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Barnes-type Daehee with λ-parameter and degenerate Euler mixed-type polynomials

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Abstract

In this paper, we consider the Barnes-type Daehee with λ-parameter and degenerate Euler mixed-type polynomials. We present several explicit formulas and recurrence relations for these polynomials. Also, we establish a connection between our polynomials and several known families of polynomials.

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1 Introduction

In this paper, we use umbral calculus techniques (see [1, 2]) to obtain several new and interesting identities of Barnes-type Daehee with λ-parameter and degenerate Euler mixed-type polynomials. To define the umbral calculus, let Π be the algebra of polynomials in a single variable x over $\mathbb{C}$ and Π^* be the vector space of all linear functionals on Π. The action of a linear functional $L \in \Pi^*$ on a polynomial $p(x)$ is denoted by $\langle L|p(x)\rangle$, and linearly extended as $\langle cL + dL'|p(x)\rangle = c\langle L|p(x)\rangle + d\langle L'|p(x)\rangle$, where $c, d \in \mathbb{C}$. Define $\mathcal{H} = \{f(t) = \sum_{k\geq 0} a_k \frac{t^k}{k!} \mid a_k \in \mathbb{C}\}$ to be the algebra of formal power series in a single variable t. The formal power series $f(t) \in \mathcal{H}$ defines a linear functional on Π by setting $\langle f(t)|x^n\rangle = a_n$ for all $n \geq 0$. Thus, we have (see [1, 2])

$$\left\langle t^k|x^n\right\rangle = n!\delta_{n,k} \quad \text{for all } n, k \geq 0, \tag{1.1}$$

where $\delta_{n,k}$ is the Kronecker symbol. Let $f_L(t) = \sum_{n\geq 0}\langle L|x^n\rangle \frac{t^n}{n!}$. By (1.1), we get that $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$. Thus, the map $L \mapsto f_L(t)$ gives a vector space isomorphism from Π^* onto $\mathcal{H}$. Therefore, $\mathcal{H}$ is thought of as a set of both formal power series and linear functionals, which is called the *umbral algebra*. The *umbral calculus* is the study of umbral algebra.

The *order* $O(f(t))$ of the non-zero power series $f(t)$ is defined to be k when $f(t) = \sum_{n\geq k} a_n t^n$ and $a_k \neq 0$. Suppose that $O(f(t)) = 1$ and $O(g(t)) = 0$. Then there exists a unique sequence $s_n(x)$ of polynomials such that $\langle g(t)f(t)^k|s_n(x)\rangle = n!\delta_{n,k}$, where $n, k \geq 0$. The sequence $s_n(x)$ is called the *Sheffer sequence* for $(g(t), f(t))$, and we write $s_n(x) \sim (g(t), f(t))$

(see [1, 2]). For $f(t) \in \mathcal{H}$ and $p(x) \in \Pi$, we have that $\langle e^{yt}|p(x)\rangle = p(y)$, $\langle f(t)g(t)|p(x)\rangle = \langle g(t)|f(t)p(x)\rangle$, $f(t) = \sum_{n\geq 0}\langle f(t)|x^n\rangle \frac{t^n}{n!}$ and $p(x) = \sum_{n\geq 0}\langle t^n|p(x)\rangle \frac{x^n}{n!}$. Therefore, $\langle t^k|p(x)\rangle = p^{(k)}(0)$, $\langle 1|p^{(k)}(x)\rangle = p^{(k)}(0)$, where $p^{(k)}(0)$ denotes the kth derivative of $p(x)$ with respect to x at $x = 0$. So, $t^k p(x) = p^{(k)}(x) = \frac{d^k}{dx^k}p(x)$ for all $k \geq 0$ (see [1, 2]).

Let $s_n(x) \sim (g(t), f(t))$. Then we have

$$\frac{1}{g(\bar{f}(t))}e^{y\bar{f}(t)} = \sum_{n\geq 0} s_n(y)\frac{t^n}{n!} \tag{1.2}$$

for all $y \in \mathbb{C}$, where $\bar{f}(t)$ is the compositional inverse of $f(t)$ (see [1, 2]). For $s_n(x) \sim (g(t), f(t))$ and $r_n(x) \sim (h(t), \ell(t))$, let $s_n(x) = \sum_{k=0}^{n} c_{n,k} r_k(x)$. Then we have

$$c_{n,k} = \frac{1}{k!}\left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))}\left(\ell\left(\bar{f}(t)\right)\right)^k \middle| x^n\right\rangle \tag{1.3}$$

(see [1, 2]).

Throughout the paper, let $r, s \in \mathbb{Z}_{>0}$, and let $\mathbf{a} = (a_1, a_2, \dots, a_r)$, $\mathbf{b} = (b_1, b_2, \dots, b_s)$ with $a_j, b_i \neq 0$ for all i, j. We define the *Barnes-type Daehee with* λ*-parameter and degenerate Euler mixed-type polynomials* $D\mathcal{E}_n(\lambda, x|\mathbf{a}; \mathbf{b})$ (for other Barnes-types, see [3–5]) as

$$P_{r,s}(t)(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n\geq 0} D\mathcal{E}_n(\lambda, x|\mathbf{a}; \mathbf{b})\frac{t^n}{n!}, \tag{1.4}$$

where we define

$$P_{r,s}(t) = \prod_{i=1}^{r}\left(\frac{\log(1+\lambda t)}{\lambda((1+\lambda t)^{\frac{a_i}{\lambda}} - 1)}\right)\prod_{i=1}^{s}\left(\frac{2}{(1+\lambda t)^{\frac{b_i}{\lambda}} + 1}\right).$$

For $x = 0$, $D\mathcal{E}_n(\lambda|\mathbf{a}; \mathbf{b}) = D\mathcal{E}_n(\lambda, 0|\mathbf{a}; \mathbf{b})$ are called the *Barnes-type Daehee with* λ*-parameter and degenerate Euler mixed-type numbers*.

