove that $\alpha = 1/2$ and $\beta = \log 3/(1+\log 2)$ are the best possible parameters such that the double inequality $H_\alpha(a,b) < X(a,b) < H_\beta(a,b)$ holds for all $a, b > 0$ with $a \neq b$.

If $p < \log 3/(1 + \log 2)$, then (3.3) leads to

$$\lim_{x\to+\infty} F(x) > 0. \tag{3.8}$$

Equation (3.1) and inequality (3.8) imply that there exists large enough $T_0 = T_0(p) > 1$ such that $X(a,b) > H_p(a,b)$ for all $a, b > 0$ with $a/b \in (T_0,\infty)$ if $p < \log 3/(1 + \log 2)$.

Let $p > 1/2$, $x > 0$, and $x \to 0$. Then elaborated computations lead to

$$\begin{aligned}
&H_p(1,1+x) - X(1,1+x) \\
&\quad = \left[\frac{1 + (1+x)^{p/2} + (1+x)^p}{3}\right]^{1/p} - \left(1 + \frac{x}{2}\right)e^{\frac{2\sqrt{1+x}\arcsin(\frac{x}{2+x})}{x} - 1} \\
&\quad = \frac{2p-1}{24}x^2 + o\left(x^2\right).
\end{aligned} \tag{3.9}$$

Equation (3.9) implies that there exists small enough $\delta_0 = \delta_0(p) > 0$ such that $X(1,1+x) < H_p(1,1+x)$ for $x \in (0,\delta_0)$ if $p > 1/2$. □

**Theorem 3.2** *The double inequality*

$$H_\lambda(a,b) < U(a,b) < H_\mu(a,b)$$

*holds for all $a, b > 0$ with $a \neq b$ if and only if $\lambda \leq 2\log 3/(2\log \pi - \log 2) = 1.3764\cdots$ and $\mu \geq 2$.*

*Proof* Since $U(a,b)$ and $H_p(a,b)$ are symmetric and homogeneous of degree one, we assume that $a > b$. Let $x = \sqrt{a/b} \in (1,\infty)$ and $p \in \mathbb{R}$. Then (1.1) and (1.3) lead to

$$\begin{aligned}
&\log\left[U(a,b)\right] - \log\left[H_p(a,b)\right] \\
&\quad = \log\left[\frac{x^2-1}{\sqrt{2}\arctan(\frac{x^2-1}{\sqrt{2}x})}\right] - \frac{1}{p}\log\left(\frac{x^{2p}+x^p+1}{3}\right) := G(x).
\end{aligned} \tag{3.10}$$

Simple computations lead to

$$G(1) = 0, \tag{3.11}$$

$$\lim_{x\to+\infty} G(x) = \frac{1}{p}\log 3 + \frac{1}{2}\log 2 - \log\pi, \tag{3.12}$$

$$G'(x) = \frac{2x^{2p} + x^{p+2} + x^p + 2x^2}{x(x^2-1)(x^{2p}+x^p+1)\arctan(\frac{x^2-1}{\sqrt{2}x})} G_1(x), \tag{3.13}$$

where

$$G_1(x) = \arctan\left(\frac{x^2-1}{\sqrt{2}x}\right) - \frac{\sqrt{2}x(x^4-1)(x^{2p}+x^p+1)}{(x^4+1)(2x^{2p}+x^{p+2}+x^p+2x^2)},$$

$$G_1(1) = 0, \qquad \lim_{x\to+\infty} G_1(x) = -\infty, \tag{3.14}$$

$$G_1'(x) = -\frac{\sqrt{2}x^2(x^2-1)}{(x^4+1)^2(2x^{2p}+x^{p+2}+x^p+2x^2)^2} g(x), \tag{3.15}$$

where $g(x)$ is defined as in Lemma 2.2.

If $p = 2\log 3/(2\log\pi - \log 2)$, then (3.15) and Lemma 2.2(2) lead to the conclusion that there exists $\mu \in (1,\infty)$ such that $G_1(x)$ is strictly increasing on $[1,\mu]$ and strictly decreasing on $[\mu,\infty)$.

It follows from (3.13) and (3.14) together with the piecewise monotonicity of $G_1$ that there exists $\mu^* \in (1,\infty)$ such that $G(x)$ is strictly increasing on $[1,\mu^*]$ and strictly decreasing on $[\mu^*,\infty)$.

Note that (3.12) becomes

$$\lim_{x\to+\infty} G(x) = 0. \tag{3.16}$$

Therefore,

$$U(a,b) > H_{2\log 3/(2\log\pi-\log 2)}(a,b)$$

for all $a,b > 0$ with $a \neq b$ follows from (3.10), (3.11), and (3.16) together with the piecewise monotonicity of $G$.

If $p = 2$, then

$$U(a,b) < H_2(a,b)$$

for all $a,b > 0$ with $a \neq b$ follows easily from (3.10), (3.11), and (3.13)-(3.15) together with Lemma 2.2(1).

Next, we prove that $\lambda = 2\log 3/(2\log\pi - \log 2)$ and $\mu = 2$ are the best possible parameters such that the double inequality

$$H_\lambda(a,b) < U(a,b) < H_\mu(a,b)$$

holds for all $a,b > 0$ with $a \neq b$.

If $p > 2\log 3/(2\log\pi - \log 2)$, then (3.12) leads to

$$\lim_{x\to+\infty} G(x) < 0. \tag{3.17}$$

Equation (3.10) and inequality (3.17) imply that there exists large enough $T_1 = T_1(p) > 1$ such that $U(a,b) < H_p(a,b)$ for all $a, b > 0$ with $a/b \in (T_1, \infty)$.

Let $p < 2$, $x > 0$, and $x \to 0$. Then elaborated computations lead to

$$\begin{aligned} &U(1,1+x) - H_p(1,1+x) \\ &\quad = \frac{x}{\sqrt{2}\arctan(\frac{x}{\sqrt{2(1+x)}})} - \left[\frac{1+(1+x)^{p/2}+(1+x)^p}{3}\right]^{1/p} \\ &\quad = \frac{2-p}{12}x^2 + o(x^2). \end{aligned} \tag{3.18}$$

Inequality (3.18) implies that there exists small enough $\delta_1 = \delta_1(p) > 0$ such that $U(1,1+x) > H_p(1,1+x)$ for $x \in (0, \delta_1)$. □

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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