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Convergence of three-step iterations for nearly asymptotically nonexpansive mappings in CAT(*k*) spaces

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Abstract

In this paper, we study strong and Δ-convergence of a newly defined three-step iteration process for nearly asymptotically nonexpansive mappings in the setting of CAT(*k*) spaces. Our results generalize, unify and extend many known results from the existing literature.

MSC: 54H25; 54E40

Keywords: nearly asymptotically nonexpansive mapping; three-step iteration scheme; fixed point; strong convergence; Δ-convergence; CAT(*k*) space

1 Introduction

For a real number k, a CAT(k) space is a geodesic metric space whose geodesic triangle is thinner than the corresponding comparison triangle in a model space with curvature k. The precise definition is given below. The term 'CAT(k)' was coined by Gromov ([1], p.119). The initials are in honor of Cartan, Alexandrov and Toponogov, each of whom considered similar conditions in varying degrees of generality.

Fixed point theory in CAT(k) spaces was first studied by Kirk (see [2, 3]). His works were followed by a series of new works by many authors, mainly focusing on CAT(0) spaces (see, *e.g.*, [4–11]). It is worth mentioning that the results in CAT(0) spaces can be applied to any CAT(k) space with $k \leq 0$ since any CAT(k) space is a CAT(m) space for every $m \geq k$ (see [12], 'Metric spaces of non-positive curvature').

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [13] in 1972, as an important generalization of the class of nonexpansive mappings, and they proved that if C is a nonempty closed and bounded subset of a uniformly convex Banach space, then every asymptotically nonexpansive self-mapping of C has a fixed point.

There are many papers dealing with the approximation of fixed points of asymptotically nonexpansive mappings and asymptotically quasi-nonexpansive mappings in uniformly convex Banach spaces, using modified Mann, Ishikawa and three-step iteration processes (see, *e.g.*, [14–22]; see also [23–27]).

The concept of Δ-convergence in a general metric space was introduced by Lim [28]. In 2008, Kirk and Panyanak [29] used the notion of Δ-convergence introduced by Lim [28] to prove in the CAT(0) space and analogous of some Banach space results which involve weak convergence. Further, Dhompongsa and Panyanak [30] obtained Δ-convergence theorems for the Picard, Mann and Ishikawa iterations in a CAT(0) space. Since then, the existence problem and the Δ-convergence problem of iterative sequences to a fixed point for nonexpansive mapping, asymptotically nonexpansive mapping, nearly asymptotically nonexpansive mapping, asymptotically nonexpansive mapping in the intermediate sense, asymptotically quasi-nonexpansive mapping in the intermediate sense, total asymptotically nonexpansive mapping and asymptotically quasi-nonexpansive mapping through Picard, Mann [31], Ishikawa [32], modified Agarwal *et al.* [33] have been rapidly developed in the framework of CAT(0) spaces and many papers have appeared in this direction (see, *e.g.*, [5, 30, 34–39]).

The aim of this article is to establish Δ-convergence and strong convergence of a modified three-step iteration process which contains a modified S-iteration process for a class of mappings which is wider than that of asymptotically nonexpansive mappings in CAT(k) spaces. Our results extend and improve the corresponding results of Abbas *et al.* [34], Dhompongsa and Panyanak [30], Khan and Abbas [35] and many other results of this direction.

2 Preliminaries

Let $F(T) = \{x \in K : Tx = x\}$ denote the set of fixed points of the mapping T. We begin with the following definitions.

Definition 2.1 Let (X, d) be a metric space and K be its nonempty subset. Then the mapping $T: K \to K$ is said to be:

(1) *nonexpansive* if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in K$;

(2) *asymptotically nonexpansive* if there exists a sequence $\{u_n\} \subset [0, \infty)$, with $\lim_{n\to\infty} u_n = 0$, such that $d(T^n x, T^n y) \leq (1 + u_n)d(x, y)$ for all $x, y \in K$ and $n \geq 1$;

(3) *asymptotically quasi-nonexpansive* if $F(T) \neq \emptyset$, and there exists a sequence $\{u_n\} \subset [0, \infty)$, with $\lim_{n\to\infty} u_n = 0$, such that $d(T^n x, p) \leq (1 + u_n)d(x, p)$ for all $x \in K$, $p \in F(T)$ and $n \geq 1$;

(4) *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that $d(T^n x, T^n y) \leq Ld(x, y)$ for all $x, y \in K$ and $n \geq 1$;

(5) *semi-compact* if for a sequence $\{x_n\}$ in K, with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in K$ as $k \to \infty$;

(6) a sequence $\{x_n\}$ in K is called *approximate fixed point sequence* for T (AFPS, in short) if $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

The class of nearly Lipschitzian mappings is an important generalization of the class of Lipschitzian mappings and was introduced by Sahu [40].

Definition 2.2 Let K be a nonempty subset of a metric space (X, d) and fix a sequence $\{a_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} a_n = 0$. A mapping $T: K \to K$ is said to be *nearly Lipschitzian* with respect to $\{a_n\}$ if, for all $n \geq 1$, there exists a constant $k_n \geq 0$ such that

$$d\big(T^n x, T^n y\big) \leq k_n\big[d(x, y) + a_n\big] \quad \text{for all } x, y \in K.$$

The infimum of the constants k_n, for which the above inequality holds, is denoted by $\eta(T^n)$ and is called nearly Lipschitz constant of T^n.

A nearly Lipschitzian mapping T with sequence $\{a_n, \eta(T^n)\}$ is said to be:

(i) *nearly nonexpansive* if $\eta(T^n) = 1$ for all $n \geq 1$;

(ii) *nearly asymptotically nonexpansive* if $\eta(T^n) \geq 1$ for all $n \geq 1$ and $\lim_{n\to\infty} \eta(T^n) = 1$;

(iii) *nearly uniformly k-Lipschitzian* if $\eta(T^n) \leq k$ for all $n \geq 1$.