We recall here that the polynomials $D_{n,\lambda}(x|\mathbf{a})$ given by

$$P_{r,0}(t)(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n\geq 0} D_{n,\lambda}(x|\mathbf{a})\frac{t^n}{n!}$$

are called the *Barnes-type Daehee polynomials* with λ-parameter (see [6, 7]). Also, the polynomials $\mathcal{E}_n(\lambda, x|\mathbf{b})$ given by

$$P_{0,s}(t)(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n\geq 0} \mathcal{E}_n(\lambda, x|\mathbf{b})\frac{t^n}{n!} \tag{1.5}$$

are called the *Barnes-type degenerate Euler polynomials* which are studied in [8–11]. In the case $x = 0$, we write $\mathcal{E}_n(\lambda|\mathbf{b}) = \mathcal{E}_n(\lambda, 0|\mathbf{b})$, which are called the *Barnes-type degenerate Euler numbers*. Note that $\lim_{\lambda\to 0}\mathcal{E}_n(\lambda, x|\mathbf{b}) = E_n(x|\mathbf{b})$ and $\lim_{\lambda\to\infty}\lambda^{-n}\mathcal{E}_n(\lambda, \lambda x|\mathbf{b}) = (x)_n$, where $(x)_n = \prod_{i=0}^{n-1}(x - i)$ with $(x)_0 = 1$ and $E_n(x|\mathbf{b})$ are the *Barnes-type degenerate Euler polynomials* given by

$$\prod_{i=1}^{s}\left(\frac{2}{e^{b_i t} + 1}\right)e^{xt} = \sum_{n\geq 0} E_n(x|\mathbf{b})\frac{t^n}{n!}.$$

It is immediate from (1.2) and (1.4) to see that $D\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b})$ is the Sheffer sequence for the pair $g(t) = \prod_{i=1}^{r}(\frac{e^{a_i t}-1}{t})\prod_{i=1}^{s}(\frac{e^{b_i t}+1}{2})$ and $f(t) = \frac{e^{\lambda t}-1}{\lambda}$. Thus,

$$D\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) \sim \left(\prod_{i=1}^{r}\left(\frac{e^{a_i t}-1}{t}\right)\prod_{i=1}^{s}\left(\frac{e^{b_i t}+1}{2}\right), \frac{e^{\lambda t}-1}{\lambda}\right). \tag{1.6}$$

The aim of the present paper is to present several new identities for Barnes-type Daehee with λ-parameter and degenerate Euler mixed-type polynomials by the use of umbral calculus. For some of the related works, one is referred to the papers [12–20].

2 Explicit formulas

In this section we suggest several explicit formulas for the Barnes-type Daehee with λ-parameter and degenerate Euler mixed-type polynomials. To do that, we recall that the Stirling numbers $S_1(n,m)$ of the first kind are defined as $(x)_n = \sum_{m=0}^{n} S_1(n,m)x^m \sim (1, e^t-1)$ or $\frac{1}{j!}(\log(1+t))^j = \sum_{\ell\geq j} S_1(\ell,j)\frac{t^\ell}{\ell!}$. Let $(x|\lambda)_n$ be the *generalized falling factorials* defined by $(x|\lambda)_n = \prod_{i=0}^{n-1}(x-i\lambda)$ with $(x|\lambda)_0 = 1$, namely $(x|\lambda)_n = \lambda^n(x/\lambda)_n$.

Let $BE_n(x|\mathbf{a};\mathbf{b})$ be the *Barnes-type Bernoulli and Euler mixed-type polynomials* given by

$$\prod_{i=1}^{r}\left(\frac{t}{e^{a_i t}-1}\right)\prod_{i=1}^{s}\left(\frac{2}{e^{b_i t}+1}\right)e^{xt} = \sum_{n\geq 0} BE_n(x|\mathbf{a};\mathbf{b})\frac{t^n}{n!}. \tag{2.1}$$

Note that $BE_n^{r,s}(x)$ denotes the special case $BE_n(x|\underbrace{1,1,\ldots,1}_{r};\underbrace{1,1,\ldots,1}_{s})$ and was treated in [21, 22] by using p-adic integrals on $\mathbb{Z}_p$.

Theorem 2.1 *For all* $n \geq 0$,

$$D\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) = \sum_{m=0}^{n} S_1(n,m)\lambda^{n-m}BE_m(x|\mathbf{a};\mathbf{b}).$$

Proof By (1.6), we have that

$$\prod_{i=1}^{r}\left(\frac{e^{a_i t}-1}{t}\right)\prod_{i=1}^{s}\left(\frac{e^{b_i t}+1}{2}\right)D\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) \sim \left(1, \frac{e^{\lambda t}-1}{\lambda}\right). \tag{2.2}$$

Thus,

$$\begin{aligned} D\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) &= \sum_{m=0}^{n} S_1(n,m)\lambda^{n-m}\prod_{i=1}^{r}\left(\frac{t}{e^{a_i t}-1}\right)\prod_{i=1}^{s}\left(\frac{2}{e^{b_i t}+1}\right)x^m \\ &= \sum_{m=0}^{n} S_1(n,m)\lambda^{n-m}BE_m(x|\mathbf{a};\mathbf{b}), \end{aligned}$$

as claimed. □

Theorem 2.2 *For all* $n \geq 0$,

$$D\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) = \sum_{j=0}^{n}\left(\sum_{\ell=j}^{n}\binom{n}{\ell}S_1(\ell,j)\lambda^{\ell-j}D\mathcal{E}_{n-\ell}(\lambda|\mathbf{a};\mathbf{b})\right)x^j.$$

Proof We proceed the proof by applying the conjugate representation: for $s_n(x) \sim (g(t), f(t))$, we have $S_n(x) = \sum_{j=0}^{n} \frac{1}{j!} \langle g(\bar{f}(t))^{-1}\bar{f}(t)^j | x^n \rangle x^j$. By (1.6), we obtain

$$\begin{aligned}
&\langle g(\bar{f}(t))^{-1}\bar{f}(t)^j | x^n \rangle \\
&= \left\langle P_{r,s}(t) \frac{\log^j(1+\lambda t)}{\lambda^j} \middle| x^n \right\rangle = \lambda^{-j} \left\langle P_{r,s}(t) \middle| j! \sum_{\ell \geq j} S_1(\ell, j) \frac{\lambda^\ell t^\ell}{\ell!} x^n \right\rangle \\
&= \lambda^{-j} j! \sum_{\ell=j}^{n} \binom{n}{\ell} S_1(\ell, j) \lambda^\ell \langle P_{r,s}(t) | x^{n-\ell} \rangle = \lambda^{-j} j! \sum_{\ell=j}^{n} \binom{n}{\ell} S_1(\ell, j) \lambda^\ell D\mathcal{E}_{n-\ell}(\lambda | \mathbf{a}; \mathbf{b}).
\end{aligned}$$

Therefore, $D\mathcal{E}_n(\lambda, x | \mathbf{a}; \mathbf{b}) = \sum_{j=0}^{n} (\sum_{\ell=j}^{n} \binom{n}{\ell} S_1(\ell, j) \lambda^{\ell-j} D\mathcal{E}_{n-\ell}(\lambda | \mathbf{a}; \mathbf{b})) x^j$, as claimed. □

Theorem 2.3 *For all $n \geq 1$,*

$$D\mathcal{E}_n(\lambda, x | \mathbf{a}; \mathbf{b}) = \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} \lambda^\ell B_\ell^{(n)} BE_{n-\ell}(x | \mathbf{a}; \mathbf{b}),$$

where $B_\ell^{(n)}$ is the ℓth Bernoulli number of order n (see [23]).

