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A hybrid iterative method for a common solution of variational inequalities, generalized mixed equilibrium problems, and hierarchical fixed point problems

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Abstract

In this paper, we suggest and analyze an iterative method for finding a common solution of variational inequalities, a generalized mixed equilibrium problem, and a hierarchical fixed point problem in the setting of a real Hilbert space. Under suitable conditions, we prove the strong convergence theorem. Several special cases are also discussed. The results presented in this paper extend and improve some well-known results in the literature.

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Keywords: variational inequalities; generalized mixed equilibrium problems; hierarchical fixed point problems; fixed point problems

1 Introduction

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$. Let C be a nonempty, closed, and convex subset of H. Let $F: C\times C\to R$ be a bifunction, $D: C\to H$ be a nonlinear mapping, and $\varphi: C\to R$ be a function. Recently, Peng and Yao [1] considered the generalized mixed equilibrium problem (GMEP) which involves finding $x\in C$ such that

$$F(x,y)+\varphi(y)-\varphi(x)+\langle Dx,y-x\rangle\geq 0,\quad \forall y\in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $GMEP(F,\varphi,D)$. The GMEP is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, and Nash equilibrium problems; see, for example, [2–5]. For instance, we refer to [6] for a general system generalized equilibrium problems.

If $D=0$, then the generalized mixed equilibrium problem (GMEP) (1.1) becomes the following mixed equilibrium problem (MEP): Find $x\in C$ such that

$$F(x,y)+\varphi(y)-\varphi(x)\geq 0,\quad \forall y\in C. \tag{1.2}$$

Problem (1.2) was studied by Ceng and Yao [7]. The set of solutions of (1.2) is denoted by $MEP(F,\varphi)$.

If $\varphi = 0$, then the generalized mixed equilibrium problem (GMEP) (1.1) becomes the following generalized equilibrium problem (GEP): Find $x \in C$ such that

$$F(x,y) + \langle Dx, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.3}$$

Problem (1.3) was studied by Takahashi and Takahashi [8]. The set of solutions of (1.3) is denoted by $GEP(F,D)$.

If $\varphi = 0$ and $D = 0$, then the generalized mixed equilibrium problem (GMEP) (1.1) becomes the following equilibrium problem (EP): Find $x \in C$ such that

$$F(x,y) \geq 0, \quad \forall y \in C. \tag{1.4}$$

The solution set of (1.4) is denoted by $EP(F)$. Numerous problems in physics, optimization, and economics reduce to finding a solution of (1.4); see [9, 10].

Let $A : C \to H$, and let $F(x,y) = \langle Ax, y - x\rangle$, $\forall x, y \in C$. Then $x \in EP(F)$ if and only if $\langle Ax, y - x\rangle \geq 0$, $\forall y \in C$, which is a classical variational inequality problem (VIP): Find a vector $u \in C$ such that

$$\langle v - u, Au\rangle, \quad \forall v \in C. \tag{1.5}$$

The solution set of (1.5) is denoted by $VI(C,A)$. It is easy to observe that

$$u^* \in VI(C,A) \quad \Longleftrightarrow \quad u^* = P_C\bigl[u^* - \rho Au^*\bigr], \quad \text{where } \rho > 0.$$

We now have a variety of techniques to suggest and analyze various iterative algorithms for solving variational inequalities and related optimization problems; see [1–31]. The fixed point theory has played an important role in the development of various algorithms for solving variational inequalities. Using the projection operator technique, one usually establishes an equivalence between variational inequalities and fixed point problems. We introduce the following definitions, which are useful in the following analysis.

Definition 1.1 The mapping $T : C \to H$ is said to be

(a) monotone if

$$\langle Tx - Ty, x - y\rangle \geq 0, \quad \forall x, y \in C;$$

(b) strongly monotone if there exists $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in C;$$

(c) α-inverse strongly monotone if there exists $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha\|Tx - Ty\|^2, \quad \forall x, y \in C;$$

(d) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C;$$

(e) k-Lipschitz continuous if there exists a constant $k > 0$ such that

$$\|Tx - Ty\| \le k\|x - y\|, \quad \forall x, y \in C;$$

(f) a contraction on C if there exists a constant $0 \le k \le 1$ such that

$$\|Tx - Ty\| \le k\|x - y\|, \quad \forall x, y \in C.$$

It is easy to observe that every α-inverse strongly monotone T is monotone and Lipschitz continuous. It is well known that every nonexpansive operator $T : H \to H$ satisfies, for all $(x, y) \in H \times H$, the inequality

$$\bigl\langle (x - Tx) - (y - Ty), Ty - Tx \bigr\rangle \le \frac{1}{2} \bigl\| (Tx - x) - (Ty - y) \bigr\|^2, \tag{1.6}$$

and therefore, we get, for all $(x, y) \in H \times \mathrm{Fix}(T)$,

$$\langle x - Tx, y - Tx \rangle \le \frac{1}{2} \|Tx - x\|^2. \tag{1.7}$$

The fixed point problem for the mapping T is to find $x \in C$ such that

$$Tx = x. \tag{1.8}$$

We denote by $F(T)$ the set of solutions of (1.8). It is well known that $F(T)$ is closed and convex, and $P_F(T)$ is well defined.

Recently, many researchers studied various iterative algorithms for finding an element of $VI(C, A) \cap F(S)$. Takahashi and Toyoda [11] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C (I - \lambda_n B) x_n, \quad \forall n \ge 0. \tag{1.9}$$

They proved that the sequence $\{x_n\}$ converges weakly to a point $q \in VI(C, B) \cap F(S)$. Yao and Yao [12] introduced the following scheme:

$$\begin{cases} x_1 = u \in C, \\ y_n = P_C (I - \lambda_n A) x_n, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C (I - \lambda_n A) y_n, \end{cases} \tag{1.10}$$

and obtain some convergence theorems. Later, Chang *et al.* [9] introduced the following iterative scheme:

$$\begin{cases} \phi(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \ge 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n k_n, \\ k_n = P_C (I - \lambda_n B) y_n, \\ y_n = P_C (I - \lambda_n B) u_n, \end{cases} \tag{1.11}$$

and obtained some convergence theorems. In 2014, Zhou *et al.* [13] introduced the following iterative scheme:

$$\begin{cases} F(y_n,\eta) + \langle Dy_n, \eta - y_n\rangle + \frac{1}{r_n}\langle \eta - y_n, y_n - x_n\rangle \geq 0, \quad \forall \eta \in C, \\ \rho_n = \sum_{m=1}^{r} \eta_n^m P_C(I - \mu_m B_m)y_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)W_n\rho_n, \end{cases} \tag{1.12}$$

where A is a strongly positive bounded linear operator, f is a contraction on H, and W_n is the W-mapping of C into itself which is generated by a family of nonexpansive mappings $S_n, S_{n-1}, \ldots, S_1$, and a sequence of positive numbers in $[0,1]$, $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$, then they obtained some strong convergence theorems.

On the other hand, let $S : C \to H$ be a nonexpansive mapping. The following problem is called a hierarchical fixed point problem (in short, HFPP): Find $x \in F(T)$ such that

$$\langle x - Sx, y - x\rangle \geq 0, \quad \forall y \in F(T). \tag{1.13}$$

It is well known that the hierarchical fixed point problem (1.13) links with some monotone variational inequalities and convex programming problems; see [14]. Various methods have been proposed to solve the hierarchical fixed point problem; see [15–19]. In 2010, Yao *et al.* [14] introduced the following strong convergence iterative algorithm to solve problem (1.13):

$$\begin{cases} y_n = \beta_n Sx_n + (1-\beta_n)x_n, \\ x_{n+1} = P_C[\alpha_n f(x_n) + (1-\alpha_n)Ty_n], \quad \forall n \geq 0, \end{cases} \tag{1.14}$$

where $f : C \to H$ is a contraction mapping and $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in $(0,1)$. Under certain restrictions on the parameters, Yao *et al.* proved that the sequence $\{x_n\}$ generated by (1.14) converges strongly to $z \in F(T)$, which is the unique solution of the following variational inequality:

$$\bigl\langle (I-f)z, y - z\bigr\rangle \geq 0, \quad \forall y \in F(T). \tag{1.15}$$

In 2011, Ceng *et al.* [20] investigated the following iterative method:

$$x_{n+1} = P_C\bigl[\alpha_n \rho U(x_n) + (I - \alpha_n \mu F)T(y_n)\bigr], \quad \forall n \geq 0, \tag{1.16}$$

where U is a Lipschitzian mapping, and F is a Lipschitzian and strongly monotone mapping. They proved that under some assumptions as regards approximations on the operators and parameters, the sequence generated by (1.16) converges strongly to the unique solution of the variational inequality

$$\bigl\langle \rho U(z) - \mu F(z), x - z\bigr\rangle \leq 0, \quad \forall x \in F(T).$$

Very recently, Ceng *et al.* [21] introduced and analyzed hybrid implicit and explicit viscosity iterative algorithms for solving a general system of variational inequalities with a hierarchical fixed point problem constraint for a countable family of nonexpansive mapping in a real Banach space, which can be viewed as an extension and improvement of the recent results in the literature. In 2014, Bnouhachem *et al.* [22] introduced the following iterative method:

$$\begin{cases} F(u_n,y) + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \quad \forall y \in C; \\ y_n = \beta_n Sx_n + (1-\beta_n)u_n; \\ x_{n+1} = P_C[\alpha_n \rho U(x_n) + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n \mu F)T(y_n)], \quad \forall n \geq 0, \end{cases} \tag{1.17}$$

where U and F are the same as above. They proved that under some assumptions as regards approximations on the operators and parameters, the sequence $\{x_n\}$ generated by (1.17) converges strongly to the unique solution of the variational inequality

$$\langle \rho U(z) - \mu F(z), x - z\rangle \leq 0, \quad \forall x \in F(T) \cap EP(F).$$

In the same year, Bnouhachem and Chen [23] introduced the following iterative method:

$$\begin{cases} F(u_n,y) + \langle Dx_n, y-u_n\rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \quad \forall y \in C; \\ z_n = P_C[u_n - \lambda_n Au_n]; \\ y_n = \beta_n Sx_n + (1-\beta_n)z_n; \\ x_{n+1} = P_C[\alpha_n \rho U(x_n) + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n \mu F)T(y_n)], \quad \forall n \geq 0, \end{cases} \tag{1.18}$$

where U and F are the same as above. They proved that under some assumptions as regards approximations on the operators and parameters, the sequence $\{x_n\}$ generated by (1.18) converges strongly to the unique solution of variational inequality

$$\langle \rho U(z) - \mu F(z), x - z\rangle \leq 0, \quad \forall x \in VI(C,A) \cap GMEP(F,\varphi,D) \cap F(T).$$

In this paper, motivated by the work of Zhou *et al.* [13], Bnouhachem *et al.* [22, 23] and others, we give an iterative method for finding the approximate element of the common set of solutions of GMEP (1.1), VIP (1.5) and HFPP (1.13) in real Hilbert space. We establish a strong convergence theorem for the sequence generated by the proposed method. The proposed method is quite general and flexible and includes several well-known methods for solving variational inequality problems, mixed equilibrium problems, and hierarchical fixed point problems; see, *e.g.*, [6, 13, 22–27] and the references therein.

2 Preliminaries

In this section, we list some fundamental lemmas that are useful in the consequent analysis. The first lemma provides some basic properties of the projection onto C.

Lemma 2.1 *Let P_C denote the projection of H onto C. Then we have the following inequalities*:

$$\langle z - P_C[z], P_C[z] - v\rangle \geq 0, \quad \forall z \in H, v \in C; \tag{2.1}$$

$$\langle u - v, P_C[u] - P_C[v]\rangle \geq \|P_C[u] - P_C[v]\|^2, \quad \forall u, v \in H; \tag{2.2}$$

$$\|P_C[u] - P_C[v]\| \leq \|u - v\|, \quad \forall u, v \in H; \tag{2.3}$$

$$\|u - P_C[z]\|^2 \leq \|z - u\|^2 - \|z - P_C[z]\|^2, \quad \forall z \in H, u \in C. \tag{2.4}$$

Assumption 2.1 [1] Let $F : C \times C \to R$ be a bifunction and $\varphi : C \to R$ be a function satisfying the following assumptions:

(A_1) $F(x,x) = 0$, $\forall x \in C$;
(A_2) F is monotone, *i.e.*, $F(x,y) + F(y,x) \le 0$, $\forall x,y \in C$;
(A_3) for each $x,y,z \in C$, $\lim_{t\to 0} F(tz + (1-t)x, y) \le F(x,y)$;
(A_4) for each $x \in C$, $y \to F(x,y)$ is convex and lower semicontinuous;
(B_1) for each $x \in H$ and $r > 0$, there exists a bounded sunset K of C and $y_x \in C \cap \operatorname{dom}(\varphi)$ such that

$$F(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle \le 0, \quad \forall z \in C \setminus K;$$

(B_2) C is a bounded set.