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry, and $d(x, y) = l$. The image α of c is called a geodesic (or metric) *segment* joining x and y. We say that X is (i) a *geodesic space* if any two points of X are joined by a geodesic, and (ii) *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$, which we will denote by $[x, y]$, called the segment joining x to y. This means that $z \in [x, y]$ if and only if there exists $\alpha \in [0, 1]$ such that $d(x, z) = (1 - \alpha)d(x, y)$ and $d(y, z) = \alpha d(x, y)$.

In this case, we write $z = \alpha x \oplus (1 - \alpha)y$. The space (X, d) is said to be a geodesic space (D-geodesic space) if every two points of X (every two points of distance smaller than D) are joined by a geodesic, and X is said to be uniquely geodesic (D-uniquely geodesic) if there is exactly one geodesic joining x and y for each $x, y \in X$ (for $x, y \in X$ with $d(x, y) < D$). A subset K of X is said to be convex if K includes every geodesic segment joining any two of its points. The set K is said to be bounded if $\operatorname{diam}(K) := \sup\{d(x, y) : x, y \in K\} < \infty$.

The model spaces M_k^2 are defined as follows.

Given a real number k, we denote by M_k^2 the following metric spaces:

(i) if $k = 0$, then M_k^2 is an Euclidean space $\mathbb{E}^n$;

(ii) if $k > 0$, then M_k^2 is obtained from the sphere $\mathbb{S}^n$ by multiplying the distance function by $\frac{1}{\sqrt{k}}$;

(iii) if $k < 0$, then M_k^2 is obtained from a hyperbolic space $\mathbb{H}^n$ by multiplying the distance function by $\frac{1}{\sqrt{-k}}$.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in M_k^2 such that $d(x_1, x_2) = d_{M_k^2}(\overline{x_1}, \overline{x_2})$, $d(x_2, x_3) = d_{M_k^2}(\overline{x_2}, \overline{x_3})$ and $d(x_3, x_1) = d_{M_k^2}(\overline{x_3}, \overline{x_1})$. If $k \leq 0$, then such a comparison triangle always exists in M_k^2. If $k > 0$, then such a triangle exists whenever $d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_1) < 2D_k$, where $D_k = \pi/\sqrt{k}$. A point $\bar{p} \in [\bar{x}, \bar{y}]$ is called a comparison point for $p \in [x, y]$ if $d(x, p) = d_{M_k^2}(\bar{x}, \bar{p})$.

A geodesic triangle $\triangle(x_1, x_2, x_3)$ in X is said to satisfy the CAT(k) inequality if for any $p, q \in \triangle(x_1, x_2, x_3)$ and for their comparison points $\bar{p}, \bar{q} \in \overline{\triangle}(\bar{x_1}, \bar{x_2}, \bar{x_3})$, one has $d(p, q) = d_{M_k^2}(\overline{p}, \overline{q})$.

Definition 2.3 If $k \leq 0$, then X is called a CAT(k) *space* if and only if X is a geodesic space such that all of its geodesic triangles satisfy the CAT(k) inequality.

If $k > 0$, then X is called a CAT(k) *space* if and only if X is D_k-geodesic and any geodesic triangle $\triangle(x_1, x_2, x_3)$ in X with $d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_1) < 2D_k$ satisfies the CAT(k) inequality.

Notice that in a CAT(0) space (X, d) if $x, y, z \in X$, then the CAT(0) inequality implies

$$\text{(CN)} \quad d^2\left(x, \frac{y \oplus z}{2}\right) \leq \frac{1}{2}d^2(x, y) + \frac{1}{2}d^2(x, z) - \frac{1}{4}d^2(y, z).$$

This is the (CN) inequality of Bruhat and Tits [41]. This inequality is extended by Dhompongsa and Panyanak in [30] as

$$(\mathrm{CN}^*) \quad d^2\big(z, \alpha x \oplus (1-\alpha)y\big) \le \alpha d^2(z,x) + (1-\alpha)d^2(z,y) - \alpha(1-\alpha)d^2(x,y)$$

for all $\alpha \in [0,1]$ and $x, y, z \in X$. In fact, if X is a geodesic space, then the following statements are equivalent:

(i) X is a CAT(0) space;

(ii) X satisfies the (CN) inequality;

(iii) X satisfies the (CN*) inequality.

Let $R \in (0,2]$. Recall that a geodesic space (X,d) is said to be *R-convex* for R (see [42]) if for any three points $x, y, z \in X$, we have

$$\begin{aligned} d^2\big(z, \alpha x \oplus (1-\alpha)y\big) \le{}& \alpha d^2(z,x) + (1-\alpha)d^2(z,y) \\ &- \frac{R}{2}\alpha(1-\alpha)d^2(x,y). \end{aligned} \tag{2.1}$$

It follows from (CN*) that a geodesic space (X,d) is a CAT(0) space if and only if (X,d) is R-convex for R=2.

In the sequel we need the following lemma.

Lemma 2.1 ([12], p.176) *Let $k > 0$ and (X,d) be a complete* CAT(k) *space with* $\operatorname{diam}(X) = \frac{\pi/2-\varepsilon}{\sqrt{k}}$ *for some $\varepsilon \in (0,\pi/2)$. Then*

$$d\big((1-\alpha)x \oplus \alpha y, z\big) \le (1-\alpha)d(x,z) + \alpha d(y,z)$$

for all $x, y, z \in X$ and $\alpha \in [0,1]$.

We now recall some elementary facts about CAT(k) spaces. Most of them are proved in the framework of CAT(1) spaces. For completeness, we state the results in a CAT(k) space with $k > 0$.