Proof We proceed the proof by using the following transfer formula: for $p_n(x) \sim (1, f(t))$ and $q_n(x) \sim (1, g(t))$, we have that $q_n(x) = x(\frac{f(t)}{g(t)})^n x^{-1} p_n(x)$ for all $n \geq 1$. So, by the fact that $x^n \sim (1, t)$ and (2.2), we obtain

$$\begin{aligned}
&\prod_{i=1}^{r} \left(\frac{e^{a_i t} - 1}{t} \right) \prod_{i=1}^{s} \left(\frac{e^{b_i t} + 1}{2} \right) D\mathcal{E}_n(\lambda, x | \mathbf{a}; \mathbf{b}) \\
&= x \left(\frac{\lambda t}{e^{\lambda t} - 1} \right)^n x^{n-1} = x \sum_{\ell \geq 0} B_\ell^{(n)} \frac{\lambda^\ell t^\ell}{\ell!} x^{n-1} = \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} \lambda^\ell B_\ell^{(n)} x^{n-\ell},
\end{aligned}$$

which, by (2.1), implies

$$\begin{aligned}
D\mathcal{E}_n(\lambda, x | \mathbf{a}; \mathbf{b}) &= \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} \lambda^\ell B_\ell^{(n)} \prod_{i=1}^{r} \left(\frac{t}{e^{a_i t} - 1} \right) \prod_{i=1}^{s} \left(\frac{2}{e^{b_i t} + 1} \right) x^{n-\ell} \\
&= \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} \lambda^\ell B_\ell^{(n)} BE_{n-\ell}(x | \mathbf{a}; \mathbf{b}),
\end{aligned}$$

as required. □

In order to state our next theorem, we recall the polynomials $\beta\mathcal{E}_n(\lambda, x | \mathbf{a}; \mathbf{b})$, which are called the *Barnes-type degenerate Bernoulli and Euler mixed-type polynomials*. They are defined as

$$Q_{r,s}(t)(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n \geq 0} \beta\mathcal{E}_n(\lambda, x | \mathbf{a}; \mathbf{b}) \frac{t^n}{n!}, \tag{2.3}$$

where $Q_{r,s}(t) = \prod_{i=1}^{r} (\frac{t}{(1+\lambda t)^{\frac{a_i}{\lambda}} - 1}) \prod_{i=1}^{s} (\frac{2}{(1+\lambda t)^{\frac{b_i}{\lambda}} + 1})$, for example, see [3].

Theorem 2.4 *For all* $n \geq 0$,

$$D\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) = \sum_{\ell=0}^{n} \frac{\binom{n}{\ell}}{\binom{\ell+r}{r}} \lambda^\ell S_1(\ell+r, r)\beta\mathcal{E}_{n-\ell}(\lambda, x|\mathbf{a};\mathbf{b}).$$

Proof By (1.4), we have

$$\begin{aligned} D\mathcal{E}_n(\lambda, y|\mathbf{a};\mathbf{b}) &= \left\langle \sum_{\ell\geq 0} D\mathcal{E}_\ell(\lambda, y|\mathbf{a};\mathbf{b})\frac{t^\ell}{\ell!} \middle| x^n \right\rangle = \left\langle P_{r,s}(t)(1+\lambda t)^{\frac{y}{\lambda}} | x^n \right\rangle \\ &= \left\langle Q_{r,s}(t)(1+\lambda t)^{\frac{y}{\lambda}} \middle| \frac{\log^r(1+\lambda t)}{\lambda^r t^r} x^n \right\rangle \\ &= \left\langle Q_{r,s}(t)(1+\lambda t)^{\frac{y}{\lambda}} \middle| r! \sum_{\ell\geq 0} \frac{S_1(\ell+r,r)\lambda^\ell t^\ell}{(\ell+r)!} x^n \right\rangle \\ &= \sum_{\ell=0}^{n} \frac{\binom{n}{\ell}}{\binom{\ell+r}{r}} \lambda^\ell S_1(\ell+r,r) \left\langle \sum_{m\geq 0} \beta\mathcal{E}_m(\lambda, y|\mathbf{a};\mathbf{b}) \frac{t^m}{m!} \middle| x^{n-\ell} \right\rangle, \end{aligned}$$

which, by (2.3), implies $D\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) = \sum_{\ell=0}^{n} \frac{\binom{n}{\ell}}{\binom{\ell+r}{r}} \lambda^\ell S_1(\ell+r,r)\beta\mathcal{E}_{n-\ell}(\lambda, x|\mathbf{a};\mathbf{b})$, as required. □

In order to present our next theorem, we recall the polynomials $\beta_n(\lambda, x|\mathbf{a})$, which are called the *Barnes-type degenerate Bernoulli polynomials*. They are given by

$$Q_{r,0}(t)(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n\geq 0} \beta_n(\lambda, x|\mathbf{a}) \frac{t^n}{n!}, \tag{2.4}$$

for example, see [8, 9, 23].

Theorem 2.5 *For all* $n \geq 0$,

$$\begin{aligned} D\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) &= \sum_{\ell=0}^{n}\sum_{m=0}^{n-\ell} \frac{\binom{n}{\ell}\binom{n-\ell}{m}}{\binom{\ell+r}{r}} \lambda^\ell S_1(\ell+r,r)\mathcal{E}_{n-\ell-m}(\lambda|\mathbf{b})\beta_m(\lambda, x|\mathbf{a}) \\ &= \sum_{\ell=0}^{n}\sum_{m=0}^{n-\ell} \frac{\binom{n}{\ell}\binom{n-\ell}{m}}{\binom{\ell+r}{r}} \lambda^\ell S_1(\ell+r,r)\beta_{n-\ell-m}(\lambda|\mathbf{a})\mathcal{E}_m(\lambda, x|\mathbf{b}). \end{aligned}$$

Proof By the proof of Theorem 2.4, we have

$$\begin{aligned} D\mathcal{E}_n(\lambda, y|\mathbf{a};\mathbf{b}) &= \sum_{\ell=0}^{n} \frac{\binom{n}{\ell}}{\binom{\ell+r}{r}} \lambda^\ell S_1(\ell+r,r) \left\langle Q_{r,s}(t)(1+\lambda t)^{\frac{y}{\lambda}} | x^{n-\ell} \right\rangle \\ &= \sum_{\ell=0}^{n} \frac{\binom{n}{\ell}}{\binom{\ell+r}{r}} \lambda^\ell S_1(\ell+r,r) \left\langle Q_{0,s}(t) | Q_{r,0}(t)(1+\lambda t)^{\frac{y}{\lambda}} x^{n-\ell} \right\rangle. \end{aligned}$$

Thus, by (1.5) and (2.4), we obtain

$$\begin{aligned} &D\mathcal{E}_n(\lambda, y|\mathbf{a};\mathbf{b}) \\ &\quad = \sum_{\ell=0}^{n} \frac{\binom{n}{\ell}}{\binom{\ell+r}{r}} \lambda^\ell S_1(\ell+r,r) \left\langle Q_{0,s}(t) \middle| \sum_{m=0}^{n-\ell} \binom{n-\ell}{m} \beta_m(\lambda, y|\mathbf{a}) x^{n-\ell-m} \right\rangle \end{aligned}$$

$$= \sum_{\ell=0}^{n} \frac{\binom{n}{\ell}}{\binom{\ell+r}{r}} \lambda^{\ell} S_1(\ell+r,r) \sum_{m=0}^{n-\ell} \binom{n-\ell}{m} \beta_m(\lambda,y|\mathbf{a}) \langle Q_{0,s}(t)|x^{n-\ell-m}\rangle$$

$$= \sum_{\ell=0}^{n} \frac{\binom{n}{\ell}}{\binom{\ell+r}{r}} \lambda^{\ell} S_1(\ell+r,r) \sum_{m=0}^{n-\ell} \binom{n-\ell}{m} \beta_m(\lambda,y|\mathbf{a}) \mathcal{E}_{n-\ell-m}(\lambda|\mathbf{b}),$$

which completes the proof of the first formula.