Lemma 2.2 [1] *Let C be a nonempty, closed, and convex subset of H. Let $F : C \times C \to R$ satisfy* (A_1)-(A_4), *and let $\varphi : C \to R$ be a proper lower semicontinuous and convex function. Assume that either* (B_1) *or* (B_2) *holds. For $r > 0$ and $\forall x \in H$, define a mapping $T_r : H \to C$ as follows*:

$$T_r(x) = \left\{ z \in C : F(z,y) + \varphi(y) + \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \forall y \in C \right\}.$$

Then the following hold:

(i) *T_r is nonempty and single-valued*;
(ii) *T_r is firmly nonexpansive, i.e.*,

$$\| T_r(x) - T_r(y) \|^2 \le \langle T_r(x) - T_r(y), x - y\rangle, \quad \forall x,y \in H;$$

(iii) *$F(T_r(I - rD)) = GMEP(F,\varphi,D)$*;
(iv) *$GMEP(F,\varphi,D)$ is closed and convex.*

Lemma 2.3 [28] (Demiclosedness principle) *Let $T : C \to C$ be a nonexpansive mapping with* $\operatorname{Fix}(T) \ne \emptyset$. *If $\{x_n\}$ is a sequence in C that converges weakly to x and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$; in particular, if $y = 0$, then $x \in \operatorname{Fix}(T)$.*

Lemma 2.4 [20] *Let $U : C \to H$ be a τ-Lipschitzian mapping, and let $F : C \to H$ be a κ-Lipschitzian and η-strongly monotone mapping, then for $0 \le \rho\tau < \mu\eta$, $\mu F - \rho U$ is $\mu\eta - \rho\tau$-strongly monotone, i.e.*,

$$\langle (\mu F - \rho U)x - (\mu F - \rho U)y, x - y\rangle \ge (\mu\eta - \rho\tau)\|x - y\|^2, \quad \forall x,y \in C.$$

Lemma 2.5 [29] *Suppose that $\lambda \in (0,1)$ and $\mu > 0$. Let $F : C \to H$ be a κ-Lipschitzian and η-strongly monotone operator. In association with a nonexpansive mapping $T : C \to C$, define the mapping $T^\lambda : C \to H$ by*

$$T^\lambda x = Tx - \lambda\mu FT(x), \quad \forall x \in C.$$

Then T^λ is a contraction provided $\mu < \frac{2\eta}{\kappa^2}$, that is,

$$\|T^\lambda x - T^\lambda y\| \le (1 - \lambda\nu)\|x - y\|, \quad \forall x, y \in C,$$

where $\nu = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$.

Lemma 2.6 [30] *Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \le (1 - \omega_n)s_n + \omega_n\delta_n + \gamma_n, \quad \forall n \ge 0,$$

where $\{\omega_n\}$, $\{\delta_n\}$, and $\{\gamma_n\}$ satisfying the following conditions:
(i) $\{\omega_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty} \omega_n = \infty$,
(ii) $\limsup_{n\to\infty} \delta_n \le 0$ *or* $\sum_{n=0}^{\infty} \omega_n|\delta_n| < \infty$,
(iii) $\gamma_n \ge 0$ $(n \ge 0)$, $\sum_{n=0}^{\infty} \gamma_n < \infty$.
Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.7 [31] *Let C be a closed convex subset of a real Hilbert H. Let $\{T_m : 1 \le m \le r\}$ be a sequence of nonexpansive mappings on C. Suppose that $\bigcap_{m=1}^{r} F(T_m)$ is nonempty. Let $\{\lambda_m\}$ be a sequence of positive numbers with $\sum_{m=1}^{r} \lambda_m = 1$. Then a mapping S on C defined by*

$$Sx = \sum_{m=1}^{r} \lambda_m T_m x$$

for all $x \in C$ is well defined, nonexpansive, and $F(S) = \bigcap_{m=1}^{r} F(T_m)$ holds.

3 Main result

In this section, we suggest and analyze our method for finding common solutions of the generalized mixed equilibrium problem (1.1), the variational problem (1.5), and the hierarchical fixed point problem (1.13).

Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $B_m : C \to H$ be a l_m-inverse strongly monotone mapping for each $1 \le m \le r$, where r is some positive integer. Let $D : C \to H$ be a θ-inverse strongly monotone mapping. Let $F : C \times C \to R$ satisfy (A_1)-(A_4), and let $\varphi : C \to R$ be a proper lower semicontinuous and convex function. Let $S, T : C \to C$ be nonexpansive mappings and such that $\mathscr{F} = F(T) \cap VI(C, B_m) \cap GMEP(F, \varphi, D) \neq \emptyset$. Let $F : C \to H$ be a κ-Lipschitzian mapping and η-strongly monotone, and let $U : C \to H$ be a τ-Lipschitzian mapping.

Algorithm 3.1 For an arbitrary given $x_0 \in C$, let the iterative sequences $\{u_n\}$, $\{v_n\}$, $\{x_n\}$, and $\{y_n\}$ be generated by

$$\begin{cases} F(u_n, y) + \langle Dx_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C; \\ v_n = \delta_n u_n + (1 - \delta_n)\sum_{m=1}^{r} \eta_n^m P_C(I - \mu_m B_m)u_n; \\ y_n = \beta_n S x_n + (1 - \beta_n)v_n; \\ x_{n+1} = P_C[\alpha_n\rho U(x_n) + \gamma_n x_n + ((1 - \gamma_n)I - \alpha_n\mu F)T(y_n)], \quad \forall n \ge 0, \end{cases}$$

where $\mu_m \in (0, 2l_m)$, $\{r_n\} \subset (0, 2\theta)$. Suppose that the parameters satisfy $0 < \mu < \frac{2\eta}{\kappa^2}$, $0 \leq \rho\tau < \upsilon$, where $\upsilon = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Also $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ are sequences in $(0,1)$ satisfying the following conditions:

(a) $\lim_{n\to\infty} \alpha_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(b) $\lim_{n\to\infty}(\beta_n/\alpha_n) = 0$;
(c) $\lim_{n\to\infty} \gamma_n = 0$, and $\gamma_n + \alpha_n < 1$;
(d) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$, $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$, $\sum_{n=1}^{\infty} |\gamma_n - \gamma_{n-1}| < \infty$, and $\sum_{n=1}^{\infty} |\delta_n - \delta_{n-1}| < \infty$;
(e) $\liminf_{n\to\infty} r_n > 0$, and $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$;
(f) $\lim_{n\to\infty} \eta_n^m = \eta^m \in (0,1)$ for each m, where $1 \leq m \leq r$;
(g) $\sum_{m=1}^{r} \eta_n^m = 1$, $\forall n \geq 1$.

Remark 3.1 Our method can be reviewed as an extension and improvement for some well-known results, for example, the following:

(i) The (self-)contraction mapping $f : H \to H$ in [13], Theorem 10 is extended to the case of a Lipschitzian (possibly nonself-)mapping $U : C \to H$ on a nonempty, closed, and convex subset C of H.
(ii) The strongly positive linear bounded operator A in [13], Theorem 10 is extended to the case of the κ-Lipschitzian mapping and η-strongly monotone (possibly nonself-)mapping $F : C \to H$.
(iii) The contractive coefficient $h \in (0,1)$ in [13], Theorem 10 is extended to the case where the Lipschitzian constant τ lies in $[0, \infty)$.
(iv) The equilibrium problem in [13], Theorem 10 is extended to the case of the generalized mixed equilibrium problem.
(v) If $D = \varphi = 0$, $B_m = 0$ for each m, and $\delta_n = 0$, then the proposed method is an extension and improvement of a method studied in [22].
(vi) If $\delta_n = 0$, $m = 1$, then we obtain an extension and improvement of a method in [23].
(vii) If $\rho = \mu = 1$, $\beta_n = \delta_n = 0$, $\varphi = 0$, $U = f$ a contraction mapping, $F = A$ a strongly positive linear bounded operator, and $T = W_n$, where W_n is the W-mapping of C into itself which is generated by a family of nonexpansive mappings $S_n, S_{n-1}, \ldots, S_1$, and a sequence of positive numbers in $[0,1]$ $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$, then the proposed method is an extension and improvement of a method studied in [13].

This shows that Algorithm 3.1 is quite general and unifying.

Lemma 3.1 *Let $x^* \in \mathscr{F} = F(T) \cap VI(C, B_m) \cap GMEP(F, \varphi, D)$. Then $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, and $\{y_n\}$ are bounded.*

Proof First, we show that the mapping $I - r_nD$ is nonexpansive. For any $x, y \in C$,

$$\begin{aligned} \|(I - r_nD)x - (I - r_nD)y\|^2 &= \|(x - y) - r_n(Dx - Dy)\|^2 \\ &= \|x - y\|^2 - 2r_n\langle x - y, Dx - Dy\rangle + r_n^2\|Dx - Dy\|^2 \\ &\leq \|x - y\|^2 - r_n(2\theta - r_n)\|Dx - Dy\|^2 \\ &\leq \|x - y\|^2. \end{aligned}$$

Similarly, we can show that the mapping $I - \mu_m B_m$ is nonexpansive for each $1 \leq m \leq r$. For each $1 \leq m \leq r$, put

$$w_n^m = P_C(I - \mu_m B_m)u_n \quad \text{and} \quad z_n = \sum_{m=1}^{r} (\eta_n^m w_n^m).$$

Then Algorithm 3.1 can be rewritten as

$$\begin{cases} F(u_n, y) + \langle Dx_n, y - u_n \rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C; \\ v_n = \delta_n u_n + (1 - \delta_n) z_n; \\ y_n = \beta_n S x_n + (1 - \beta_n) v_n; \\ x_{n+1} = P_C[\alpha_n \rho U(x_n) + \gamma_n x_n + ((1 - \gamma_n)I - \alpha_n \mu F)T(y_n)], \quad \forall n \geq 0. \end{cases} \tag{3.1}$$

Fixing $x \in \mathscr{F}$, we have

$$\begin{aligned} \|w_n^m - x^*\| &= \|P_C(I - \mu_m B_m)u_n - P_C(I - \mu_m B_m)x^*\| \\ &\leq \|u_n - x^*\|, \quad 1 \leq \forall m \leq r. \end{aligned}$$

It follows that

$$\begin{aligned} \|z_n - x^*\| &= \left\| \sum_{m=1}^{r} (\eta_n^m w_n^m) - x^* \right\| \leq \sum_{m=1}^{r} \eta_n^m \|w_n^m - x^*\| \\ &\leq \|u_n - x^*\|. \end{aligned}$$

It follows from Lemma 2.2 that $u_n = T_{r_n}(x_n - r_n D x_n)$ and $x^* = T_{r_n}(x^* - r_n D x^*)$, we have

$$\begin{aligned} \|u_n - x^*\|^2 &= \|T_{r_n}(x_n - r_n D x_n) - T_{r_n}(x^* - r_n D x^*)\|^2 \\ &\leq \|(x_n - r_n D x_n) - (x^* - r_n D x^*)\|^2 \\ &\leq \|x_n - x^*\|^2 - r_n(2\theta - r_n)\|Dx_n - Dx^*\|^2 \\ &\leq \|x_n - x^*\|^2. \end{aligned}$$

From (3.1) and the above inequalities, we have

$$\begin{aligned} \|v_n - x^*\|^2 &\leq \delta_n \|u_n - x^*\|^2 + (1 - \delta_n)\|z_n - x^*\|^2 \\ &\leq \delta_n \|u_n - x^*\|^2 + (1 - \delta_n)\|u_n - x^*\|^2 \\ &\leq \|u_n - x^*\|^2 \\ &\leq \|x_n - x^*\|^2 - r_n(2\theta - r_n)\|Dx_n - Dx^*\|^2 \\ &\leq \|x_n - x^*\|. \end{aligned}$$

Then we have

$$\begin{aligned} &\|z_n - x^*\|^2 \leq \|u_n - x^*\|^2 \leq \|x_n - x^*\|^2 - r_n(2\theta - r_n)\|Dx_n - Dx^*\|^2 \leq \|x_n - x^*\|^2, \\ &\|v_n - x^*\|^2 \leq \|u_n - x^*\|^2 \leq \|x_n - x^*\|^2 - r_n(2\theta - r_n)\|Dx_n - Dx^*\|^2 \leq \|x_n - x^*\|^2. \end{aligned} \tag{3.2}$$