Let $\{x_n\}$ be a bounded sequence in a CAT(k) space (X,d). For $x \in X$, set

$$r\big(x, \{x_n\}\big) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r\big(\{x_n\}\big) = \inf\big\{r\big(x, \{x_n\}\big) : x \in X\big\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A\big(\{x_n\}\big) = \big\{x \in X : r\big(\{x_n\}\big) = r\big(x, \{x_n\}\big)\big\}.$$

It is known from Proposition 4.1 of [8] that a CAT(k) space with $\operatorname{diam}(X) = \frac{\pi}{2\sqrt{k}}$, $A(\{x_n\})$ consists of exactly one point. We now give the concept of Δ-convergence and collect some of its basic properties.

Definition 2.4 ([28, 29]) A sequence $\{x_n\}$ in X is said to Δ-*converge* to $x \in X$ if x is the unique asymptotic center of $\{x_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta\text{-}\lim_n x_n = x$ and call x the Δ-limit of $\{x_n\}$.

Lemma 2.2 *Let $k > 0$ and (X,d) be a complete* CAT(k) *space with* $\operatorname{diam}(X) = \frac{\pi/2-\varepsilon}{\sqrt{k}}$ *for some $\varepsilon \in (0,\pi/2)$. Then the following statements hold*:

(i) ([8], Corollary 4.4) *Every sequence in X has a Δ-convergent subsequence.*

(ii) ([8], Proposition 4.5) *If $\{x_n\} \subseteq X$ and $\Delta\text{-}\lim_{n\to\infty} x_n = x$, then* $x \in \bigcap_{k=1}^{\infty} \overline{\operatorname{conv}}\{x_k, x_{k+1}, \dots\}$,

where $\overline{\operatorname{conv}}(A) = \bigcap\{B : B \supseteq A$ *and B is closed and convex*$\}$.

By the uniqueness of asymptotic center, we can obtain the following lemma in [30].

Lemma 2.3 ([30], Lemma 2.8) *Let $k > 0$ and (X,d) be a complete* CAT(k) *space with* $\operatorname{diam}(X) = \frac{\pi/2-\varepsilon}{\sqrt{k}}$ *for some $\varepsilon \in (0,\pi/2)$. If $\{x_n\}$ is a bounded sequence in X with $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n,u)\}$ converges, then $x = u$.*

Lemma 2.4 (see [20]) *Let $\{p_n\}_{n=1}^{\infty}$, $\{q_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ be sequences of nonnegative numbers satisfying the inequality*

$$p_{n+1} \le (1+q_n)p_n + r_n, \quad \forall n \ge 1.$$

If $\sum_{n=1}^{\infty} q_n < \infty$ and $\sum_{n=1}^{\infty} r_n < \infty$, then $\lim_{n\to\infty} p_n$ exists.

Proposition 2.1 ([37], Proposition 3.12) *Let $\{x_n\}$ be a bounded sequence in a* CAT(0) *space X, and let C be a closed convex subset of X which contains $\{x_n\}$. Then*

(i) *$\Delta\text{-}\lim_{n\to\infty} x_n = x$ implies that $\{x_n\} \rightharpoonup x$,*

(ii) *the converse is true if $\{x_n\}$ is regular.*

Algorithm 1 The sequence $\{x_n\}$ defined by $x_1 \in K$ and

$$\begin{aligned} y_n &= (1-\beta_n)x_n \oplus \beta_n T^n x_n, \\ x_{n+1} &= (1-\alpha_n)T^n x_n \oplus \alpha_n T^n y_n, \quad n \ge 1, \end{aligned} \tag{2.2}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are appropriate sequences in $(0,1)$, is called a modified S-iterative sequence (see [33]).

If $T^n = T$ for all $n \ge 1$, then Algorithm 1 reduces to the following.

Algorithm 2 The sequence $\{x_n\}$ defined by $x_1 \in K$ and

$$\begin{aligned} y_n &= (1-\beta_n)x_n \oplus \beta_n T x_n, \\ x_{n+1} &= (1-\alpha_n)T x_n \oplus \alpha_n T y_n, \quad n \ge 1, \end{aligned} \tag{2.3}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are appropriate sequences in $(0,1)$, is called an S-iterative sequence (see [33]).

Algorithm 3 The sequence $\{x_n\}$ defined by $x_1 \in K$ and

$$\begin{aligned} y_n &= (1-\beta_n)x_n \oplus \beta_n T^n x_n, \\ x_{n+1} &= (1-\alpha_n)x_n \oplus \alpha_n T^n y_n, \quad n \geq 1, \end{aligned} \tag{2.4}$$

where $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ are appropriate sequences in $[0,1]$, is called an Ishikawa iterative sequence (see [32]).

If $\beta_n = 0$ for all $n \geq 1$, then Algorithm 3 reduces to the following.

Algorithm 4 The sequence $\{x_n\}$ defined by $x_1 \in K$ and

$$x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n T^n x_n, \quad n \geq 1, \tag{2.5}$$

where $\{\alpha_n\}_{n=1}^\infty$ is a sequence in $(0,1)$, is called a Mann iterative sequence (see [31]).

Motivated and inspired by [33] and some others, we modify iteration scheme (2.2) as follows.

Algorithm 5 The sequence $\{x_n\}$ defined by $x_1 \in K$ and

$$\begin{aligned} z_n &= (1-\gamma_n)x_n \oplus \gamma_n T^n x_n, \\ y_n &= (1-\beta_n)x_n \oplus \beta_n T^n z_n, \\ x_{n+1} &= (1-\alpha_n)T^n x_n \oplus \alpha_n T^n y_n, \quad n \geq 1, \end{aligned} \tag{2.6}$$

where $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$, $\{\gamma_n\}_{n=1}^\infty$ are appropriate sequences in $(0,1)$, is called a modified three-step iterative sequence. Iteration scheme (2.6) is independent of modified Noor iteration, modified Ishikawa iteration and modified Mann iteration schemes.