The second formula can be obtained by using very similar techniques. □

3 Recurrence relations

In this section, we present several recurrence relations for Barnes-type Daehee with λ-parameter and degenerate Euler mixed-type polynomials. Our first recurrence is based on the polynomials $(x|\lambda)_n$.

Theorem 3.1 *For all* $n \geq 0$,

$$D\mathcal{E}_n(\lambda, x+y|\mathbf{a};\mathbf{b}) = \sum_{j=0}^{n} \binom{n}{j} D\mathcal{E}_j(\lambda,x|\mathbf{a};\mathbf{b})(y|\lambda)_{n-j}.$$

Proof Let $p_n(x) = \prod_{i=1}^{r}(\frac{e^{a_i t}-1}{t})\prod_{i=1}^{s}(\frac{e^{b_i t}+1}{2})D\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b})$. By (2.2) we have that $p_n(x) = (x|\lambda)_n \sim (1, \frac{e^{\lambda t}-1}{\lambda})$, which leads to the required recurrence. □

The second recurrence is obtained from the fact that $f(t)s_n(x) = ns_{n-1}(x)$ for all $s_n(x) \sim (g(t),f(t))$ (see [1, 2]).

Theorem 3.2 *For all* $n \geq 1$,

$$D\mathcal{E}_n(\lambda, x+\lambda|\mathbf{a};\mathbf{b}) - D\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b}) = n\lambda D\mathcal{E}_{n-1}(\lambda,x|\mathbf{a};\mathbf{b}).$$

Proof By (1.6) and $f(t)s_n(x) = ns_{n-1}(x)$ whenever $s_n(x) \sim (g(t),f(t))$, we have

$$\frac{e^{\lambda t}-1}{\lambda} D\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b}) = nD\mathcal{E}_{n-1}(\lambda,x|\mathbf{a};\mathbf{b}),$$

which implies $D\mathcal{E}_n(\lambda, x+\lambda|\mathbf{a};\mathbf{b}) - D\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b}) = n\lambda D\mathcal{E}_{n-1}(\lambda,x|\mathbf{a};\mathbf{b})$, as required. □

The next result gives an explicit formula for $\frac{d}{dx}D\mathcal{E}_n(\lambda, x+\lambda|\mathbf{a};\mathbf{b})$.

Theorem 3.3 *For all* $n \geq 1$,

$$\frac{d}{dx} D\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b}) = n! \sum_{\ell=0}^{n-1} \frac{(-\lambda)^{n-\ell-1}}{\ell!(n-\ell)} D\mathcal{E}_\ell(\lambda,x|\mathbf{a};\mathbf{b}).$$

Proof It is well known that for $s_n(x) \sim (g(t),f(t))$, $\frac{d}{dx}s_n(x) = \sum_{\ell=0}^{n-1}\binom{n}{\ell}\langle\bar{f}(t)|x^{n-\ell}\rangle s_\ell(x)$ (see [1, 2]). In our case, by (1.6), we have

$$\begin{aligned}\langle\bar{f}(t)|x^{n-\ell}\rangle &= \left\langle \frac{1}{\lambda}\log(1+\lambda t) \middle| x^{n-\ell}\right\rangle \\ &= \lambda^{-1}\left\langle \sum_{m\geq 1} \frac{(-1)^{m-1}(m-1)!\lambda^m t^m}{m!} \middle| x^{n-\ell}\right\rangle\end{aligned}$$

$$= \lambda^{-1}(-1)^{n-\ell-1}\lambda^{n-\ell}(n-\ell-1)!$$
$$= (-\lambda)^{n-\ell-1}(n-\ell-1)!.$$

Thus $\frac{d}{dx}D\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b}) = n!\sum_{\ell=0}^{n-1}\frac{(-\lambda)^{n-\ell-1}}{\ell!(n-\ell)}D\mathcal{E}_\ell(\lambda,x|\mathbf{a};\mathbf{b})$, as required. □

Another recurrence relation can be stated as follows.

Theorem 3.4 *For all $n \geq 1$,*

$$\begin{aligned}
&D\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b})\\
&\quad= \left(x - \sum_{i=1}^{r}a_i - \sum_{j=1}^{s}b_j\right)D\mathcal{E}_{n-1}(\lambda,x-\lambda|\mathbf{a};\mathbf{b}) + \frac{r}{n}\sum_{\ell=0}^{n}\binom{n}{\ell}\lambda^\ell\mathfrak{b}_\ell D\mathcal{E}_{n-\ell}(\lambda,x-\lambda|\mathbf{a};\mathbf{b})\\
&\qquad - \frac{1}{n}\sum_{i=1}^{r}a_i\sum_{\ell=0}^{n}\binom{n}{\ell}\lambda^\ell\mathfrak{b}_\ell D\mathcal{E}_{n-\ell}(\lambda,x-\lambda|a_i,a_1,\ldots,a_r;\mathbf{b})\\
&\qquad + \frac{1}{2}\sum_{j=1}^{s}b_j D\mathcal{E}_{n-1}(\lambda,x-\lambda|\mathbf{a};b_j,b_1,\ldots,b_s),
\end{aligned}$$

where $\mathfrak{b}_n$ is the nth Bernoulli number of the second kind, which is defined by $\frac{t}{\log(1+t)} = \sum_{n\geq 0}\mathfrak{b}_n\frac{t^n}{n!}$.