We define $V_n = \alpha_n\rho U(x_n) + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n\mu F)T(y_n)$. Next, we prove that the sequence $\{x_n\}$ is bounded, and without loss of generality, we can assume that $\beta_n \le \alpha_n$ for all $n \ge 0$. From (3.1), we have

$$
\begin{aligned}
&\|x_{n+1} - x^*\| \\
&\quad = \|P_C[V_n] - P_C[x^*]\| \\
&\quad \le \|\alpha_n\rho U(x_n) + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n\mu F)T(y_n) - x^*\| \\
&\quad = \|\alpha_n(\rho U(x_n) - \mu F(x^*)) + \gamma_n(x_n - x^*) + ((1-\gamma_n)I - \alpha_n\mu F)T(y_n) \\
&\qquad - ((1-\gamma_n)I - \alpha_n\mu F)T(x^*)\| \\
&\quad \le \alpha_n\|\rho U(x_n) - \mu F(x^*)\| + \gamma_n\|x_n - x^*\| \\
&\qquad + (1-\gamma_n)\left\|\left(I - \frac{\alpha_n\mu}{1-\gamma_n}F\right)T(y_n) - \left(I - \frac{\alpha_n\mu}{1-\gamma_n}F\right)T(x^*)\right\| \\
&\quad = \alpha_n\|\rho U(x_n) - \rho U(x^*) + (\rho U - \mu F)x^*\| + \gamma_n\|x_n - x^*\| \\
&\qquad + (1-\gamma_n)\left\|\left(I - \frac{\alpha_n\mu}{1-\gamma_n}F\right)T(y_n) - \left(I - \frac{\alpha_n\mu}{1-\gamma_n}F\right)T(x^*)\right\| \\
&\quad \le \alpha_n\rho\tau\|x_n - x^*\| + \alpha_n\|(\rho U - \mu F)x^*\| + \gamma_n\|x_n - x^*\| \\
&\qquad + (1-\gamma_n)\left(1 - \frac{\alpha_n\upsilon}{1-\gamma_n}\right)\|y_n - x^*\| \\
&\quad \le \alpha_n\rho\tau\|x_n - x^*\| + \alpha_n\|(\rho U - \mu F)x^*\| + \gamma_n\|x_n - x^*\| \\
&\qquad + (1-\gamma_n-\alpha_n\upsilon)\|\beta_n Sx_n + (1-\beta_n)v_n - x^*\| \\
&\quad \le \alpha_n\rho\tau\|x_n - x^*\| + \alpha_n\|(\rho U - \mu F)x^*\| + \gamma_n\|x_n - x^*\| \\
&\qquad + (1-\gamma_n-\alpha_n\upsilon)(\beta_n\|Sx_n - Sx^*\| + \beta_n\|Sx^* - x^*\| + (1-\beta_n)\|v_n - x^*\|) \\
&\quad \le \alpha_n\rho\tau\|x_n - x^*\| + \alpha_n\|(\rho U - \mu F)x^*\| + \gamma_n\|x_n - x^*\| \\
&\qquad + (1-\gamma_n-\alpha_n\upsilon)(\beta_n\|Sx_n - Sx^*\| + \beta_n\|Sx^* - x^*\| + (1-\beta_n)\|x_n - x^*\|) \\
&\quad \le \alpha_n\rho\tau\|x_n - x^*\| + \alpha_n\|(\rho U - \mu F)x^*\| + \gamma_n\|x_n - x^*\| \\
&\qquad + (1-\gamma_n-\alpha_n\upsilon)(\beta_n\|x_n - x^*\| + \beta_n\|Sx^* - x^*\| + (1-\beta_n)\|x_n - x^*\|) \\
&\quad \le (1-\alpha_n(\upsilon-\rho\tau))\|x_n - x^*\| + \alpha_n\|(\rho U - \mu F)x^*\| \\
&\qquad + (1-\gamma_n-\alpha_n\upsilon)\beta_n\|Sx^* - x^*\| \\
&\quad \le (1-\alpha_n(\upsilon-\rho\tau))\|x_n - x^*\| + \alpha_n\|(\rho U - \mu F)x^*\| + \beta_n\|Sx^* - x^*\| \\
&\quad \le (1-\alpha_n(\upsilon-\rho\tau))\|x_n - x^*\| + \alpha_n(\|(\rho U - \mu F)x^*\| + \|Sx^* - x^*\|) \\
&\quad \le (1-\alpha_n(\upsilon-\rho\tau))\|x_n - x^*\| + \frac{\alpha_n(\upsilon-\rho\tau)}{\upsilon-\rho\tau}(\|(\rho U - \mu F)x^*\| + \|Sx^* - x^*\|) \\
&\quad \le \max\left\{\|x_n - x^*\|, \frac{1}{\upsilon-\rho\tau}(\|(\rho U - \mu F)x^*\| + \|Sx^* - x^*\|)\right\},
\end{aligned}
$$

where the third inequality follows from Lemma 2.5. By induction on n, we obtain $\|x_n - x_0\| \leq \max\{\|x_0 - x^*\|, \frac{1}{\upsilon-\rho\tau}(\|(\rho U - \mu F)x^*\| + \|Sx^* - x^*\|)\}$ for $n \geq 0$ and $x_0 \in C$. Hence $\{x_n\}$ is bounded, and, consequently, we deduce that $\{u_n\}$, $\{v_n\}$, and $\{y_n\}$ are bounded. □

Lemma 3.2 *Let $x^* \in \mathscr{F} = F(T) \cap VI(C, B_m) \cap GMEP(F, \varphi, D)$ and $\{x_n\}$ be generated by Algorithm* 3.1. *Then we have*:

(a) $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$,

(b) *the weak w-limit set* $w_w(x_n) \subset F(T)$ ($w_w(x_n) = \{x : x_{n_i} \rightharpoonup x\}$).

Proof Note that

$$\begin{aligned}\left\|w_n^m - w_{n-1}^m\right\| &= \left\|P_C(I - \mu_m B_m)u_n - P_C(I - \mu_m B_m)u_{n-1}\right\| \\ &\leq \|u_n - u_{n-1}\|, \quad 1 \leq \forall m \leq r.\end{aligned} \tag{3.3}$$

On the other hand, we have

$$v_n - v_{n-1} = \delta_n(u_n - u_{n-1}) + (1 - \delta_n)(z_n - z_{n-1}) + (\delta_n - \delta_{n-1})(u_{n-1} - z_{n-1}).$$

It follows from (3.3) that

$$\begin{aligned}&\|v_n - v_{n-1}\| \\ &\quad\leq \delta_n\|u_n - u_{n-1}\| + (1 - \delta_n)\|z_n - z_{n-1}\| + |\delta_n - \delta_{n-1}|\|u_{n-1} - z_{n-1}\| \\ &\quad\leq \delta_n\|u_n - u_{n-1}\| + (1 - \delta_n)\left\|\sum_{m=1}^{r}\left(\eta_n^m w_n^m\right) - \sum_{m=1}^{r}\left(\eta_{n-1}^m w_{n-1}^m\right)\right\| + |\delta_n - \delta_{n-1}|\|u_{n-1} - z_{n-1}\| \\ &\quad\leq (1 - \delta_n)\left\|\sum_{m=1}^{r}\left(\eta_n^m w_n^m\right) - \sum_{m=1}^{r}\left(\eta_n^m w_{n-1}^m\right) + \sum_{m=1}^{r}\left(\eta_n^m w_{n-1}^m\right) - \sum_{m=1}^{r}\left(\eta_{n-1}^m w_{n-1}^m\right)\right\| \\ &\qquad + \delta_n\|u_n - u_{n-1}\| + |\delta_n - \delta_{n-1}|\|u_{n-1} - z_{n-1}\| \\ &\quad\leq \|u_n - u_{n-1}\| + M \cdot \sum_{m=1}^{r}\left|\eta_n^m - \eta_{n-1}^m\right| + |\delta_n - \delta_{n-1}|\|u_{n-1} - z_{n-1}\|,\end{aligned} \tag{3.4}$$

where $M = \max\{\sup\{\|P_C(I - \mu_m B_m)u_n\| : n \geq 1\} : 1 \leq m \leq r\}$. Next we estimate that

$$\begin{aligned}&\|y_n - y_{n-1}\| \\ &\quad\leq \left\|\beta_n Sx_n + (1 - \beta_n)v_n - \beta_{n-1}Sx_{n-1} - (1 - \beta_{n-1})v_{n-1}\right\| \\ &\quad= \left\|\beta_n(Sx_n - Sx_{n-1}) + (1 - \beta_n)(v_n - v_{n-1}) + (\beta_n - \beta_{n-1})(Sx_{n-1} - v_{n-1})\right\| \\ &\quad\leq \beta_n\|x_n - x_{n-1}\| + (1 - \beta_n)\|v_n - v_{n-1}\| + |\beta_n - \beta_{n-1}|\left(\|Sx_{n-1}\| + \|v_{n-1}\|\right).\end{aligned}$$

It follows from (3.4) and the above inequality that

$$\begin{aligned}\|y_n - y_{n-1}\| \leq \beta_n\|x_n - x_{n-1}\| + (1 - \beta_n)\Bigg\{&\|u_n - u_{n-1}\| + M \cdot \sum_{m=1}^{r}\left|\eta_n^m - \eta_{n-1}^m\right| \\ &+ |\delta_n - \delta_{n-1}|\|u_{n-1} - z_{n-1}\|\Bigg\} + |\beta_n - \beta_{n-1}|\left(\|Sx_{n-1}\| + \|v_{n-1}\|\right).\end{aligned} \tag{3.5}$$

On the other hand, $u_n = T_{r_n}(x_n - r_n D x_n)$ and $u_{n-1} = T_{r_{n-1}}(x_{n-1} - r_{n-1} D x_{n-1})$, we have

$$F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Dx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \tag{3.6}$$

and

$$\begin{aligned} &F(u_{n-1}, y) + \varphi(y) - \varphi(u_{n-1}) + \langle Dx_{n-1}, y - u_{n-1}\rangle + \frac{1}{r_{n-1}}\langle y - u_{n-1}, u_{n-1} - x_{n-1}\rangle \geq 0, \\ &\forall y \in C, \end{aligned} \tag{3.7}$$

Taking $y = u_{n-1}$ in (3.6) and $y = u_n$ in (3.7), we get

$$F(u_n, u_{n-1}) + \varphi(u_{n-1}) - \varphi(u_n) + \langle Dx_n, u_{n-1} - u_n\rangle + \frac{1}{r_n}\langle u_{n-1} - u_n, u_n - x_n\rangle \geq 0 \tag{3.8}$$

and

$$\begin{aligned} &F(u_{n-1}, u_n) + \varphi(u_n) - \varphi(u_{n-1}) + \langle Dx_{n-1}, u_n - u_{n-1}\rangle \\ &\quad + \frac{1}{r_{n-1}}\langle u_n - u_{n-1}, u_{n-1} - x_{n-1}\rangle \geq 0. \end{aligned} \tag{3.9}$$

Adding (3.8) and (3.9) and using the monotonicity of F, we have

$$\langle Dx_{n-1} - Dx_n, u_n - u_{n-1}\rangle + \left\langle u_n - u_{n-1}, \frac{u_{n-1} - x_{n-1}}{r_{n-1}} - \frac{u_n - x_n}{r_n}\right\rangle \geq 0,$$

which implies that

$$\begin{aligned} 0 &\leq \left\langle u_n - u_{n-1}, r_n(Dx_{n-1} - Dx_n) + \frac{r_n}{r_{n-1}}(u_{n-1} - x_{n-1}) - (u_n - x_n)\right\rangle \\ &= \left\langle u_{n-1} - u_n, u_n - u_{n-1} + \left(1 - \frac{r_n}{r_{n-1}}\right)u_{n-1} \right. \\ &\quad \left. + (x_{n-1} - r_n Dx_{n-1}) - (x_n - r_n Dx_n) - x_{n-1} + \frac{r_n}{r_{n-1}}x_{n-1}\right\rangle \\ &= \left\langle u_{n-1} - u_n, \left(1 - \frac{r_n}{r_{n-1}}\right)u_{n-1} + (x_{n-1} - r_n Dx_{n-1}) - (x_n - r_n Dx_n) - x_{n-1} + \frac{r_n}{r_{n-1}}x_{n-1}\right\rangle \\ &\quad - \|u_n - u_{n-1}\|^2 \\ &= \left\langle u_{n-1} - u_n, \left(1 - \frac{r_n}{r_{n-1}}\right)(u_{n-1} - x_{n-1}) + (x_{n-1} - r_n Dx_{n-1}) - (x_n - r_n Dx_n)\right\rangle \\ &\quad - \|u_n - u_{n-1}\|^2 \\ &\leq \|u_n - u_{n-1}\|\left\{\left|1 - \frac{r_n}{r_{n-1}}\right|\|u_{n-1} - x_{n-1}\| + \big\|(x_{n-1} - r_n Dx_{n-1}) - (x_n - r_n Dx_n)\big\|\right\} \\ &\quad - \|u_n - u_{n-1}\|^2 \\ &= \|u_n - u_{n-1}\|\left\{\left|1 - \frac{r_n}{r_{n-1}}\right|\|u_{n-1} - x_{n-1}\| + \|x_{n-1} - x_n\|\right\} - \|u_n - u_{n-1}\|^2, \end{aligned}$$

and then

$$\|u_n - u_{n-1}\| \le \left|1 - \frac{r_n}{r_{n-1}}\right| \|u_{n-1} - x_{n-1}\| + \|x_{n-1} - x_n\|.$$