If $\gamma_n = 0$ for all $n \geq 1$, then Algorithm 5 reduces to Algorithm 1.

Iteration procedures in fixed point theory are led by considerations in summability theory. For example, if a given sequence converges, then we do not look for the convergence of the sequence of its arithmetic means. Similarly, if the sequence of Picard iterates of any mapping T converges, then we do not look for the convergence of other iteration procedures.

The three-step iterative approximation problems were studied extensively by Noor [43, 44], Glowinski and Le Tallec [45], and Haubruge *et al.* [46]. The three-step iterations lead to highly parallelized algorithms under certain conditions. They are also a natural generalization of the splitting methods for solving partial differential equations. It has been shown [45] that a three-step iterative scheme gives better numerical results than the two-step and one-step approximate iterations. Thus we conclude that a three-step scheme plays an important and significant role in solving various problems which arise in pure and applied sciences. These facts motivated us to study a class of three-step iterative schemes in the setting of CAT(k) spaces with $k > 0$.

In this paper, we study a newly defined modified three-step iteration scheme to approximate a fixed point for nearly asymptotically nonexpansive mappings in the setting of a

CAT(k) space with $k > 0$ and also establish Δ-convergence and strong convergence results for the above mentioned iteration scheme and mappings.

3 Main results

Now, we shall introduce existence theorems.

Theorem 3.1 *Let $k > 0$ and (X, d) be a complete* CAT(k) *space with* diam$(X) = \frac{\pi/2-\varepsilon}{\sqrt{k}}$ *for some $\varepsilon \in (0, \pi/2)$. Let K be a nonempty closed convex subset of X, and let $T : K \to K$ be a continuous nearly asymptotically nonexpansive mapping. Then T has a fixed point.*

Proof Fix $x \in K$. We can consider the sequence $\{T^n x\}_{n=1}^{\infty}$ as a bounded sequence in K. Let ϕ be a function defined by

$$\phi : K \to [0, \infty), \quad \phi(u) = \limsup_{n\to\infty} d\bigl(T^n x, u\bigr) \quad \text{for all } u \in K.$$

Then there exists $z \in K$ such that $\phi(z) = \inf\{\Phi(u) : u \in K\}$. Since T is a nearly asymptotically nonexpansive mapping, for each $n, m \in \mathbb{N}$, we have

$$d\bigl(T^{n+m}x, T^m z\bigr) \le \eta\bigl(T^m\bigr)\bigl(d\bigl(T^n x, z\bigr) + a_m\bigr).$$

On taking limit as $n \to \infty$, we obtain

$$\phi\bigl(T^m z\bigr) \le \eta\bigl(T^m\bigr)\phi(z) + \eta\bigl(T^m\bigr)a_m \tag{3.1}$$

for any $m \in \mathbb{N}$. This implies that

$$\lim_{m\to\infty} \phi\bigl(T^m z\bigr) \le \phi(z). \tag{3.2}$$

In view of inequality (2.1), we obtain

$$\begin{aligned} d\left(T^n x, \frac{T^m z \oplus T^h z}{2}\right)^2 &\le \frac{1}{2} d\bigl(T^n x, T^m z\bigr)^2 + \frac{1}{2} d\bigl(T^n x, T^h z\bigr)^2 \\ &\quad - \frac{R}{8} d\bigl(T^m z, T^h z\bigr)^2, \end{aligned}$$

which, on taking limit as $n \to \infty$, gives

$$\begin{aligned} \phi(z)^2 &\le \Phi\left(\frac{T^m z \oplus T^h z}{2}\right)^2 \\ &\le \frac{1}{2}\phi\bigl(T^m z\bigr)^2 + \frac{1}{2}\phi\bigl(T^h z\bigr)^2 - \frac{R}{8} d\bigl(T^m z, T^h z\bigr)^2. \end{aligned} \tag{3.3}$$

The above inequality yields

$$\frac{R}{8} d\bigl(T^m z, T^h z\bigr)^2 \le \frac{1}{2}\phi\bigl(T^m z\bigr)^2 + \frac{1}{2}\phi\bigl(T^h z\bigr)^2 - \phi(z)^2. \tag{3.4}$$

By (3.2) and (3.4), we have $\limsup_{m,h\to\infty} d(T^m z, T^h z) \le 0$. Therefore, $\{T^n z\}_{n=1}^{\infty}$ is a Cauchy sequence in K and hence converges to some point $v \in K$. Since T is continuous,

$$Tv = T\Bigl(\lim_{n\to\infty} T^n z\Bigr) = \lim_{n\to\infty} T^{n+1} z = v.$$

This shows that T has a fixed point in K. This completes the proof. □

From Theorem 3.1 we shall now derive a result for a CAT(0) space as follows.