Proof Let $n \geq 1$. Then

$$\begin{aligned}
&D\mathcal{E}_n(\lambda,y|\mathbf{a};\mathbf{b})\\
&\quad= \left\langle \sum_{\ell\geq 0}D\mathcal{E}_\ell(\lambda,y|\mathbf{a};\mathbf{b})\frac{t^\ell}{\ell!}\middle| x^n\right\rangle\\
&\quad= \left\langle P_{r,s}(t)(1+\lambda t)^{y/\lambda}|x^n\right\rangle = \left\langle \frac{d}{dt}\left(P_{r,s}(t)(1+\lambda t)^{y/\lambda}\right)\middle| x^{n-1}\right\rangle\\
&\quad= \left\langle \frac{d}{dt}\prod_{i=1}^{r}\left(\frac{\log(1+\lambda t)}{\lambda((1+\lambda t)^{\frac{a_i}{\lambda}}-1)}\right)\prod_{i=1}^{s}\left(\frac{2}{(1+\lambda t)^{\frac{b_i}{\lambda}}+1}\right)(1+\lambda t)^{y/\lambda}\middle| x^{n-1}\right\rangle &(3.1)\\
&\qquad+ \left\langle \prod_{i=1}^{r}\left(\frac{\log(1+\lambda t)}{\lambda((1+\lambda t)^{\frac{a_i}{\lambda}}-1)}\right)\frac{d}{dt}\prod_{i=1}^{s}\left(\frac{2}{(1+\lambda t)^{\frac{b_i}{\lambda}}+1}\right)(1+\lambda t)^{y/\lambda}\middle| x^{n-1}\right\rangle &(3.2)\\
&\qquad+ \left\langle P_{r,s}(t)\frac{d}{dt}(1+\lambda t)^{y/\lambda}\middle| x^{n-1}\right\rangle. &(3.3)
\end{aligned}$$

By (1.6), the term in (3.3) equals

$$y\left\langle P_{r,s}(t)(1+\lambda t)^{(y-\lambda)/\lambda}|x^{n-1}\right\rangle = yD\mathcal{E}_{n-1}(\lambda,y-\lambda|\mathbf{a};\mathbf{b}). \tag{3.4}$$

For the term in (3.2), we observe that

$$\frac{d}{dt}\prod_{i=1}^{s}\left(\frac{2}{(1+\lambda t)^{\frac{b_i}{\lambda}}+1}\right) = \prod_{i=1}^{s}\left(\frac{2}{(1+\lambda t)^{\frac{b_i}{\lambda}}+1}\right)\sum_{i=1}^{s}\left(\frac{-b_i}{1+\lambda t} + \frac{b_i}{2(1+\lambda t)}\frac{2}{(1+\lambda t)^{b_i/\lambda}+1}\right).$$

So the term in (3.2) is

$$-\sum_{j=1}^{s} b_j\langle P_{r,s}(t)(1+\lambda t)^{(y-\lambda)/\lambda}|x^{n-1}\rangle + \frac{1}{2}\sum_{j=1}^{s} b_j\left\langle P_{r,s}(t)\frac{2(1+\lambda t)^{(y-\lambda)/\lambda}}{(1+\lambda t)^{b_j/\lambda}+1}\middle|x^{n-1}\right\rangle$$
$$= -\sum_{j=1}^{s} b_j D\mathcal{E}_{n-1}(\lambda, y-\lambda|\mathbf{a};\mathbf{b}) + \frac{1}{2}\sum_{j=1}^{s} b_j D\mathcal{E}_{n-1}(\lambda, y-\lambda|\mathbf{a}; b_j, b_1,\ldots,b_s). \tag{3.5}$$

For the term in (3.1), we note that

$$(1+\lambda t)\frac{d}{dt}\prod_{i=1}^{r}\left(\frac{\log(1+\lambda t)}{\lambda((1+\lambda t)^{\frac{a_i}{\lambda}}-1)}\right)$$
$$= \prod_{i=1}^{r}\left(\frac{\log(1+\lambda t)}{\lambda((1+\lambda t)^{\frac{a_i}{\lambda}}-1)}\right)\left(-\sum_{i=1}^{r} a_i + \frac{1}{t}\sum_{i=1}^{r}\left(\frac{\lambda t}{\log(1+\lambda t)} - \frac{a_i t}{(1+\lambda t)^{a_i/\lambda}-1}\right)\right),$$

where $\frac{\lambda t}{\log(1+\lambda t)} - \frac{a_i t}{(1+\lambda t)^{a_i/\lambda}-1}$ has order at least 1. Thus, the term in (3.1) equals

$$-\sum_{i=1}^{r} a_i\langle P_{r,s}(t)(1+\lambda t)^{(y-\lambda)/\lambda}|x^{n-1}\rangle$$
$$+\left\langle P_{r,s}(t)(1+\lambda t)^{(y-\lambda)/\lambda}\middle|\frac{1}{t}\sum_{i=1}^{r}\left(\frac{\lambda t}{\log(1+\lambda t)} - \frac{a_i t}{(1+\lambda t)^{a_i/\lambda}-1}\right)x^{n-1}\right\rangle$$
$$= -\sum_{i=1}^{r} a_i D\mathcal{E}_{n-1}(\lambda, y-\lambda|\mathbf{a};\mathbf{b})$$
$$+\frac{1}{n}\left\langle P_{r,s}(t)(1+\lambda t)^{(y-\lambda)/\lambda}\middle|\sum_{i=1}^{r}\left(\frac{\lambda t}{\log(1+\lambda t)} - \frac{a_i t}{(1+\lambda t)^{a_i/\lambda}-1}\right)x^{n}\right\rangle$$
$$= -\sum_{i=1}^{r} a_i D\mathcal{E}_{n-1}(\lambda, y-\lambda|\mathbf{a};\mathbf{b})$$
$$+\frac{r}{n}\left\langle P_{r,s}(t)(1+\lambda t)^{(y-\lambda)/\lambda}\middle|\sum_{\ell\geq 0}\mathfrak{b}_\ell\frac{\lambda^\ell t^\ell}{\ell!}x^{n}\right\rangle$$
$$-\frac{1}{n}\sum_{i=1}^{r} a_i\left\langle\frac{\log(1+\lambda t)}{\lambda((1+\lambda t)^{a_i/\lambda}-1)}P_{r,s}(t)(1+\lambda t)^{(y-\lambda)/\lambda}\middle|\sum_{\ell\geq 0}\mathfrak{b}_\ell\frac{\lambda^\ell t^\ell}{\ell!}x^{n}\right\rangle,$$

which is equal to

$$-\sum_{i=1}^{r} a_i D\mathcal{E}_{n-1}(\lambda, y-\lambda|\mathbf{a};\mathbf{b}) + \frac{r}{n}\sum_{\ell=0}^{n}\binom{n}{\ell}\lambda^\ell\mathfrak{b}_\ell D\mathcal{E}_{n-\ell}(\lambda, y-\lambda|\mathbf{a};\mathbf{b})$$
$$-\frac{1}{n}\sum_{i=1}^{r} a_i\sum_{\ell=0}^{n}\binom{n}{\ell}\lambda^\ell\mathfrak{b}_\ell D\mathcal{E}_{n-\ell}(\lambda, y-\lambda|a_i, a_1,\ldots,a_r;\mathbf{b}). \tag{3.6}$$

By using (3.4), (3.5) and (3.6) instead of (3.3), (3.2) and (3.1), respectively, we complete the proof. □

Theorem 3.5 *For all* $n \geq 0$,

$$\begin{aligned} & D\mathcal{E}_{n+1}(\lambda, x|\mathbf{a};\mathbf{b}) \\ & \quad = xD\mathcal{E}_n(\lambda, x-\lambda|\mathbf{a};\mathbf{b}) - \sum_{i=1}^{r} a_i \sum_{m=0}^{n} S_1(n,m)\lambda^{n-m} BE_m(x-\lambda|\mathbf{a};\mathbf{b}) \\ & \qquad - \sum_{m=0}^{n}\sum_{\ell=0}^{m} S_1(n,m)\lambda^{n-m}\binom{n}{m}\left(\frac{B_{\ell+1}}{\ell+1}\sum_{i=1}^{r} a_i^{\ell+1} + \frac{E_\ell(1)}{2}\sum_{j=1}^{s} b_j^{\ell+1}\right) \\ & \qquad \times BE_{m-\ell}(x-\lambda|\mathbf{a};\mathbf{b}), \end{aligned}$$

where B_ℓ *is the* ℓ*th Bernoulli number and* $E_\ell(1)$ *is the* ℓ*th Euler polynomial evaluated at* 1.