Without loss of generality, let us assume that there exists a real number μ such that $r_n > \mu > 0$ for all positive integers n. Then we get

$$\|u_{n-1} - u_n\| \le \|x_{n-1} - x_n\| + \frac{1}{\mu}|r_{n-1} - r_n| \|u_{n-1} - x_{n-1}\|. \tag{3.10}$$

It follows from (3.5) and (3.10) that

$$\begin{aligned}
&\|y_n - y_{n-1}\| \\
&\quad \le \beta_n \|x_n - x_{n-1}\| + (1 - \beta_n) \left\{ \|x_n - x_{n-1}\| + \frac{1}{\mu}|r_n - r_{n-1}| \|u_{n-1} - x_{n-1}\| \right. \\
&\qquad \left. + M \cdot \sum_{m=1}^{r} \left|\eta_n^m - \eta_{n-1}^m\right| + |\delta_n - \delta_{n-1}| \|u_{n-1} - z_{n-1}\| \right\} + |\beta_n - \beta_{n-1}| \bigl(\|Sx_{n-1}\| + \|v_{n-1}\|\bigr) \\
&\quad = \|x_n - x_{n-1}\| + (1 - \beta_n) \left\{ \frac{1}{\mu}|r_n - r_{n-1}| \|u_{n-1} - x_{n-1}\| + M \cdot \sum_{m=1}^{r} \left|\eta_n^m - \eta_{n-1}^m\right| \right. \\
&\qquad \left. + |\delta_n - \delta_{n-1}| \|u_{n-1} - z_{n-1}\| \right\} + |\beta_n - \beta_{n-1}| \bigl(\|Sx_{n-1}\| + \|v_{n-1}\|\bigr) \\
&\quad \le \|x_n - x_{n-1}\| + \frac{1}{\mu}|r_n - r_{n-1}| \|u_{n-1} - x_{n-1}\| + M \cdot \sum_{m=1}^{r} \left|\eta_n^m - \eta_{n-1}^m\right| \\
&\qquad + |\delta_n - \delta_{n-1}| \|u_{n-1} - z_{n-1}\| + |\beta_n - \beta_{n-1}| \bigl(\|Sx_{n-1}\| + \|v_{n-1}\|\bigr).
\end{aligned} \tag{3.11}$$

Next, we estimate

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\quad = \bigl\|P_C[V_n] - P_C[V_{n-1}]\bigr\| \\
&\quad \le \Bigl\| \alpha_n \rho \bigl(U(x_n) - U(x_{n-1})\bigr) + (\alpha_n - \alpha_{n-1}) \rho U(x_{n-1}) \\
&\qquad + \gamma_n (x_n - x_{n-1}) + (\gamma_n - \gamma_{n-1}) x_{n-1} \\
&\qquad + (1 - \gamma_n) \left[\left(I - \frac{\alpha_n \mu}{1 - \gamma_n} F \right) T(y_n) - \left(I - \frac{\alpha_n \mu}{1 - \gamma_n} F \right) T(y_{n-1}) \right] \\
&\qquad + \bigl((1 - \gamma_n) I - \alpha_n \mu F\bigr) T(y_{n-1}) - \bigl((1 - \gamma_{n-1}) I - \alpha_{n-1} \mu F\bigr) T(y_{n-1}) \Bigr\| \\
&\quad \le \alpha_n \rho \tau \|x_n - x_{n-1}\| + \gamma_n \|x_n - x_{n-1}\| + (1 - \gamma_n) \left(1 - \frac{\alpha_n \upsilon}{1 - \gamma_n} \right) \|y_n - y_{n-1}\| \\
&\qquad + |\gamma_n - \gamma_{n-1}| \bigl(\|x_{n-1}\| + \bigl\|T(y_{n-1})\bigr\|\bigr) \\
&\qquad + |\alpha_n - \alpha_{n-1}| \bigl(\rho \bigl\|U(x_{n-1})\bigr\| + \bigl\|\mu F\bigl(T(y_{n-1})\bigr)\bigr\|\bigr),
\end{aligned} \tag{3.12}$$

which the second inequality follows from Lemma 2.5. From (3.11) and (3.12), we have

$$
\begin{aligned}
&\|x_{n+1}-x_n\| \\
&\le \alpha_n\rho\tau\|x_n-x_{n-1}\|+\gamma_n\|x_n-x_{n-1}\|+(1-\gamma_n-\alpha_n\upsilon) \\
&\quad\times\Biggl\{\|x_n-x_{n-1}\|+\frac{1}{\mu}|r_n-r_{n-1}|\|u_{n-1}-x_{n-1}\|+M\cdot\sum_{m=1}^{r}\bigl|\eta_n^m-\eta_{n-1}^m\bigr| \\
&\quad+|\delta_n-\delta_{n-1}|\|u_{n-1}-z_{n-1}\|+|\beta_n-\beta_{n-1}|\bigl(\|Sx_{n-1}\|+\|v_{n-1}\|\bigr)\Biggr\} \\
&\quad+|\gamma_n-\gamma_{n-1}|\bigl(\|x_{n-1}\|+\bigl\|T(y_{n-1})\bigr\|\bigr)+|\alpha_n-\alpha_{n-1}|\bigl(\rho\bigl\|U(x_{n-1})\bigr\|+\bigl\|\mu F\bigl(T(y_{n-1})\bigr)\bigr\|\bigr) \\
&\le\bigl(1-(\upsilon-\rho\tau)\bigr)\|x_n-x_{n-1}\|+\frac{1}{\mu}|r_n-r_{n-1}|\|u_{n-1}-x_{n-1}\|+M\cdot\sum_{m=1}^{r}\bigl|\eta_n^m-\eta_{n-1}^m\bigr| \\
&\quad+|\delta_n-\delta_{n-1}|\|u_{n-1}-z_{n-1}\|+|\beta_n-\beta_{n-1}|\bigl(\|Sx_{n-1}\|+\|v_{n-1}\|\bigr) \\
&\quad+|\gamma_n-\gamma_{n-1}|\bigl(\|x_{n-1}\|+\bigl\|T(y_{n-1})\bigr\|\bigr) \\
&\quad+|\alpha_n-\alpha_{n-1}|\bigl(\rho\bigl\|U(x_{n-1})\bigr\|+\bigl\|\mu F\bigl(T(y_{n-1})\bigr)\bigr\|\bigr) \\
&\le\bigl(1-(\upsilon-\rho\tau)\bigr)\|x_n-x_{n-1}\|+M\cdot\sum_{m=1}^{r}\bigl|\eta_n^m-\eta_{n-1}^m\bigr|+M_1\biggl(\frac{1}{\mu}|r_n-r_{n-1}| \\
&\quad+|\delta_n-\delta_{n-1}|+|\beta_n-\beta_{n-1}|+|\gamma_n-\gamma_{n-1}|+|\alpha_n-\alpha_{n-1}|\biggr),
\end{aligned}
\tag{3.13}
$$

where

$$
\begin{aligned}
M_1=\max\Bigl\{&\sup_{n\ge1}\|u_{n-1}-x_{n-1}\|,\sup_{n\ge1}\|u_{n-1}-z_{n-1}\|,\sup_{n\ge1}\bigl(\|Sx_{n-1}\|+\|v_{n-1}\|\bigr), \\
&\sup_{n\ge1}\bigl(\|x_{n-1}\|+\bigl\|T(y_{n-1})\bigr\|\bigr),\bigl(\rho\bigl\|U(x_{n-1})\bigr\|+\bigl\|\mu F\bigl(T(y_{n-1})\bigr)\bigr\|\bigr)\Bigr\}.
\end{aligned}
$$

It follows by condition (a)-(e) of Algorithm 3.1 and Lemma 2.6 that

$$
\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.
$$

Next, we show that $\lim_{n\to\infty}\|u_n-x_n\|=0$. Since $x^*\in\mathscr{F}=F(T)\cap VI(C,B_m)\cap GMEP(F,\varphi,D)$, by using (3.1) and (3.2), we obtain

$$
\begin{aligned}
\bigl\|x_{n+1}-x^*\bigr\|^2&=\bigl\langle P_C[V_n]-x^*,x_{n+1}-x^*\bigr\rangle \\
&=\bigl\langle P_C[V_n]-V_n,P_C[V_n]-x^*\bigr\rangle+\bigl\langle V_n-x^*,x_{n+1}-x^*\bigr\rangle \\
&\le\bigl\langle\alpha_n\bigl(\rho U(x_n)-\mu F\bigl(x^*\bigr)\bigr)+\gamma_n\bigl(x_n-x^*\bigr)+\bigl((1-\gamma_n)I-\alpha_n\mu F\bigr)T(y_n) \\
&\quad-\bigl((1-\gamma_n)I-\alpha_n\mu F\bigr)T\bigl(x^*\bigr),x_{n+1}-x^*\bigr\rangle \\
&=\bigl\langle\alpha_n\rho\bigl(U(x_n)-U\bigl(x^*\bigr)\bigr),x_{n+1}-x^*\bigr\rangle+\alpha_n\bigl\langle\rho U\bigl(x^*\bigr)-\mu F\bigl(x^*\bigr),x_{n+1}-x^*\bigr\rangle \\
&\quad+\bigl\langle\gamma_n\bigl(x_n-x^*\bigr),x_{n+1}-x^*\bigr\rangle \\
&\quad+(1-\gamma_n)\biggl\langle\biggl(I-\frac{\alpha_n\mu}{1-\gamma_n}F\biggr)T(y_n)-\biggl(I-\frac{\alpha_n\mu}{1-\gamma_n}F\biggr)T\bigl(x^*\bigr),x_{n+1}-x^*\biggr\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\le (\alpha_n\rho\tau + \gamma_n)\|x_n - x^*\|\|x_{n+1} - x^*\| + \alpha_n\langle\rho U(x^*) - \mu F(x^*), x_{n+1} - x^*\rangle \\
&\quad + (1 - \gamma_n - \alpha_n\upsilon)\|y_n - x^*\|\|x_{n+1} - x^*\| \\
&\le \frac{\gamma_n + \alpha_n\rho\tau}{2}\left(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2\right) \\
&\quad + \alpha_n\langle\rho U(x^*) - \mu F(x^*), x_{n+1} - x^*\rangle \\
&\quad + \frac{1 - \gamma_n - \alpha_n\upsilon}{2}\left(\|y_n - x^*\|^2 + \|x_{n+1} - x^*\|^2\right) \\
&\le \frac{1 - \alpha_n(\upsilon - \rho\tau)}{2}\|x_{n+1} - x^*\|^2 + \frac{\gamma_n + \alpha_n\rho\tau}{2}\|x_n - x^*\|^2 \\
&\quad + \alpha_n\langle\rho U(x^*) - \mu F(x^*), x_{n+1} - x^*\rangle \\
&\quad + \frac{1 - \gamma_n - \alpha_n\upsilon}{2}\left(\beta_n\|Sx_n - x^*\|^2 + (1 - \beta_n)\|v_n - x^*\|^2\right) \\
&\le \frac{1 - \alpha_n(\upsilon - \rho\tau)}{2}\|x_{n+1} - x^*\|^2 + \frac{\gamma_n + \alpha_n\rho\tau}{2}\|x_n - x^*\|^2 \\
&\quad + \alpha_n\langle\rho U(x^*) - \mu F(x^*), x_{n+1} - x^*\rangle + \frac{(1 - \gamma_n - \alpha_n\upsilon)\beta_n}{2}\|Sx_n - x^*\|^2 \\
&\quad + \frac{(1 - \gamma_n - \alpha_n\upsilon)(1 - \beta_n)}{2} \\
&\quad \times \left\{\|x_n - x^*\|^2 - r_n(2\theta - r_n)\|Dx_n - Dx^*\|^2\right\},
\end{aligned}
\tag{3.14}
$$

which implies that

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \frac{\gamma_n + \alpha_n\rho\tau}{1 + \alpha_n(\upsilon - \rho\tau)}\|x_n - x^*\|^2 \\
&\quad + \frac{2\alpha_n}{1 + \alpha_n(\upsilon - \rho\tau)}\langle\rho U(x^*) - \mu F(x^*), x_{n+1} - x^*\rangle \\
&\quad + \frac{(1 - \gamma_n - \alpha_n\upsilon)\beta_n}{1 + \alpha_n(\upsilon - \rho\tau)}\|Sx_n - x^*\|^2 \\
&\quad + \frac{(1 - \gamma_n - \alpha_n\upsilon)(1 - \beta_n)}{1 + \alpha_n(\upsilon - \rho\tau)}\left\{\|x_n - x^*\|^2 - r_n(2\theta - r_n)\|Dx_n - Dx^*\|^2\right\} \\
&\le \frac{\gamma_n + \alpha_n\rho\tau}{1 + \alpha_n(\upsilon - \rho\tau)}\|x_n - x^*\|^2 \\
&\quad + \frac{2\alpha_n}{1 + \alpha_n(\upsilon - \rho\tau)}\langle\rho U(x^*) - \mu F(x^*), x_{n+1} - x^*\rangle \\
&\quad + \|x_n - x^*\|^2 + \frac{(1 - \gamma_n - \alpha_n\upsilon)\beta_n}{1 + \alpha_n(\upsilon - \rho\tau)}\|Sx_n - x^*\|^2 \\
&\quad - \frac{(1 - \gamma_n - \alpha_n\upsilon)(1 - \beta_n)}{1 + \alpha_n(\upsilon - \rho\tau)}\left\{r_n(2\theta - r_n)\|Dx_n - Dx^*\|^2\right\}.
\end{aligned}
$$