Corollary 3.1 *Let (X,d) be a complete* CAT(0) *space and K be a nonempty bounded, closed convex subset of X. If $T: K \to K$ is a continuous nearly asymptotically nonexpansive mapping, then T has a fixed point.*

Proof It is well known that every convex subset of a CAT(0) space, equipped with the induced metric, is a CAT(k) space (see [12]). Then (K,d) is a CAT(0) space and hence it is a CAT(k) space for all $k > 0$. Also note that K is R-convex for $R = 2$. Since K is bounded, we can chose $\varepsilon \in (0, \pi/2)$ and $k > 0$ so that $\operatorname{diam}(K) \le \frac{\pi/2-\varepsilon}{\sqrt{k}}$. The conclusion follows from Theorem 3.1. This completes the proof. □

Theorem 3.2 *Let $k > 0$ and (X,d) be a complete* CAT(k) *space with* $\operatorname{diam}(X) = \frac{\pi/2-\varepsilon}{\sqrt{k}}$ *for some $\varepsilon \in (0, \pi/2)$. Let K be a nonempty closed convex subset of X, and let $T: K \to K$ be a uniformly continuous nearly asymptotically nonexpansive mapping. If $\{x_n\}$ is an AFPS for T such that $\Delta\text{-}\lim_{n\to\infty} x_n = z$, then $z \in K$ and $z = Tz$.*

Proof By Lemma 2.2, we get that $z \in K$. As in Theorem 3.1, we define

$$\phi(u) = \limsup_{n\to\infty} d(x_n, u)$$

for each $u \in K$. Since $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, by induction we can show that

$$\lim_{n\to\infty} d\bigl(x_n, T^m x_n\bigr) = 0$$

for some $m \in \mathbb{N}$. This implies that

$$\phi(u) = \limsup_{n\to\infty} d\bigl(T^m x_n, u\bigr) \quad \text{for each } u \in K \text{ and } m \in \mathbb{N}. \tag{3.5}$$

Taking $u = T^m z$ in (3.5), we have

$$\begin{aligned} \phi\bigl(T^m z\bigr) &= \limsup_{n\to\infty} d\bigl(T^m x_n, T^m z\bigr) \\ &\le \limsup_{n\to\infty} \bigl[\eta\bigl(T^m\bigr)\bigl(d(x_n, z) + a_m\bigr)\bigr]. \end{aligned} \tag{3.6}$$

Hence

$$\limsup_{m\to\infty} \phi\bigl(T^m z\bigr) \le \phi(z). \tag{3.7}$$

In view of inequality (2.1), we have

$$d\left(x_n, \frac{z \oplus T^m z}{2}\right)^2 \leq \frac{1}{2}d(x_n, z)^2 + \frac{1}{2}d\left(x_n, T^m z\right)^2 - \frac{R}{8}d\left(z, T^m z\right)^2,$$

where $R = (\pi - 2\varepsilon)\tan(\varepsilon)$. Since $\Delta\text{-}\lim_{n\to\infty} x_n = z$, letting $n \to \infty$, we get

$$\begin{aligned}\phi(z)^2 &\leq \Phi\left(\frac{z \oplus T^m z}{2}\right)^2 \\ &\leq \frac{1}{2}\phi(z)^2 + \frac{1}{2}\phi\left(T^m z\right)^2 - \frac{R}{8}d\left(z, T^m z\right)^2.\end{aligned} \tag{3.8}$$

This yields

$$d\left(z, T^m z\right)^2 \leq \frac{4}{R}\left[\phi\left(T^m z\right)^2 - \phi(z)^2\right]. \tag{3.9}$$

By (3.7) and (3.9), we have $\lim_{m\to\infty} d(z, T^m z) = 0$. Since T is continuous,

$$Tz = T\left(\lim_{m\to\infty} T^m z\right) = \lim_{n\to\infty} T^{m+1} z = z.$$

This shows that T has a fixed point in K. This completes the proof. □

From Theorem 3.2 we can derive the following result as follows.

Corollary 3.2 *Let (X, d) be a complete* CAT(0) *space, K be a nonempty bounded, closed convex subset of X and $T: K \to K$ be a uniformly continuous nearly asymptotically nonexpansive mapping. If $\{x_n\}$ is an AFPS for T such that $\Delta\text{-}\lim_{n\to\infty} x_n = z$, then $z \in K$ and $z = Tz$.*

Now, we prove the following lemma using iteration scheme (2.6) needed in the sequel.

Lemma 3.1 *Let $k > 0$ and (X, d) be a complete* CAT(k) *space with* $\operatorname{diam}(X) = \frac{\pi/2-\varepsilon}{\sqrt{k}}$ *for some $\varepsilon \in (0, \pi/2)$. Let K be a nonempty closed and convex subset of X, and let $T: K \to K$ be a uniformly continuous nearly asymptotically nonexpansive mapping with sequence $\{(a_n, \eta(T^n))\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty}(\eta(T^n) - 1) < \infty$. Let $\{x_n\}$ be a sequence in K defined by* (2.6). *Then $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F(T)$.*

Proof It follows from Theorem 3.1 that $F(T) \neq \emptyset$. Let $p \in F(T)$ and since T is nearly asymptotically nonexpansive, by (2.6) and Lemma 2.1, we have

$$\begin{aligned}d(z_n, p) &= d\left((1-\gamma_n)x_n \oplus \gamma_n T^n x_n, p\right) \\ &\leq (1-\gamma_n)d(x_n, p) + \gamma_n d\left(T^n x_n, p\right) \\ &\leq (1-\gamma_n)d(x_n, p) + \gamma_n\left[\eta\left(T^n\right)\left(d(x_n, p) + a_n\right)\right] \\ &\leq \eta\left(T^n\right)\left[(1-\gamma_n)d(x_n, p) + \gamma_n d(x_n, p)\right] + \gamma_n \eta\left(T^n\right)a_n \\ &\leq \eta\left(T^n\right)d(x_n, p) + \eta\left(T^n\right)a_n.\end{aligned} \tag{3.10}$$