Proof It is well known that for $s_n(x) \sim (g(t), f(t))$, $s_{n+1}(x) = (x - g'(t)/g(t))\frac{1}{f'(t)}s_n(x)$ (see [1, 2]). In our case, by (1.6), we have

$$D\mathcal{E}_{n+1}(\lambda, x|\mathbf{a};\mathbf{b}) = xD\mathcal{E}_n(\lambda, x-\lambda|\mathbf{a};\mathbf{b}) - e^{-\lambda t}\frac{g'(t)}{g(t)} D\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}),$$

and by Theorem 2.1, we obtain

$$\begin{aligned} D\mathcal{E}_{n+1}(\lambda, x|\mathbf{a};\mathbf{b}) &= xD\mathcal{E}_n(\lambda, x-\lambda|\mathbf{a};\mathbf{b}) \\ &\quad - \sum_{m=0}^{n} S_1(n,m)\lambda^{n-m} e^{-\lambda t}\frac{g'(t)}{g(t)} BE_m(x|\mathbf{a};\mathbf{b}). \end{aligned} \tag{3.7}$$

Note that

$$\begin{aligned} \frac{g'(t)}{g(t)} &= \big(\log g(t)\big)' = \sum_{i=1}^{r}\frac{a_i e^{a_i t}}{e^{a_i t}-1} - \frac{r}{t} + \sum_{j=1}^{s}\frac{b_j e^{b_j t}}{e^{b_j t}+1} \\ &= \sum_{i=1}^{r} a_i + \frac{1}{t}\sum_{i=1}^{r}\left(\frac{a_i t}{e^{a_i t}-1} - 1\right) + \frac{1}{2}\sum_{j=1}^{s}\frac{2b_j e^{b_j t}}{e^{b_j t}+1} \\ &= \sum_{i=1}^{r} a_i + \frac{1}{t}\sum_{i=1}^{r}\sum_{\ell\geq 0}\beta_\ell a_i^\ell\frac{t^\ell}{\ell!} + \frac{1}{2}\sum_{j=1}^{s}\sum_{\ell\geq 0}E_\ell(1) b_j^{\ell+1}\frac{t^\ell}{\ell!} \\ &= \sum_{i=1}^{r} a_i + \sum_{\ell\geq 0}\frac{\beta_{\ell+1}}{(\ell+1)!}\sum_{i=1}^{r} a_i^{\ell+1}t^\ell + \frac{1}{2}\sum_{\ell\geq 0}\frac{E_\ell(1)}{\ell!}\sum_{j=1}^{s} b_j^{\ell+1}t^\ell. \end{aligned}$$

So

$$\begin{aligned} \frac{g'(t)}{g(t)} BE_m(x|\mathbf{a};\mathbf{b}) &= \sum_{i=1}^{r} a_i BE_m(x|\mathbf{a};\mathbf{b}) + \sum_{\ell=0}^{m}\binom{m}{\ell}\frac{\beta_{\ell+1}}{\ell+1}\sum_{i=1}^{r} a_i^{\ell+1} BE_{m-\ell}(x|\mathbf{a};\mathbf{b}) \\ &\quad + \frac{1}{2}\sum_{\ell=0}^{m}\binom{m}{\ell} E_\ell(1)\sum_{j=1}^{s} b_j^{\ell+1} BE_{m-\ell}(x|\mathbf{a};\mathbf{b}). \end{aligned}$$

Hence, by substituting into (3.7), we complete the proof. □

4 Relations with other families of polynomials

In this section, we establish a connection between Barnes-type Daehee with λ-parameter and degenerate Euler mixed-type polynomials and several known families of polynomials.

Theorem 4.1 *For all* $n \geq 0$,

$$D\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) = \sum_{m=0}^{n} \binom{n}{m} D\mathcal{E}_{n-m}(\lambda|\mathbf{a};\mathbf{b})(x|\lambda)_m.$$

Proof Note that $(x|\lambda)_n \sim (1, \frac{e^{\lambda t}-1}{\lambda})$. Let $D\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) = \sum_{m=0}^{n} c_{n,m}(x|\lambda)_m$. By (1.3) and (1.6), we have

$$\begin{aligned} c_{n,m} &= \frac{1}{m!}\langle P_{r,s}(t)|t^m x^n\rangle = \binom{n}{m}\langle P_{r,s}(t)|x^{n-m}\rangle \\ &= \binom{n}{m} D\mathcal{E}_{n-m}(\lambda|\mathbf{a};\mathbf{b}), \end{aligned}$$

which completes the proof. □

For the following, we note that $B_n^{(\alpha)}(x) \sim (\frac{(e^t-1)^\alpha}{t^\alpha}, t)$.

Theorem 4.2 *For all* $n \geq 0$, *the polynomial* $D\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b})$ *is given by*

$$\sum_{m=0}^{n}\left(\sum_{\ell=m}^{n}\sum_{k=0}^{n-\ell}\sum_{q=0}^{n-\ell-k}\sum_{p=0}^{q} \frac{\binom{n}{\ell}\binom{n-\ell}{k}\binom{n-\ell-k}{q}}{\binom{q+\alpha}{\alpha}} a_{\ell,k,q,p} D\mathcal{E}_{n-\ell-k-q}(\lambda|\mathbf{a};\mathbf{b})\right) B_m^{(\alpha)}(x),$$

where $a_{\ell,k,q,p} = S_1(\ell,m)S_1(q+\alpha, q-p+\alpha)S_2(q-p+\alpha,\alpha)\lambda^{k+\ell+p-m} b_\ell^{(\alpha)}$ *and* $b_\ell^{(\alpha)}$ *is the* ℓ*th Bernoulli number of the second kind of order* α *given by* $(\frac{t}{\log(1+t)})^\alpha = \sum_{\ell\geq 0} b_\ell^{(\alpha)} \frac{t^\ell}{k!}$.