Then from the above inequality, we get

$$
\begin{aligned}
&\frac{(1 - \gamma_n - \alpha_n\upsilon)(1 - \beta_n)}{1 + \alpha_n(\upsilon - \rho\tau)}\left\{r_n(2\theta - r_n)\|Dx_n - Dx^*\|^2\right\} \\
&\quad \le \frac{\gamma_n + \alpha_n\rho\tau}{1 + \alpha_n(\upsilon - \rho\tau)}\|x_n - x^*\|^2 + \frac{2\alpha_n}{1 + \alpha_n(\upsilon - \rho\tau)}\langle\rho U(x^*) - \mu F(x^*), x_{n+1} - x^*\rangle \\
&\quad\quad + \beta_n\|Sx_n - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2
\end{aligned}
$$

$$\begin{aligned}
&\le \frac{\gamma_n + \alpha_n\rho\tau}{1+\alpha_n(\upsilon-\rho\tau)}\|x_n - x^*\|^2 + \frac{2\alpha_n}{1+\alpha_n(\upsilon-\rho\tau)}\langle \rho U(x^*) - \mu F(x^*), x_{n+1}-x^*\rangle \\
&\quad + \beta_n\|Sx_n - x^*\|^2 + \left(\|x_n - x^*\| + \|x_{n+1}-x^*\|\right)\|x_{n+1}-x_n\|.
\end{aligned}$$

Since $\{r_n\} \subset (0,2\theta)$, $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$, $\gamma_n \to 0$, $\alpha_n \to 0$, and $\beta_n \to 0$, we obtain $\lim_{n\to\infty}\|Dx_n - Dx^*\| = 0$.

Since T_{r_n} is firmly nonexpansive, we have

$$\begin{aligned}
\|u_n - x^*\|^2 &= \left\|T_{r_n}(x_n - r_nDx_n) - T_{r_n}(x^* - r_nDx^*)\right\|^2 \\
&\le \langle u_n - x^*, (x_n - r_nDx_n) - (x^* - r_nDx^*)\rangle \\
&= \frac{1}{2}\left\{\|u_n - x^*\|^2 + \left\|(x_n - r_nDx_n) - (x^* - r_nDx^*)\right\|^2 \right.\\
&\quad \left. - \left\|u_n - x^* - \left[(x_n - r_nDx_n) - (x^* - r_nDx^*)\right]\right\|^2\right\}.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\|u_n - x^*\|^2 &\le \left\|(x_n - r_nDx_n) - (x^* - r_nDx^*)\right\|^2 - \left\|u_n - x_n + r_n(Dx_n - Dx^*)\right\|^2 \\
&\le \|x_n - x^*\|^2 - \left\|u_n - x_n + r_n(Dx_n - Dx^*)\right\|^2 \\
&\le \|x_n - x^*\|^2 - \|u_n - x_n\|^2 + 2r_n\|u_n - x_n\|\|Dx_n - Dx^*\|.
\end{aligned}$$

From (3.14), (3.2), and the above inequality, we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le \frac{1-\alpha_n(\upsilon-\rho\tau)}{2}\|x_{n+1}-x^*\|^2 + \frac{\gamma_n+\alpha_n\rho\tau}{2}\|x_n - x^*\|^2 \\
&\quad + \alpha_n\langle \rho U(x^*) - \mu F(x^*), x_{n+1}-x^*\rangle \\
&\quad + \frac{1-\gamma_n-\alpha_n\upsilon}{2}\left(\beta_n\|Sx_n - x^*\|^2 + (1-\beta_n)\|v_n - x^*\|^2\right) \\
&\le \frac{1-\alpha_n(\upsilon-\rho\tau)}{2}\|x_{n+1}-x^*\|^2 + \frac{\gamma_n+\alpha_n\rho\tau}{2}\|x_n - x^*\|^2 \\
&\quad + \alpha_n\langle \rho U(x^*) - \mu F(x^*), x_{n+1}-x^*\rangle \\
&\quad + \frac{1-\gamma_n-\alpha_n\upsilon}{2}\left(\beta_n\|Sx_n - x^*\|^2 + (1-\beta_n)\|u_n - x^*\|^2\right) \\
&\le \frac{1-\alpha_n(\upsilon-\rho\tau)}{2}\|x_{n+1}-x^*\|^2 + \frac{\gamma_n+\alpha_n\rho\tau}{2}\|x_n - x^*\|^2 \\
&\quad + \alpha_n\langle \rho U(x^*) - \mu F(x^*), x_{n+1}-x^*\rangle \\
&\quad + \frac{1-\gamma_n-\alpha_n\upsilon}{2}\left\{\beta_n\|Sx_n - x^*\|^2 + (1-\beta_n)\left(\|x_n - x^*\|^2 - \|u_n - x_n\|^2\right.\right. \\
&\quad \left.\left. + 2r_n\|u_n - x_n\|\|Dx_n - Dx^*\|\right)\right\},
\end{aligned}$$

which implies

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le \frac{\gamma_n+\alpha_n\rho\tau}{1+\alpha_n(\upsilon-\rho\tau)}\|x_n - x^*\|^2 \\
&\quad + \frac{2\alpha_n}{1+\alpha_n(\upsilon-\rho\tau)}\langle \rho U(x^*) - \mu F(x^*), x_{n+1}-x^*\rangle
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{(1-\gamma_n-\alpha_n\upsilon)\beta_n}{1+\alpha_n(\upsilon-\rho\tau)}\left\|Sx_n-x^*\right\|^2\\
&+\frac{(1-\gamma_n-\alpha_n\upsilon)(1-\beta_n)}{1+\alpha_n(\upsilon-\rho\tau)}\left\{\left\|x_n-x^*\right\|^2-\|u_n-x_n\|^2\right.\\
&\left.+2r_n\|u_n-x_n\|\left\|Dx_n-Dx^*\right\|\right\}\\
\le{}& \frac{\gamma_n+\alpha_n\rho\tau}{1+\alpha_n(\upsilon-\rho\tau)}\left\|x_n-x^*\right\|^2\\
&+\frac{2\alpha_n}{1+\alpha_n(\upsilon-\rho\tau)}\left\langle\rho U\left(x^*\right)-\mu F\left(x^*\right),x_{n+1}-x^*\right\rangle\\
&+ \frac{(1-\gamma_n-\alpha_n\upsilon)\beta_n}{1+\alpha_n(\upsilon-\rho\tau)}\left\|Sx_n-x^*\right\|^2\\
&+\left\|x_n-x^*\right\|^2+\frac{(1-\gamma_n-\alpha_n\upsilon)(1-\beta_n)}{1+\alpha_n(\upsilon-\rho\tau)}\\
&\times\left\{-\|u_n-x_n\|^2+2r_n\|u_n-x_n\|\left\|Dx_n-Dx^*\right\|\right\}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\frac{(1-\gamma_n-\alpha_n\upsilon)(1-\beta_n)}{1+\alpha_n(\upsilon-\rho\tau)}\|u_n-x_n\|^2\\
&\quad\le \frac{\gamma_n+\alpha_n\rho\tau}{1+\alpha_n(\upsilon-\rho\tau)}\left\|x_n-x^*\right\|^2+\frac{2\alpha_n}{1+\alpha_n(\upsilon-\rho\tau)}\left\langle\rho U\left(x^*\right)-\mu F\left(x^*\right),x_{n+1}-x^*\right\rangle\\
&\qquad+ \frac{(1-\gamma_n-\alpha_n\upsilon)\beta_n}{1+\alpha_n(\upsilon-\rho\tau)}\left\|Sx_n-x^*\right\|^2\\
&\qquad+\frac{2(1-\gamma_n-\alpha_n\upsilon)(1-\beta_n)r_n}{1+\alpha_n(\upsilon-\rho\tau)}\|u_n-x_n\|\left\|Dx_n-Dx^*\right\|+\left\|x_n-x^*\right\|^2-\left\|x_{n+1}-x^*\right\|^2\\
&\quad\le \frac{\gamma_n+\alpha_n\rho\tau}{1+\alpha_n(\upsilon-\rho\tau)}\left\|x_n-x^*\right\|^2+\frac{2\alpha_n}{1+\alpha_n(\upsilon-\rho\tau)}\left\langle\rho U\left(x^*\right)-\mu F\left(x^*\right),x_{n+1}-x^*\right\rangle\\
&\qquad+ \frac{(1-\gamma_n-\alpha_n\upsilon)\beta_n}{1+\alpha_n(\upsilon-\rho\tau)}\left\|Sx_n-x^*\right\|^2\\
&\qquad+\frac{2(1-\gamma_n-\alpha_n\upsilon)(1-\beta_n)r_n}{1+\alpha_n(\upsilon-\rho\tau)}\|u_n-x_n\|\left\|Dx_n-Dx^*\right\|\\
&\qquad+\left(\left\|x_n-x^*\right\|+\left\|x_{n+1}-x^*\right\|\right)\left(\|x_{n+1}-x_n\|\right).
\end{aligned}
$$

Since $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, $\gamma_n\to 0$, $\alpha_n\to 0$, $\beta_n\to 0$, and $\lim_{n\to\infty}\|Dx_n-Dx^*\|=0$, we obtain

$$
\lim_{n\to\infty}\|u_n-x_n\|=0. \tag{3.15}
$$

Consider

$$
\begin{aligned}
\left\|w_n^m-x^*\right\|^2&=\left\|P_C(I-\mu_mB_m)u_n-P_C(I-\mu_mB_m)x^*\right\|^2\\
&\le\left\|\left(u_n-x^*\right)-\mu_m\left(B_mu_n-B_mx^*\right)\right\|^2\\
&=\left\|u_n-x^*\right\|^2+\mu_m^2\left\|B_mu_n-B_mx^*\right\|^2-2\mu_m\left\langle u_n-x^*,B_mu_n-B_mx^*\right\rangle\\
&\le\left\|u_n-x^*\right\|^2+\mu_m^2\left\|B_mu_n-B_mx^*\right\|^2-2\mu_ml_m\left\|B_mu_n-B_mx^*\right\|^2\\
&\le\left\|u_n-x^*\right\|^2-\mu_m(2l_m-\mu_m)\left\|B_mu_n-B_mx^*\right\|^2,\quad 1\le\forall m\le r.
\end{aligned}
$$

It follows that

$$\begin{aligned}\|z_n - x^*\|^2 &= \Big\|\sum_{m=1}^{r}(\eta_n^m w_n^m) - x^*\Big\|^2 \le \sum_{m=1}^{r}\eta_n^m\|w_n^m - x^*\|^2 \\ &\le \|u_n - x^*\|^2 - \sum_{m=1}^{r}\eta_n^m\mu_m(2l_m - \mu_m)\|B_m u_n - B_m x^*\|^2.\end{aligned}$$