Again using (2.6), (3.10) and Lemma 2.1, we have

$$\begin{aligned} d(y_n,p) &= d\big((1-\beta_n)x_n \oplus \beta_n T^n z_n, p\big) \\ &\le (1-\beta_n)d(x_n,p) + \beta_n d\big(T^n z_n, p\big) \\ &\le (1-\beta_n)d(x_n,p) + \beta_n\big[\eta\big(T^n\big)\big(d(z_n,p)+a_n\big)\big] \\ &\le (1-\beta_n)d(x_n,p) + \beta_n\eta\big(T^n\big)d(z_n,p) + \eta\big(T^n\big)a_n \\ &\le (1-\beta_n)d(x_n,p) + \beta_n\eta\big(T^n\big)\big[\eta\big(T^n\big)\big(d(x_n,p)+\eta\big(T^n\big)a_n\big)\big] + \eta\big(T^n\big)a_n \\ &\le \eta\big(T^n\big)^2\big[(1-\beta_n)d(x_n,p) + \beta_n d(x_n,p)\big] + \big(\eta\big(T^n\big)+\eta\big(T^n\big)^2\big)a_n \\ &\le \eta\big(T^n\big)^2 d(x_n,p) + \big(\eta\big(T^n\big)+\eta\big(T^n\big)^2\big)a_n. \end{aligned} \tag{3.11}$$

Finally, using (2.6), (3.11) and Lemma 2.1, we get

$$\begin{aligned} d(x_{n+1},p) &= d\big((1-\alpha_n)T^n x_n \oplus \alpha_n T^n y_n, p\big) \\ &\le (1-\alpha_n)d\big(T^n x_n, p\big) + \alpha_n d\big(T^n y_n, p\big) \\ &\le (1-\alpha_n)\big[\eta\big(T^n\big)\big(d(x_n,p)+a_n\big)\big] + \alpha_n\big[\eta\big(T^n\big)\big(d(y_n,p)+a_n\big)\big] \\ &= (1-\alpha_n)\eta\big(T^n\big)d(x_n,p) + \alpha_n\eta\big(T^n\big)d(y_n,p) + \eta\big(T^n\big)a_n \\ &\le (1-\alpha_n)\eta\big(T^n\big)d(x_n,p) + \alpha_n\eta\big(T^n\big) \\ &\quad \times \big[\eta\big(T^n\big)^2 d(x_n,p) + \big(\eta\big(T^n\big)+\eta\big(T^n\big)^2\big)a_n\big] + \eta\big(T^n\big)a_n \\ &\le \eta\big(T^n\big)^3\big[(1-\alpha_n)d(x_n,p) + \alpha_n d(x_n,p)\big] \\ &\quad + \eta\big(T^n\big)\big(\eta\big(T^n\big)+\eta\big(T^n\big)^2\big)a_n + \eta\big(T^n\big)a_n \\ &= \eta\big(T^n\big)^3 d(x_n,p) + \big(\eta\big(T^n\big)+\eta\big(T^n\big)^2+\eta\big(T^n\big)^3\big)a_n \\ &= (1+w_n)d(x_n,p) + v_n, \end{aligned} \tag{3.12}$$

where $w_n = (\eta(T^n)^3 - 1) = (\eta(T^n)^2 + \eta(T^n) + 1)(\eta(T^n) - 1)$ and $v_n = (\eta(T^n) + \eta(T^n)^2 + \eta(T^n)^3)a_n$. Since $\sum_{n=1}^{\infty}(\eta(T^n) - 1) < \infty$ and $\sum_{n=1}^{\infty} a_n < \infty$, it follows that $\sum_{n=1}^{\infty} w_n < \infty$ and $\sum_{n=1}^{\infty} v_n < \infty$. Hence, by Lemma 2.4, we get that $\lim_{n\to\infty} d(x_n,p)$ exists. This completes the proof. □

Lemma 3.2 *Let $k > 0$ and (X,d) be a complete* CAT(k) *space with* $\operatorname{diam}(X) = \frac{\pi/2-\varepsilon}{\sqrt{k}}$ *for some $\varepsilon \in (0,\pi/2)$. Let K be a nonempty closed convex subset of X, and let $T\colon K \to K$ be a uniformly continuous nearly asymptotically nonexpansive mapping with sequence $\{(a_n,\eta(T^n))\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty}(\eta(T^n) - 1) < \infty$. Let $\{x_n\}$ be a sequence in K defined by* (2.6). *Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0,1)$ such that* $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$, $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$ *and* $\liminf_{n\to\infty}\gamma_n(1-\gamma_n) > 0$. *Then* $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Proof It follows from Theorem 3.1 that $F(T) \neq \emptyset$. Let $p \in F(T)$. From Lemma 3.1, we obtain that $\lim_{n\to\infty} d(x_n,p)$ exists for each $p \in F(T)$. We claim that $\lim_{n\to\infty} d(Tx_n,x_n) = 0$.

Since $\{x_n\}$ is bounded, there exists $R > 0$ such that $\{x_n\}, \{y_n\}, \{z_n\} \subset B'_R(p)$ for all $n \geq 1$ with $R' < D_k/2$. In view of (2.1), we have

$$\begin{aligned} d(z_n,p)^2 &= d\bigl((1-\gamma_n)x_n \oplus \gamma_n T^n x_n, p\bigr)^2 \\ &\leq \gamma_n d\bigl(T^n x_n, p\bigr)^2 + (1-\gamma_n) d(x_n,p)^2 - \frac{R}{2}\gamma_n(1-\gamma_n) d\bigl(T^n x_n, x_n\bigr) \\ &\leq \gamma_n\bigl[\eta\bigl(T^n\bigr)\bigl(d(x_n,p)+a_n\bigr)\bigr]^2 + (1-\gamma_n) d(x_n,p)^2 - \frac{R}{2}\gamma_n(1-\gamma_n) d\bigl(T^n x_n, x_n\bigr) \\ &\leq \eta\bigl(T^n\bigr)^2 d(x_n,p)^2 + P a_n - \frac{R}{2}\gamma_n(1-\gamma_n) d\bigl(T^n x_n, x_n\bigr) \end{aligned} \tag{3.13}$$