Proof Let $D\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) = \sum_{m=0}^{n} c_{n,m} B_m^{(\alpha)}(x)$. By (1.3) and (1.6), we have

$$\begin{aligned} c_{n,m} &= \frac{1}{m!\lambda^m}\left\langle P_{r,s}(t)\left(\frac{(1+\lambda t)^{1/\lambda}-1}{t}\right)^\alpha \left(\frac{\lambda t}{\log(1+\lambda t)}\right)^\alpha \middle| \left(\log(1+\lambda t)\right)^m x^n\right\rangle \\ &= \frac{1}{\lambda^m}\sum_{\ell=m}^{n}\binom{n}{\ell}\lambda^\ell S_1(\ell,m)\left\langle P_{r,s}(t)\left(\frac{(1+\lambda t)^{1/\lambda}-1}{t}\right)^\alpha \middle| \left(\frac{\lambda t}{\log(1+\lambda t)}\right)^\alpha x^{n-\ell}\right\rangle \\ &= \frac{1}{\lambda^m}\sum_{\ell=m}^{n}\sum_{k=0}^{n-\ell}\binom{n}{\ell}\binom{n-\ell}{k} S_1(\ell,m)\lambda^{\ell+k} b_k^{(\alpha)}\left\langle P_{r,s}(t)\left(\frac{(1+\lambda t)^{1/\lambda}-1}{t}\right)^\alpha \middle| x^{n-\ell-k}\right\rangle. \end{aligned}$$

One can show that

$$\begin{aligned} \left(\frac{(1+\lambda t)^{1/\lambda}-1}{t}\right)^\alpha &= \left(\frac{e^{\frac{1}{\lambda}\log(1+\lambda t)}-1}{t}\right)^\alpha \\ &= \sum_{q\geq 0}\sum_{p=0}^{q}\binom{q+\alpha}{\alpha}^{-1} S_1(q+\alpha, q-p+\alpha)S_2(q-p+\alpha,\alpha)\lambda^p\frac{t^q}{q!}, \end{aligned}$$

where $S_2(n,m)$ is the Stirling number of the second kind. Thus,

$$\begin{aligned}&\left\langle P_{r,s}(t)\left(\frac{(1+\lambda t)^{1/\lambda}-1}{t}\right)^{\alpha}\middle| x^{n-\ell-k}\right\rangle\\&\quad=\sum_{q=0}^{n-\ell-k}\sum_{p=0}^{q}\frac{\binom{n-\ell-k}{q}}{\binom{q+\alpha}{\alpha}}S_1(q+\alpha,q-p+\alpha)S_2(q-p+\alpha,\alpha)\lambda^p\langle P_{r,s}(t)|x^{n-\ell-k-q}\rangle,\end{aligned}$$

where $\langle P_{r,s}(t)|x^{n-\ell-k-q}\rangle = D\mathcal{E}_{n-\ell-k-q}(\lambda|\mathbf{a};\mathbf{b})$. Hence,

$$c_{n,m}=\sum_{\ell=m}^{n}\sum_{k=0}^{n-\ell}\sum_{q=0}^{n-\ell-k}\sum_{p=0}^{q}\frac{\binom{n}{\ell}\binom{n-\ell}{k}\binom{n-\ell-k}{q}}{\binom{q+\alpha}{\alpha}}a_{\ell,k,q,p}D\mathcal{E}_{n-\ell-k-q}(\lambda|\mathbf{a};\mathbf{b}),$$

which completes the proof. □

By similar techniques as in the proof of the last theorem, we can express our polynomials $D\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b})$ in terms of the degenerate Bernoulli polynomials $\beta_n^{(\alpha)}(\lambda,x)$ of order α. These polynomials are the Sheffer sequence which is given by $\beta_n^{(\alpha)}(\lambda,x)\sim((\frac{\lambda(e^t-1)}{e^{\lambda t}-1})^{\alpha},\frac{e^{\lambda t}-1}{\lambda})$.

Theorem 4.3 *For all $n\geq 0$, the polynomial $D\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b})$ is given by*

$$\sum_{m=0}^{n}\binom{n}{m}c_{n,m}\beta_m^{(\alpha)}(\lambda,x),$$

where $c_{n,m}=\sum_{q=0}^{n-m}\sum_{p=0}^{q}\frac{\binom{n-m}{q}}{\binom{q+\alpha}{\alpha}}S_1(q+\alpha,q-p+\alpha)S_2(q-p+\alpha,\alpha)\lambda^p D\mathcal{E}_{n-m-q}(\lambda|\mathbf{a};\mathbf{b})$.

Now we are interested in expressing our polynomials in terms of $H_n^{(\alpha)}(x|\mu)$ which are called the Frobenius-Euler polynomials of order α. Note that $H_n^{(\alpha)}(x|\mu)\sim((\frac{e^t-\mu}{1-\mu})^{\alpha},t)$ (see [10, 24]).

Theorem 4.4 *For all $n\geq 0$,*

$$D\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b})=\sum_{m=0}^{n}\left(\frac{a_{n,m}}{(1-\mu)^{\alpha}\lambda^m}\right)H_m^{(\alpha)}(x|\mu),$$

where

$$a_{n,m}=\sum_{\ell=m}^{n}\sum_{k=0}^{n-\ell}\sum_{p=0}^{\alpha}\binom{n}{\ell}\binom{n-\ell}{k}\binom{\alpha}{p}S_1(\ell,m)\lambda^{\ell}(-\mu)^{\alpha-p}D\mathcal{E}_k(\lambda|\mathbf{a};\mathbf{b})(p|\lambda)_{n-\ell-k}.$$

Proof Let $D\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b})=\sum_{m=0}^{n}c_{n,m}H_m^{(\alpha)}(x|\mu)$. By (1.3) and (1.6), we have

$$\begin{aligned}c_{n,m}&=\frac{1}{m!(1-\mu)^{\alpha}\lambda^m}\langle P_{r,s}(t)\big((1+\lambda t)^{1/\lambda}-\mu\big)^{\alpha}|\big(\log(1+\lambda t)\big)^m x^n\rangle\\&=\frac{1}{m!(1-\mu)^{\alpha}\lambda^m}\left\langle P_{r,s}(t)\big((1+\lambda t)^{1/\lambda}-\mu\big)^{\alpha}\middle| m!\sum_{\ell\geq m}S_1(\ell,m)\frac{\lambda^{\ell}}{\ell!}t^{\ell}x^n\right\rangle\end{aligned}$$

$$= \frac{1}{(1-\mu)^{\alpha}\lambda^{m}} \sum_{\ell=m}^{n} \binom{n}{\ell} S_1(\ell,m)\lambda^{\ell} \langle \left((1+\lambda t)^{1/\lambda}-\mu\right)^{\alpha} | P_{r,s}(t) x^{n-\ell} \rangle$$

$$= \frac{1}{(1-\mu)^{\alpha}\lambda^{m}} \sum_{\ell=m}^{n} \sum_{k=0}^{n-\ell} \binom{n}{\ell}\binom{n-\ell}{k} S_1(\ell,m)\lambda^{\ell} D\mathcal{E}_k(\lambda|\mathbf{a};\mathbf{b}) w_{n,\ell,k},$$

where

$$\begin{aligned} w_{n,\ell,k} &= \langle \left((1+\lambda t)^{1/\lambda}-\mu\right)^{\alpha} | x^{n-\ell-k} \rangle \\ &= \left\langle \sum_{p=0}^{\alpha} \binom{\alpha}{p} (-\mu)^{\alpha-p} (1+\lambda t)^{p/\lambda} \middle| x^{n-\ell-k} \right\rangle \\ &= \sum_{p=0}^{\alpha} \binom{\alpha}{p} (-\mu)^{\alpha-p} \left\langle \sum_{q\geq 0} (p|\lambda)_q \frac{t^q}{q!} \middle| x^{n-\ell-k} \right\rangle \\ &= \sum_{p=0}^{\alpha} \binom{\alpha}{p} (-\mu)^{\alpha-p} (p|\lambda)_{n-\ell-k}. \end{aligned}$$