Then we have

$$\begin{aligned}\|v_n - x^*\|^2 &= \|\delta_n(u_n - x^*) + (1-\delta_n)(z_n - x^*)\|^2 \\ &\le \delta_n\|u_n - x^*\|^2 + (1-\delta_n)\|z_n - x^*\|^2 \\ &= \|u_n - x^*\|^2 - (1-\delta_n)\sum_{m=1}^{r}\eta_n^m\mu_m(2l_m - \mu_m)\|B_m u_n - B_m x^*\|^2 \\ &\le \|x_n - x^*\|^2 - (1-\delta_n)\sum_{m=1}^{r}\eta_n^m\mu_m(2l_m - \mu_m)\|B_m u_n - B_m x^*\|^2.\end{aligned}$$

From (3.14) and the above inequality, we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 \le{}& \frac{1-\alpha_n(\upsilon - \rho\tau)}{2}\|x_{n+1} - x^*\|^2 + \frac{\gamma_n + \alpha_n\rho\tau}{2}\|x_n - x^*\|^2 \\ &+ \alpha_n\langle\rho U(x^*) - \mu F(x^*), x_{n+1} - x^*\rangle \\ &+ \frac{1-\gamma_n - \alpha_n\upsilon}{2}\big(\beta_n\|Sx_n - x^*\|^2 + (1-\beta_n)\|v_n - x^*\|^2\big) \\ \le{}& \frac{1-\alpha_n(\upsilon - \rho\tau)}{2}\|x_{n+1} - x^*\|^2 + \frac{\gamma_n + \alpha_n\rho\tau}{2}\|x_n - x^*\|^2 \\ &+ \alpha_n\langle\rho U(x^*) - \mu F(x^*), x_{n+1} - x^*\rangle \\ &+ \frac{(1-\gamma_n - \alpha_n\upsilon)}{2}\Bigg\{\beta_n\|Sx_n - x^*\|^2 + (1-\beta_n)\Bigg(\|x_n - x^*\|^2 \\ &- (1-\delta_n)\sum_{m=1}^{r}\eta_n^m\mu_m(2l_m - \mu_m)\|B_m u_n - B_m x^*\|^2\Bigg)\Bigg\},\end{aligned}$$

which implies

$$\begin{aligned}\|x_{n+1} - x^*\|^2 \le{}& \frac{\gamma_n + \alpha_n\rho\tau}{1+\alpha_n(\upsilon - \rho\tau)}\|x_n - x^*\|^2 \\ &+ \frac{2\alpha_n}{1+\alpha_n(\upsilon - \rho\tau)}\langle\rho U(x^*) - \mu F(x^*), x_{n+1} - x^*\rangle \\ &+ \frac{(1-\gamma_n - \alpha_n\upsilon)\beta_n}{1+\alpha_n(\upsilon - \rho\tau)}\|Sx_n - x^*\|^2 + \|x_n - x^*\|^2 \\ &- \frac{(1-\gamma_n - \alpha_n\upsilon)(1-\beta_n)(1-\delta_n)}{1+\alpha_n(\upsilon - \rho\tau)}\sum_{m=1}^{r}\eta_n^m\mu_m(2l_m - \mu_m)\|B_m u_n - B_m x^*\|^2.\end{aligned}$$

Then from the above inequality, we have

$$\begin{aligned}
&\frac{(1-\gamma_n-\alpha_n\upsilon)(1-\beta_n)(1-\delta_n)}{1+\alpha_n(\upsilon-\rho\tau)}\sum_{m=1}^{r}\eta_n^m\mu_m(2l_m-\mu_m)\|B_mu_n-B_mx^*\|^2\\
&\quad\le\frac{\gamma_n+\alpha_n\rho\tau}{1+\alpha_n(\upsilon-\rho\tau)}\|x_n-x^*\|^2+\frac{2\alpha_n}{1+\alpha_n(\upsilon-\rho\tau)}\langle\rho U(x^*)-\mu F(x^*),x_{n+1}-x^*\rangle\\
&\qquad+\beta_n\|Sx_n-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\
&\quad\le\frac{\gamma_n+\alpha_n\rho\tau}{1+\alpha_n(\upsilon-\rho\tau)}\|x_n-x^*\|+\frac{2\alpha_n}{1+\alpha_n(\upsilon-\rho\tau)}\langle\rho U(x^*)-\mu F(x^*),x_{n+1}-x^*\rangle\\
&\qquad+\beta_n\|Sx_n-x^*\|^2+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)(\|x_{n+1}-x_n\|).
\end{aligned}$$

Since $\mu_n\in(0,2l_m)$, $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, $\gamma_n\to0$, $\alpha_n\to0$, and $\beta_n\to0$, we obtain

$$\lim_{n\to\infty}\|B_mu_n-B_mx^*\|=0,\quad 1\le\forall m\le r.$$

On the other hand, we have

$$\begin{aligned}
&\|w_n^m-x^*\|^2\\
&\quad=\|P_C(I-\mu_mB_m)u_n-P_C(I-\mu_mB_m)x^*\|^2\\
&\quad\le\langle(I-\mu_mB_m)u_n-(I-\mu_mB_m)x^*,w_n^m-x^*\rangle\\
&\quad=\frac{1}{2}\{\|(I-\mu_mB_m)u_n-(I-\mu_mB_m)x^*\|^2+\|w_n^m-x^*\|^2\\
&\qquad-\|(I-\mu_mB_m)u_n-(I-\mu_mB_m)x^*-(w_n^m-x^*)\|^2\}\\
&\quad\le\frac{1}{2}\{\|u_n-x^*\|^2+\|w_n^m-x^*\|^2-\|u_n-w_n^m-\mu_m(B_mu_n-B_mx^*)\|^2\}\\
&\quad=\frac{1}{2}\{\|u_n-x^*\|^2+\|w_n^m-x^*\|^2-\|u_n-w_n^m\|^2\\
&\qquad+2\mu_m\langle B_mu_n-B_mx^*,u_n-w_n^m\rangle-\mu_m^2\|B_mu_n-B_mx^*\|^2\},\quad 1\le\forall m\le r.
\end{aligned}$$

It follows that

$$\|w_n^m-x^*\|^2\le\|u_n-x^*\|^2-\|u_n-w_n^m\|^2+Q^m\|B_mu_n-B_mx^*\|,\quad 1\le\forall m\le r,\tag{3.16}$$

where Q^m is an approximate constant such that

$$Q^m=\max\{2\mu_m\|u_n-w_n^m\|:\forall n\ge1\},\quad 1\le\forall m\le r.$$

On the other hand, we have

$$\|z_n-u_n\|^2\le\sum_{m=1}^{r}(\eta_n^m\|w_n^m-u_n\|^2),$$

which combined with (3.16) gives

$$\begin{aligned}\|z_n - x^*\|^2 &\le \sum_{m=1}^{r}\left(\eta_n^m \|w_n^m - x^*\|^2\right) \\ &\le \|u_n - x^*\|^2 - \|z_n - u_n\|^2 + \sum_{m=1}^{r}\left(Q^m \|B_m u_n - B_m x^*\|\right).\end{aligned}$$

Hence we have

$$\begin{aligned}\|v_n - x^*\|^2 &\le \delta_n \|u_n - x^*\|^2 + (1-\delta_n)\|z_n - x^*\|^2 \\ &\le \|u_n - x^*\|^2 - \|z_n - u_n\|^2 + \sum_{m=1}^{r}\left(Q^m \|B_m u_n - B_m x^*\|\right) \\ &\le \|x_n - x^*\|^2 - \|z_n - u_n\|^2 + \sum_{m=1}^{r}\left(Q^m \|B_m u_n - B_m x^*\|\right).\end{aligned}$$

In view of (3.14) and the above inequality, we have

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ &\quad\le \frac{1-\alpha_n(\upsilon - \rho\tau)}{2}\|x_{n+1} - x^*\|^2 + \frac{\gamma_n + \alpha_n\rho\tau}{2}\|x_n - x^*\|^2 \\ &\qquad + \alpha_n\langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^*\rangle \\ &\qquad + \frac{1-\gamma_n - \alpha_n\upsilon}{2}\left(\beta_n\|Sx_n - x^*\|^2 + (1-\beta_n)\|v_n - x^*\|^2\right) \\ &\quad\le \frac{1-\alpha_n(\upsilon - \rho\tau)}{2}\|x_{n+1} - x^*\|^2 + \frac{\gamma_n + \alpha_n\rho\tau}{2}\|x_n - x^*\|^2 \\ &\qquad + \alpha_n\langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^*\rangle + \frac{1-\gamma_n - \alpha_n\upsilon}{2}\Biggl\{\beta_n\|Sx_n - x^*\|^2 \\ &\qquad + (1-\beta_n)\Biggl(\|x_n - x^*\|^2 - \|z_n - u_n\|^2 + \sum_{m=1}^{r}\left(Q^m \|B_m u_n - B_m x^*\|\right)\Biggr)\Biggr\},\end{aligned} \tag{3.17}$$

which implies that

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ &\quad\le \frac{\gamma_n + \alpha_n\rho\tau}{1+\alpha_n(\upsilon - \rho\tau)}\|x_n - x^*\|^2 \\ &\qquad + \frac{2\alpha_n}{1+\alpha_n(\upsilon - \rho\tau)}\langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^*\rangle \\ &\qquad + \frac{(1-\gamma_n - \alpha_n\upsilon)\beta_n}{1+\alpha_n(\upsilon - \rho\tau)}\|Sx_n - x^*\|^2 \\ &\qquad + \frac{(1-\gamma_n - \alpha_n\upsilon)(1-\beta_n)}{1+\alpha_n(\upsilon - \rho\tau)}\Biggl\{\|x_n - x^*\|^2 - \|z_n - u_n\|^2 + \sum_{m=1}^{r}\left(Q^m \|B_m u_n - B_m x^*\|\right)\Biggr\}.\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{(1-\gamma_n-\alpha_n\upsilon)(1-\beta_n)}{1+\alpha_n(\upsilon-\rho\tau)}\|z_n-u_n\|^2 \\
&\quad\le \frac{\gamma_n+\alpha_n\rho\tau}{1+\alpha_n(\upsilon-\rho\tau)}\|x_n-x^*\|^2+\frac{2\alpha_n}{1+\alpha_n(\upsilon-\rho\tau)}\langle\rho U(x^*)-\mu F(x^*),x_{n+1}-x^*\rangle \\
&\qquad+\frac{(1-\gamma_n-\alpha_n\upsilon)\beta_n}{1+\alpha_n(\upsilon-\rho\tau)}\|Sx_n-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2 \\
&\qquad+\sum_{m=1}^{r}\big(Q^m\|B_mu_n-B_mx^*\|\big) \\
&\quad= \frac{\gamma_n+\alpha_n\rho\tau}{1+\alpha_n(\upsilon-\rho\tau)}\|x_n-x^*\|^2+\frac{2\alpha_n}{1+\alpha_n(\upsilon-\rho\tau)}\langle\rho U(x^*)-\mu F(x^*),x_{n+1}-x^*\rangle \\
&\qquad+\frac{(1-\gamma_n-\alpha_n\upsilon)\beta_n}{1+\alpha_n(\upsilon-\rho\tau)}\|Sx_n-x^*\|^2+\big(\|x_n-x^*\|+\|x_{n+1}-x^*\|\big)\big(\|x_{n+1}-x_n\|\big) \\
&\qquad+\sum_{m=1}^{r}\big(Q^m\|B_mu_n-B_mx^*\|\big).
\end{aligned}$$

Since $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, $\gamma_n\to 0$, $\alpha_n\to 0$, $\beta_n\to 0$, and $\lim_{n\to\infty}\|B_mu_n-B_mx^*\|=0$, we get

$$\lim_{n\to\infty}\|z_n-u_n\|=0. \tag{3.18}$$

It follows from (3.15) and (3.18) that

$$\lim_{n\to\infty}\|z_n-x_n\|=0. \tag{3.19}$$

From Algorithm 3.1, we have

$$\|v_n-x_n\|\le\delta_n\|u_n-x_n\|+(1-\delta_n)\|z_n-x_n\|,$$

which implies

$$\lim_{n\to\infty}\|x_n-v_n\|=0, \tag{3.20}$$

$$\begin{aligned}
\|x_n-T(y_n)\| &\le \|x_n-x_{n+1}\|+\|x_{n+1}-T(y_n)\| \\
&= \|x_n-x_{n+1}\|+\|P_C[V_n]-P_C[T(y_n)]\| \\
&\le \|x_n-x_{n+1}\|+\|\alpha_n(\rho U(x_n)-\mu F(T(y_n)))+\gamma_n(x_n-T(y_n))\| \\
&\le \|x_n-x_{n+1}\|+\alpha_n\|\rho U(x_n)-\mu F(T(y_n))\|+\gamma_n\|x_n-T(y_n)\|,
\end{aligned}$$

and therefore

$$\|x_n-T(y_n)\|\le\frac{1}{1-\gamma_n}\|x_n-x_{n+1}\|+\frac{\alpha_n}{1-\gamma_n}\|\rho U(x_n)-\mu F(T(y_n))\|.$$