for some $P > 0$. This implies that

$$d(z_n,p)^2 \leq \eta\bigl(T^n\bigr)^2 d(x_n,p)^2 + P a_n. \tag{3.14}$$

Again from (2.1) and using (3.14), we have

$$\begin{aligned} d(y_n,p)^2 &= d^2\bigl((1-\beta_n)x_n \oplus \beta_n T^n z_n, p\bigr)^2 \\ &\leq \beta_n d\bigl(T^n z_n, p\bigr)^2 + (1-\beta_n) d^2(x_n,p)^2 \\ &\quad - \frac{R}{2}\beta_n(1-\beta_n) d\bigl(T^n z_n, x_n\bigr)^2 \\ &\leq \beta_n\bigl[\eta\bigl(T^n\bigr)\bigl(d(z_n,p)+a_n\bigr)\bigr]^2 + (1-\beta_n) d(x_n,p)^2 \\ &\quad - \frac{R}{2}\beta_n(1-\beta_n) d\bigl(T^n z_n, x_n\bigr)^2 \\ &\leq \eta\bigl(T^n\bigr)^2 \beta_n d(z_n,p)^2 + Q a_n + (1-\beta_n) d(x_n,p)^2 \\ &\quad - \frac{R}{2}\beta_n(1-\beta_n) d\bigl(T^n z_n, x_n\bigr)^2 \\ &\leq \eta\bigl(T^n\bigr)^2 \beta_n\bigl[\eta\bigl(T^n\bigr)^2 d^2(x_n,p) + P a_n\bigr] + Q a_n \\ &\quad + (1-\beta_n) d(x_n,p)^2 - \beta_n(1-\beta_n) d\bigl(T^n z_n, x_n\bigr)^2 \\ &\leq \eta\bigl(T^n\bigr)^4 d(x_n,p)^2 + (L+Q) a_n \\ &\quad - \frac{R}{2}\beta_n(1-\beta_n) d\bigl(T^n z_n, x_n\bigr)^2 \end{aligned} \tag{3.15}$$

for some $L, Q > 0$.

This implies that

$$d(y_n,p)^2 \leq \eta\bigl(T^n\bigr)^4 d(x_n,p)^2 + (L+Q) a_n. \tag{3.16}$$

Finally, from (2.1) and using (3.16), we have

$$\begin{aligned} d(x_{n+1},p)^2 &= d\bigl((1-\alpha_n)T^n x_n \oplus \alpha_n T^n y_n, p\bigr)^2 \\ &\leq \alpha_n d\bigl(T^n y_n, p\bigr)^2 + (1-\alpha_n) d\bigl(T^n x_n, p\bigr)^2 \\ &\quad - \frac{R}{2}\alpha_n(1-\alpha_n) d\bigl(T^n x_n, T^n y_n\bigr)^2 \end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n\big[\eta\big(T^n\big)\big(d(y_n,p)+a_n\big)\big]^2+(1-\alpha_n)\big[\eta\big(T^n\big)d(x_n,p)+a_n\big]^2\\
&\quad -\frac{R}{2}\alpha_n(1-\alpha_n)d\big(T^nx_n,T^ny_n\big)^2\\
&\leq \alpha_n\eta\big(T^n\big)^2d(y_n,p)^2+Ma_n+(1-\alpha_n)\eta\big(T^n\big)^2d(x_n,p)^2\\
&\quad +Na_n-\frac{R}{2}\alpha_n(1-\alpha_n)d\big(T^nx_n,T^ny_n\big)^2\\
&\leq \alpha_n\eta\big(T^n\big)^2\big[\eta\big(T^n\big)^4d(x_n,p)^2+(L+Q)a_n\big]\\
&\quad +(M+N)a_n+(1-\alpha_n)\eta\big(T^n\big)^2d(x_n,p)^2\\
&\quad -\frac{R}{2}\alpha_n(1-\alpha_n)d\big(T^nx_n,T^ny_n\big)^2\\
&\leq \eta\big(T^n\big)^6d(x_n,p)^2+(L+Q+M+N)a_n\\
&\quad -\frac{R}{2}\alpha_n(1-\alpha_n)d\big(T^nx_n,T^ny_n\big)^2\\
&= \big[1+\big(\eta\big(T^n\big)^6-1\big)\big]d(x_n,p)^2+(L+Q+M+N)a_n\\
&\quad -\frac{R}{2}\alpha_n(1-\alpha_n)d\big(T^nx_n,T^ny_n\big)^2\\
&= \big[1+\big(\eta\big(T^n\big)-1\big)\rho\big]d(x_n,p)^2+(L+Q+M+N)a_n\\
&\quad -\frac{R}{2}\alpha_n(1-\alpha_n)d\big(T^nx_n,T^ny_n\big)^2
\end{aligned}\tag{3.17}$$

for some $M, N, \rho > 0$.

This implies that

$$\begin{aligned}
\alpha_n(1-\alpha_n)d\big(T^nx_n,T^ny_n\big)^2 &\leq d(x_n,p)^2-d(x_{n+1},p)^2+\big(\eta\big(T^n\big)-1\big)\rho d(x_n,p)^2\\
&\quad +(L+Q+M+N)a_n.
\end{aligned}$$

Since $\sum_{n=1}^{\infty} a_n < \infty$, $\sum_{n=1}^{\infty}(\eta(T^n)-1) < \infty$ and $d(x_n,p) < R'$, we have

$$\frac{R}{2}\alpha_n(1-\alpha_n)d\big(T^nx_n,T^ny_n\big)^2 < \infty.$$

Hence by the fact that $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$, we have

$$\lim_{n\to\infty} d\big(T^nx_n,T^ny_n\big) = 0. \tag{3.18}$$

Now, consider (3.15), we have

$$\begin{aligned}
d(y_n,p)^2 &\leq \big[1+\big(\eta\big(T^n\big)^4-1\big)\big]d(x_n,p)^2+(L+Q)a_n\\
&\quad -\frac{R}{2}\beta_n(1-\beta_n)d\big(T^nz_n,x_n\big)^2\\
&\leq \big[1+\big(\eta\big(T^n\big)-1\big)\mu\big]d(x_n,p)^2+(L+Q)a_n\\
&\quad -\frac{R}{2}\beta_n(1-\beta_n)d\big(T^nz_n,x_n\big)^2
\end{aligned}\tag{3.19}$$

for some $\mu > 0$.