Thus, the constants $c_{n,m}$ are given by

$$\frac{1}{(1-\mu)^{\alpha}\lambda^{m}} \sum_{\ell=m}^{n} \sum_{k=0}^{n-\ell} \sum_{p=0}^{\alpha} \binom{n}{\ell}\binom{n-\ell}{k}\binom{\alpha}{p} S_1(\ell,m)\lambda^{\ell}(-\mu)^{\alpha-p} D\mathcal{E}_k(\lambda|\mathbf{a};\mathbf{b})(p|\lambda)_{n-\ell-k},$$

which completes the proof. □

Now we are interested in expressing our polynomials in terms of $\mathcal{E}_n^{(\alpha)}(\lambda,x)$ which are called the degenerate Euler polynomials of order α. Note that

$$\mathcal{E}_n^{(\alpha)}(\lambda,x) \sim \left(\left(\frac{e^t+1}{2}\right)^{\alpha}, \frac{e^{\lambda t}-1}{\lambda}\right)$$

(see [10]). Using similar techniques as in the proof of the above theorem, we obtain the following relation.

Theorem 4.5 *For all $n \geq 0$, the polynomial $D\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b})$ is given by*

$$\frac{1}{2^{\alpha}} \sum_{m=0}^{n} \binom{n}{m} \left(\sum_{q=0}^{n-m} \sum_{p=0}^{\alpha} \binom{n-m}{q}\binom{\alpha}{p} (p|\lambda)_q D\mathcal{E}_{n-m-q}(\lambda|\mathbf{a};\mathbf{b}) \right) \mathcal{E}_m^{(\alpha)}(\lambda,x).$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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References

1. Roman, S: More on the umbral calculus, with emphasis on the q-umbral calculus. J. Math. Anal. Appl. **107**, 222-254 (1985)
2. Roman, S: The Umbral Calculus. Dover, New York (2005)
3. Kim, DS, Kim, T, Kwon, HI, Mansour, T: Barnes-type degenerate Bernoulli and Euler mixed-type polynomials (submitted)
4. Kim, DS, Kim, T, Kwon, HI, Mansour, T, Seo, JJ: Barnes-type Peters polynomial with umbral calculus viewpoint. J. Inequal. Appl. **2014**, 324 (2014)
5. Kim, DS, Kim, T, Kwon, HI, Mansour, T: Barnes-type Narumi of the first kind and poly-Cauchy of the first kind mixed-type polynomials. Adv. Stud. Theor. Phys. **8**(22), 961-975 (2014)
6. Kim, DS, Kim, T, Lee, S-H, Seo, JJ: A note on the lambda-Daehee polynomials. Int. J. Math. Anal. **7**(62), 3069-3080 (2013)
7. Park, J-W: On the twisted Daehee polynomials with q-parameter. Adv. Differ. Equ. **2014**, 304 (2014)
8. Carlitz, L, Stirling, D: Bernoulli and Eulerian numbers. Util. Math. **15**, 51-88 (1979)
9. Carlitz, L: A degenerate Staudt-Clausen theorem. Arch. Math. (Basel) **7**, 28-33 (1956)
10. Kim, DS, Kim, T: Higher-order degenerate Euler polynomials. Appl. Math. Sci. (Ruse) **9**(2), 57-73 (2015)
11. Kim, DS, Kim, T: Some identities of degenerate Euler polynomials arising from fermionic integral on Z_p. Integral Transforms Spec. Funct. **26**(4), 295-302 (2015)
12. Araci, S, Acikgoz, M, Sen, E: On the extended Kim's p-adic q-deformed fermionic integrals in the p-adic integer ring. J. Number Theory **133**(10), 3348-3361 (2013)
13. Hwang, K-W, Dolgy, DV, Kim, DS, Kim, T, Lee, SH: Some theorems on Bernoulli and Euler numbers. Ars Comb. **109**, 285-297 (2013)
14. Kim, DS, Kim, T: q-Bernoulli polynomials and q-umbral calculus. Sci. China Math. **57**(9), 1867-1874 (2014)
15. Kim, T: Some identities on the q-Euler polynomials of higher order and q-Stirling numbers by the fermionic p-adic integral on Z_p. Russ. J. Math. Phys. **16**(4), 484-491 (2009)
16. Luo, Q-M, Qi, F: Relationships between generalized Bernoulli numbers and polynomials and generalized Euler numbers and polynomials. Adv. Stud. Contemp. Math. (Kyungshang) **7**(1), 11-18 (2003)
17. Ozden, H: p-Adic distribution of the unification of the Bernoulli, Euler and Genocchi polynomials. Appl. Math. Comput. **218**(3), 970-973 (2011)
18. Park, J-W, Rim, S-H, Kwon, J: The twisted Daehee numbers and polynomials. Adv. Differ. Equ. **2014**, 1 (2014)
19. Rim, S-H, Joung, J, Jin, J-H, Lee, S-J: A note on the weighted Carlitz's type q-Euler numbers and q-Bernstein polynomials. Proc. Jangjeon Math. Soc. **15**(2), 195-201 (2012)
20. Zhang, Z, Yang, H: Some closed formulas for generalized Bernoulli-Euler numbers and polynomials. Proc. Jangjeon Math. Soc. **11**(2), 191-198 (2008)
21. Kim, DS, Kim, T, Kwon, HI, Seo, JJ: Identities of some special mixed-type polynomials. Adv. Stud. Theor. Phys. **8**(17), 745-754 (2014)
22. Lim, D, Do, Y: Some identities of Barnes-type special polynomials. Adv. Differ. Equ. **2015**, 42 (2015)
23. Kim, DS, Kim, T, Dolgy, DV, Komtasu, T: Barnes-type degenerate Bernoulli polynomials. Adv. Stud. Contemp. Math. **25**, 121-146 (2015)
24. Kurt, B, Simsek, Y: On the generalized Apostol-type Frobenius-Euler polynomials. Adv. Differ. Equ. **2013**, 1 (2013)