Since $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, $\alpha_n\to 0$, we obtain

$$\lim_{n\to\infty}\|x_n-T(y_n)\|=0.$$

Since $T(x_n) \in C$, we have

$$\begin{aligned}
\|x_n - T(x_n)\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T(x_n)\| \\
&= \|x_n - x_{n+1}\| + \|P_C[V_n] - P_C[T(x_n)]\| \\
&\le \|x_n - x_{n+1}\| + \|\alpha_n(\rho U(x_n) - \mu F(T(y_n))) \\
&\quad + \gamma_n(x_n - T(y_n)) + T(y_n) - T(x_n)\| \\
&\le \|x_n - x_{n+1}\| + \alpha_n\|\rho U(x_n) - \mu F(T(y_n))\| + \gamma_n\|x_n - T(y_n)\| + \|y_n - x_n\| \\
&\le \|x_n - x_{n+1}\| + \alpha_n\|\rho U(x_n) - \mu F(T(y_n))\| + \gamma_n\|x_n - T(y_n)\| \\
&\quad + \|\beta_n S x_n + (1-\beta_n)v_n - x_n\| \\
&\le \|x_n - x_{n+1}\| + \alpha_n\|\rho U(x_n) - \mu F(T(y_n))\| + \gamma_n\|x_n - T(y_n)\| \\
&\quad + \beta_n\|Sx_n - x_n\| + (1-\beta_n)\|v_n - x_n\|.
\end{aligned}$$

Since $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, $\gamma_n \to 0$, $\alpha_n \to 0$, $\beta_n \to 0$, $\lim_{n\to\infty}\|x_n - T(y_n)\| = 0$, $\|\rho U(x_n) - \mu F(T(y_n))\|$, and $\|Sx_n - x_n\|$ are bounded and $\lim_{n\to\infty}\|x_n - v_n\| = 0$, we obtain

$$\lim_{n\to\infty}\|x_n - T(x_n)\| = 0.$$

Since $\{x_n\}$ is bounded, without loss of generality we can assume that $x_n \rightharpoonup x^* \in C$. It follows from Lemma 2.3 that $x^* \in F(T)$. Therefore $\omega_\omega(x_n) \subset F(T)$. □

Theorem 3.1 *The sequence $\{x_n\}$ generated by Algorithm* 3.1 *converges strongly to z, which is the unique solution of the variational inequality*

$$\langle \rho U(z) - \mu F(z), x - z\rangle \le 0, \quad \forall x \in \mathscr{F} = F(T) \cap VI(C, B_m) \cap GMEP(F, \varphi, D). \tag{3.21}$$

Proof Since $\{x_n\}$ is bounded, $x_n \rightharpoonup w$, and from Lemma 3.2, we have $w \in F(T)$. Next, we show that $w \in GMEP(F,\varphi,D)$. Since $u_n = T_{r_n}(x_n - r_n D x_n)$, we have

$$F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Dx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C.$$

It follows from the monotonicity of F that

$$\varphi(y) - \varphi(u_n) + \langle Dx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge F(y, u_n), \quad \forall y \in C,$$

and

$$\varphi(y) - \varphi(u_{n_k}) + \langle Dx_{n_k}, y - u_{n_k}\rangle + \left\langle y - u_{n_k}, \frac{u_{n_k} - x_{n_k}}{r_{n_k}}\right\rangle \ge F(y, u_{n_k}), \quad \forall y \in C. \tag{3.22}$$

Since $\lim_{n\to\infty}\|u_n - x_n\| = 0$ and $x_n \rightharpoonup w$, it is easy to observe that $u_{n_k} \to w$. For any $0 < t \le 1$ and $y \in C$, let $y_t = ty + (1-t)w$, and we have $y_t \in C$. Then from (3.22), we obtain

$$\begin{aligned}
\langle Dy_t, y_t - u_{n_k}\rangle &\geq \varphi(u_{n_k}) - \varphi(y_t) + \langle Dy_t, y_t - u_{n_k}\rangle \\
&\quad - \langle Dx_{n_k}, y_t - u_{n_k}\rangle - \left\langle y_t - u_{n_k}, \frac{u_{n_k} - x_{n_k}}{r_{n_k}}\right\rangle + F(y_t, u_{n_k}) \\
&= \varphi(u_{n_k}) - \varphi(y_t) + \langle Dy_t - Du_{n_k}, y_t - u_{n_k}\rangle + \langle Du_{n_k} - Dx_{n_k}, y_t - u_{n_k}\rangle \\
&\quad - \left\langle y_t - u_{n_k}, \frac{u_{n_k} - x_{n_k}}{r_{n_k}}\right\rangle + F(y_t, u_{n_k}). \qquad (3.23)
\end{aligned}$$

Since D is Lipschitz continuous and $\lim_{n\to\infty} \|u_n - x_n\| = 0$, we obtain $\lim_{n\to\infty} \|Du_{n_k} - Dx_{n_k}\| = 0$. From the monotonicity of D, the weakly lower semicontinuity of φ, and $u_{n_k} \to w$, it follows from (3.23) that

$$\langle Dy_t, y_t - w\rangle \geq \varphi(w) - \varphi(y_t) + F(y_t, w). \qquad (3.24)$$

Hence, from assumptions (A_1)-(A_4) and (3.24), we have

$$\begin{aligned}
0 &= F(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \leq tF(y_t, y) + (1-t)F(y_t, w) + t\varphi(y) + (1-t)\varphi(w) - \varphi(y_t) \\
&= t\big[F(y_t, y) + \varphi(y) - \varphi(y_t)\big] + (1-t)\big[F(y_t, w) + \varphi(w) - \varphi(y_t)\big] \\
&\leq t\big[F(y_t, y) + \varphi(y) - \varphi(y_t)\big] + (1-t)t\langle Dy_t, y - w\rangle, \qquad (3.25)
\end{aligned}$$

which implies that $F(y_t, y) + \varphi(y) - \varphi(y_t) + (1-t)\langle Dy_t, y - w\rangle \geq 0$. Letting $t \to 0_+$, we have

$$F(w, y) + \varphi(y) - \varphi(w) + \langle Dw, y - w\rangle \geq 0, \quad \forall y \in C,$$

which implies that $w \in GMEP(F, \varphi, D)$. Furthermore, we show that $w \in VI(C, B_m)$. Define a mapping $J : C \to C$ by

$$Jx = \sum_{m=1}^{r} \eta^m P_C(I - \mu_m B_m)x, \quad \forall x \in C,$$

where $\eta^m = \lim_{n\to\infty} \eta_n^m$. From Lemma 2.7, we see that J is nonexpansive such that

$$F(J) = \bigcap_{m=1}^{r} F\big(P_C(I - \mu_m B_m)\big) = \bigcap_{m=1}^{r} VI(C, B_m).$$

Note that

$$\begin{aligned}
\|u_n - Ju_n\| &\leq \|u_n - z_n\| + \|z_n - Ju_n\| \\
&\leq \|u_n - z_n\| + \left\| \sum_{m=1}^{r} \eta_n^m P_C(I - \mu_m B_m)u_n - \sum_{m=1}^{r} \eta^m P_C(I - \mu_m B_m)u_n \right\| \\
&\leq \|u_n - z_n\| + M \cdot \sum_{m=1}^{r} \big|\eta_n^m - \eta^m\big|.
\end{aligned}$$

In view of restriction (f), we find from (3.18) that

$$\lim_{n\to\infty} \|u_n - Ju_n\| = 0.$$

It follows from Lemma 2.3 that $w \in F(J) = \bigcap_{m=1}^{r} VI(C,B_m)$. Thus, we have $w \in \mathscr{F} = F(T) \cap VI(C,B_m) \cap GMEP(F,\varphi,D)$.

Observe that the constants satisfy $0 < \rho\tau < \nu$ and

$$\begin{aligned} \kappa \geq \eta \quad &\Leftrightarrow \quad \kappa^2 \geq \eta^2 \\ &\Leftrightarrow \quad 1 - 2\mu\eta + \mu^2\kappa^2 \geq 1 - 2\mu\eta + \mu^2\eta^2 \\ &\Leftrightarrow \quad \sqrt{1 - \mu\big(2\eta - \mu\kappa^2\big)} \geq 1 - \mu\eta \\ &\Leftrightarrow \quad \mu\eta \geq 1 - \sqrt{1 - \mu\big(2\eta - \mu\kappa^2\big)} \\ &\Leftrightarrow \quad \mu\eta \geq \nu, \end{aligned}$$

therefore, from Lemma 2.4, the operator $\mu F - \rho U$ is $\mu\eta - \rho\tau$-strongly monotone, and we get the uniqueness of the solution of the variation inequality (3.21) and denote it by $z \in \mathscr{F} = F(T) \cap VI(C,B_m) \cap GMEP(F,\varphi,D)$.

Next, we claim that $\limsup_{n\to\infty}\langle \rho U(z) - \mu F(z), x_n - z\rangle \leq 0$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\begin{aligned} \limsup_{n\to\infty}\big\langle \rho U(z) - \mu F(z), x_n - z\big\rangle &= \limsup_{n\to\infty}\big\langle \rho U(z) - \mu F(z), x_{n_k} - z\big\rangle \\ &= \big\langle \rho U(z) - \mu F(z), w - z\big\rangle \leq 0. \end{aligned}$$

Next, we show that $x_n \to z$. We have

$$\begin{aligned} &\|x_{n+1} - z\|^2 \\ &\quad = \big\langle P_C[V_n] - z, x_{n+1} - z\big\rangle \\ &\quad = \big\langle P_C[V_n] - V_n, P_C[V_n] - z\big\rangle + \langle V_n - z, x_{n+1} - z\rangle \\ &\quad \leq \bigg\langle \alpha_n\big(\rho U(x_n) - \mu F(z)\big) + \gamma_n(x_n - z) \\ &\qquad + (1-\gamma_n)\bigg[\bigg(I - \frac{\alpha_n\mu}{1-\gamma_n}F\bigg)T(y_n) - \bigg(I - \frac{\alpha_n\mu}{1-\gamma_n}F\bigg)T(z)\bigg], x_{n+1} - z\bigg\rangle \\ &\quad = \big\langle \alpha_n\rho\big(U(x_n) - U(z)\big), x_{n+1} - z\big\rangle + \alpha_n\big\langle \rho U(z) - \mu F(z), x_{n+1} - z\big\rangle \\ &\qquad + \gamma_n\langle x_n - z, x_{n+1} - z\rangle \\ &\qquad + (1-\gamma_n)\bigg\langle\bigg(I - \frac{\alpha_n\mu}{1-\gamma_n}F\bigg)T(y_n) - \bigg(I - \frac{\alpha_n\mu}{1-\gamma_n}F\bigg)T(z), x_{n+1} - z\bigg\rangle \\ &\quad \leq (\gamma_n + \alpha_n\rho\tau)\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\big\langle \rho U(z) - \mu F(z), x_{n+1} - z\big\rangle \\ &\qquad + (1 - \gamma_n - \alpha_n\nu)\|y_n - z\|\|x_{n+1} - z\| \\ &\quad \leq (\gamma_n + \alpha_n\rho\tau)\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\big\langle \rho U(z) - \mu F(z), x_{n+1} - z\big\rangle \\ &\qquad + (1 - \gamma_n - \alpha_n\nu)\big\{\beta_n\|Sx_n - Sz\| + \beta_n\|Sz - z\| + (1-\beta_n)\|v_n - z\|\big\} \\ &\qquad \times \|x_{n+1} - z\| \\ &\quad \leq (\gamma_n + \alpha_n\rho\tau)\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\big\langle \rho U(z) - \mu F(z), x_{n+1} - z\big\rangle \end{aligned}$$

$$
\begin{aligned}
&\quad + (1-\gamma_n-\alpha_n\nu)\big\{\beta_n\|x_n-z\| + \beta_n\|Sz-z\| + (1-\beta_n)\|x_n-z\|\big\}\|x_{n+1}-z\| \\
&= \big(1-\alpha_n(\nu-\rho\tau)\big)\|x_n-z\|\|x_{n+1}-z\| + \alpha_n\big\langle\rho U(z)-\mu F(z), x_{n+1}-z\big\rangle \\
&\quad + (1-\gamma_n-\alpha_n\nu)\beta_n\|Sz-z\|\|x_{n+1}-z\| \\
&\le \frac{1-\alpha_n(\nu-\rho\tau)}{2}\big(\|x_n-z\|^2 + \|x_{n+1}-z\|^2\big) + \alpha_n\big\langle\rho U(z)-\mu F(z), x_{n+1}-z\big\rangle \\
&\quad + (1-\gamma_n-\alpha_n\nu)\beta_n\|Sz-z\|\|x_{n+1}-z\|,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\|x_{n+1}-z\|^2 \\
&\quad\le \frac{1-\alpha_n(\nu-\rho\tau)}{1+\alpha_n(\nu-\rho\tau)}\|x_n-z\|^2 + \frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\big\langle\rho U(z)-\mu F(z), x_{n+1}-z\big\rangle \\
&\qquad + \frac{2(1-\gamma_n-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)}\|Sz-z\|\|x_{n+1}-z\| \\
&\quad\le \big(1-\alpha_n(\nu-\rho\tau)\big)\|x_n-z\|^2 + \frac{2\alpha_n(\nu-\rho\tau)}{1+\alpha_n(\nu-\rho\tau)} \\
&\qquad \times \left\{\frac{1}{\nu-\rho\tau}\big\langle\rho U(z)-\mu F(z), x_{n+1}-z\big\rangle + \frac{(1-\gamma_n-\alpha_n\nu)\beta_n}{\alpha_n(\nu-\rho\tau)}\|Sz-z\|\|x_{n+1}-z\|\right\}.
\end{aligned}
$$

It follows from Lemma 2.6 that $x_n \to z$. This completes the proof. □

4 Applications

In this section, we obtain the following results by using a special case of the proposed method for example.

Putting $D = \varphi = 0$, $B_m = 0$ for each m, and $\delta_n = 0$ in Algorithm 3.1, we obtain the following result, which can be viewed as an extension and improvement of the method of Bnouhachem *et al.* [22] for finding the approximate element of the common set of solutions of equilibrium problem and a hierarchical fixed point problem.

Theorem 4.1 *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $F : C \times C \to R$ be a bifunction that satisfy condition* (A_1)-(A_4), *and let $S, T : C \to C$ be nonexpansive mappings such that $F(T) \cap EP(F) \neq \emptyset$. Let $F : C \to C$ be a κ-Lipschitzian mapping and η-strongly monotone, and let $U : C \to C$ be a τ-Lipschitzian mapping. For an arbitrarily given $x_0 \in C$, let the iterative sequences $\{u_n\}$, $\{x_n\}$, and $\{y_n\}$ be generated by*

$$
\begin{cases}
F(u_n, y) + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \ge 0, \quad \forall y \in C; \\
y_n = \beta_n S x_n + (1-\beta_n)u_n; \\
x_{n+1} = P_C[\alpha_n\rho U(x_n) + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n\mu F)T(y_n)], \quad \forall n \ge 0.
\end{cases}
$$

Suppose that the parameters satisfy $0 < \mu < \frac{2\eta}{\kappa^2}$, $0 \le \rho\tau < \upsilon$, where $\upsilon = 1-\sqrt{1-\mu(2\eta-\mu\kappa^2)}$. Also $\{\gamma_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{r_n\}$ are sequences in $(0,1)$ satisfying the following conditions:

(a) $\lim_{n\to\infty}\gamma_n = 0$, $\gamma_n + \alpha_n < 1$;
(b) $\lim_{n\to\infty}\alpha_n = 0$, *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;
(c) $\lim_{n\to\infty}(\beta_n/\alpha_n) = 0$;
(d) $\sum_{n=1}^{\infty}|\alpha_n-\alpha_{n-1}| < \infty$, $\sum_{n=1}^{\infty}|\gamma_n-\gamma_{n-1}| < \infty$, *and* $\sum_{n=1}^{\infty}|\beta_n-\beta_{n-1}| < \infty$;
(e) $\liminf_{n\to\infty} r_n > 0$, *and* $\sum_{n=1}^{\infty}|r_n-r_{n-1}| < \infty$.

Then the sequence $\{x_n\}$ *converges strongly to* z, *which is the unique solution of the variational inequality*:

$$\langle \rho U(z) - \mu F(z), x - z \rangle \leq 0, \quad \forall x \in F(T) \cap EP(F).$$

Putting $\delta_n = 0$, $m = 1$ in Algorithm 3.1, we obtain the following result which can be viewed as an extension and improvement of the method of Bnouhachem and Chen [23] for finding the approximate element of the common set of solutions of variational inequalities, a generalized mixed equilibrium problem, and a hierarchical fixed point problem.

Theorem 4.2 *Let* C *be a nonempty, closed, and convex subset of a real Hilbert space* H. *Let* $D, A : C \to H$ *be* θ, α*-inverse strongly monotone mapping, respectively. Let* $F : C \times C \to R$ *satisfy* (A_1)-(A_4), *and let* $\varphi : C \to R$ *be a proper lower semicontinuous and convex function. Let* $S, T : C \to C$ *be nonexpansive mappings such that* $F(T) \cap VI(C, A) \cap GMEP(F, \varphi, D) \neq \emptyset$. *Let* $F : C \to C$ *be a* κ*-Lipschitzian mapping and be* η*-strongly monotone, and let* $U : C \to C$ *be a* τ*-Lipschitzian mapping. For an arbitrarily given* $x_0 \in C$, *let the iterative sequences* $\{u_n\}$, $\{x_n\}$, $\{y_n\}$, *and* $\{z_n\}$ *be generated by*

$$\begin{cases} F(u_n, y) + \langle Dx_n, y - u_n \rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C; \\ z_n = P_C[u_n - \lambda_n A u_n]; \\ y_n = \beta_n S x_n + (1 - \beta_n) z_n; \\ x_{n+1} = P_C[\alpha_n \rho U(x_n) + \gamma_n x_n + ((1 - \gamma_n) I - \alpha_n \mu F) T(y_n)], \quad \forall n \geq 0, \end{cases}$$

where $\lambda_n \in (0, 2\alpha)$, $\{r_n\} \subset (2, 2\theta)$. *Suppose that the parameters satisfy* $0 < \mu < \frac{2\eta}{\kappa^2}$, $0 \leq \rho\tau < \upsilon$, *where* $\upsilon = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. *Also* $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$ *are sequences in* $(0, 1)$ *satisfying the following conditions*:

(a) $\lim_{n\to\infty} \gamma_n = 0$, $\gamma_n + \alpha_n < 1$;
(b) $\lim_{n\to\infty} \alpha_n = 0$, *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(c) $\lim_{n\to\infty} (\beta_n/\alpha_n) = 0$;
(d) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$, $\sum_{n=1}^{\infty} |\gamma_n - \gamma_{n-1}| < \infty$, *and* $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$;
(e) $\liminf_{n\to\infty} r_n > 0$, *and* $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$;
(f) $\liminf_{n\to\infty} \lambda_n < \limsup_{n\to\infty} \lambda_n < 2\alpha$ *and* $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n-1}| < \infty$.

Then the sequence $\{x_n\}$ *converges strongly to* z, *which is the unique solution of the variational inequality*:

$$\langle \rho U(z) - \mu F(z), x - z \rangle \leq 0, \quad \forall x \in VI(C, A) \cap GMEP(F, \varphi, D) \cap F(T).$$

Putting $\rho = \mu = 1$, $\beta_n = \delta_n = 0$, $\varphi = 0$, $U = f$ a contraction mapping, and $F = A$ a strongly positive linear bounded operator, we obtain the following theorem.

Theorem 4.3 *Let* C *be a nonempty, closed, and convex subset of a real Hilbert space* H. *Let* $B_m : C \to H$ *be* l_m*-inverse strongly monotone mapping for each* $1 \leq m \leq r$, *where* r *is some positive integer. Let* $D : C \to H$ *be a* α*-inverse strongly monotone mapping. Let* $F : C \times C \to R$ *satisfy* (A_1)-(A_4). *Let* $T : C \to C$ *be nonexpansive mappings such that* $\mathscr{F} = F(T) \cap VI(C, A) \cap EP \neq \emptyset$. *Let* A *be a strongly positive linear bounded operator with coefficient* $\overline{\gamma}$

and let $f : H \to H$ *be a contraction with contraction constant* h $(0 < h < 1)$ *and* $0 < \gamma < (\overline{\gamma}/h)$. *Let* $\{x_n\}$, $\{y_n\}$, $\{\rho_n\}$ *be sequences generated by* $x_1 \in H$ *and*

$$\begin{cases} F(y_n,\eta) + \langle Dy_n, \eta - y_n\rangle + \frac{1}{r_n}\langle \eta - y_n, y_n - x_n\rangle \geq 0, \quad \forall \eta \in C; \\ \rho_n = \sum_{m=1}^{r} \eta_n^m P_C(I - \mu_m B_m)y_n; \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)T(\rho_n); \end{cases}$$

where $\mu_m \in (0, 2l_m)$, $\{\alpha\}, \{\beta\} \subset [0,1]$, *and* $\{r_n\} \subset [0,\infty]$. *If the following conditions are satisfied*:

(a) $\lim_{n\to\infty} \alpha_n = 0$, *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(b) $\lim_{n\to\infty} \eta_n^m = \eta^m \in (0,1)$;
(c) $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;
(d) $\liminf_{n\to\infty} r_n > 0$, $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(e) $\sum_{m=1}^{r} \eta_n^m = 1$, $\forall n \geq 1$;

then $\{x_n\}$ *converges strongly to* $q \in \mathscr{F}$, *where* $q = P_{\mathscr{F}}(\gamma f + (I - A))q$.

Remark If $T = W_n$ in Theorem 4.3, where W_n is the W-mapping of C into itself which is generated by a family of nonexpansive mappings $S_n, S_{n-1}, \ldots, S_1$, and a sequence of positive numbers in $[0,1]$ $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$, we can easily get Theorem 10 in Zhou *et al.* [13]. It is worth to mention two points as follows:

(1) Since we all know that W_n mapping is nonexpansive, if $T = W_n$ in Theorem 4.3, then we can easily get Theorem 10 in Zhou *et al.* [13].
(2) A family of infinite k_n-strict pseudocontractive mappings in Theorem 10 in Zhou *et al.* [13] did not work, so we should omit them. Theorem 10 in Zhou *et al.* [13] should be corrected as follows:

Theorem 4.4 *Let* C *be a nonempty, closed, and convex subset of a real Hilbert space* H. *Let* $B_m : C \to H$ *be* l_m*-inverse strongly monotone mapping for each* $1 \leq m \leq r$, *where* r *is some positive integer. Let* $D : C \to H$ *be a* α*-inverse strongly monotone mapping. Let* $F : C \times C \to R$ *satisfy* (A_1)-(A_4). *Let* $\{\lambda_n\}_{n=1}^{\infty}$ *be a sequence of positive numbers in* $[0,b]$ *for some* $b \in (0,1)$, *and let* $\{S_n\}_{n=1}^{\infty} : C \to C$ *be a family of infinitely nonexpansive mappings such that* $\mathscr{F} = F(T) \cap VI(C,A) \cap EP \neq \emptyset$. *Let* A *be a strongly positive linear bounded operator with coefficient* $\overline{\gamma}$ *and let* $f : H \to H$ *be a contraction with contraction constant* h $(0 < h < 1)$ *and* $0 < \gamma < (\overline{\gamma}/h)$. *Let* $\{x_n\}$, $\{y_n\}$, $\{\rho_n\}$ *be sequences generated by* $x_1 \in H$ *and*

$$\begin{cases} F(y_n,\eta) + \langle Dy_n, \eta - y_n\rangle + \frac{1}{r_n}\langle \eta - y_n, y_n - x_n\rangle \geq 0, \quad \forall \eta \in C; \\ \rho_n = \sum_{m=1}^{r} \eta_n^m P_C(I - \mu_m B_m)y_n; \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)W_n\rho_n; \end{cases}$$

where $\mu_m \in (0, 2l_m)$, $\{\alpha\}, \{\beta\} \subset [0,1]$, *and* $\{r_n\} \subset [0,\infty]$. *If the following conditions are satisfied*:

(a) $\lim_{n\to\infty} \alpha_n = 0$, *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(b) $\lim_{n\to\infty} \eta_n^m = \eta^m \in (0,1)$;
(c) $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;
(d) $\liminf_{n\to\infty} r_n > 0$, $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(e) $\sum_{m=1}^{r} \eta_n^m = 1$, $\forall n \geq 1$;

then $\{x_n\}$ *converges strongly to* $q \in \mathscr{F}$, *where* $q = P_{\mathscr{F}}(\gamma f + (I - A))q$.

Competing interests
The authors declare that they have no competing interest.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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