Equation (3.19) yields

$$\frac{R}{2}\beta_n(1-\beta_n)d\big(T^n z_n, x_n\big)^2 \le d(x_n,p)^2 - d(y_n,p)^2 + \big(\eta\big(T^n\big)-1\big)\mu d(x_n,p)^2 + (L+Q)a_n.$$

Since $\sum_{n=1}^{\infty} a_n < \infty$, $\sum_{n=1}^{\infty}(\eta(T^n)-1) < \infty$, $d(x_n,p) < R'$ and $d(y_n,p) < R'$, we have

$$\frac{R}{2}\beta_n(1-\beta_n)d\big(T^n z_n, x_n\big)^2 < \infty.$$

Thus by the fact that $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$, we have

$$\lim_{n\to\infty} d\big(T^n z_n, x_n\big) = 0. \tag{3.20}$$

Next, consider (3.13), we have

$$\begin{aligned} d(z_n,p)^2 &\le \eta\big(T^n\big)^2 d(x_n,p)^2 + Pa_n - \frac{R}{2}\gamma_n(1-\gamma_n)d\big(T^n x_n, x_n\big) \\ &\le \big[1+\big(\eta\big(T^n\big)-1\big)\nu\big]d(x_n,p)^2 + Pa_n - \frac{R}{2}\gamma_n(1-\gamma_n)d\big(T^n x_n, x_n\big) \end{aligned} \tag{3.21}$$

for some $\nu > 0$.

Equation (3.21) yields

$$\frac{R}{2}\gamma_n(1-\gamma_n)d\big(T^n x_n, x_n\big)^2 \le d(x_n,p)^2 - d(z_n,p)^2 + \big(\eta\big(T^n\big)-1\big)\nu d(x_n,p)^2 + Pa_n.$$

Since $\sum_{n=1}^{\infty} a_n < \infty$, $\sum_{n=1}^{\infty}(\eta(T^n)-1) < \infty$, $d(x_n,p) < R'$ and $d(z_n,p) < R'$, we have

$$\frac{R}{2}\gamma_n(1-\gamma_n)d\big(T^n x_n, x_n\big)^2 < \infty.$$

Hence by the fact that $\liminf_{n\to\infty}\gamma_n(1-\gamma_n) > 0$, we have

$$\lim_{n\to\infty} d\big(T^n x_n, x_n\big) = 0. \tag{3.22}$$

Now, we have

$$\begin{aligned} d\big(T^n y_n, x_n\big) &\le d\big(T^n y_n, T^n x_n\big) + d\big(T^n x_n, x_n\big) \\ &\to 0 \quad \text{as } n\to\infty. \end{aligned} \tag{3.23}$$

Again, note that

$$d(x_n,y_n) \le \beta_n d\big(x_n, T^n z_n\big) \to 0 \quad \text{as } n\to\infty. \tag{3.24}$$

By the definitions of x_{n+1} and y_n, we have

$$\begin{aligned} d(x_n,x_{n+1}) &\le d\big(x_n, T^n y_n\big) \\ &\le d\big(x_n, T^n x_n\big) + d\big(T^n x_n, T^n y_n\big) \end{aligned}$$

$$\begin{aligned}
&\le d\big(x_n, T^n x_n\big) + \eta\big(T^n\big)\big(d(x_n, y_n) + a_n\big) \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned} \tag{3.25}$$

By (3.22), (3.24) and the uniform continuity of T, we have

$$\begin{aligned}
d(x_n, Tx_n) &\le d(x_n, x_{n+1}) + d\big(x_{n+1}, T^{n+1}x_{n+1}\big) \\
&\quad + d\big(T^{n+1}x_{n+1}, T^{n+1}x_n\big) + d\big(T^{n+1}x_n, Tx_n\big) \\
&\le d(x_n, x_{n+1}) + d\big(x_{n+1}, T^{n+1}x_{n+1}\big) \\
&\quad + \eta\big(T^{n+1}\big)d(x_{n+1}, x_n) + a_{n+1} + d\big(T^{n+1}x_n, Tx_n\big) \\
&= \big(1 + \eta\big(T^{n+1}\big)\big)d(x_n, x_{n+1}) + d\big(x_{n+1}, T^{n+1}x_{n+1}\big) \\
&\quad + d\big(T^{n+1}x_n, Tx_n\big) + a_{n+1} \to 0 \quad \text{as } n \to \infty.
\end{aligned} \tag{3.26}$$

This completes the proof. □

Now, we are in a position to prove the Δ-convergence and strong convergence theorems.

Theorem 3.3 *Let $k > 0$ and (X,d) be a complete* $\mathrm{CAT}(k)$ *space, with* $\operatorname{diam}(X) = \frac{\pi/2-\varepsilon}{\sqrt{k}}$, *for some $\varepsilon \in (0,\pi/2)$. Let K be a nonempty closed convex subset of X, and let $T\colon K \to K$ be a uniformly continuous nearly asymptotically nonexpansive mapping with sequence $\{(a_n, \eta(T^n))\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty}(\eta(T^n) - 1) < \infty$. Let $\{x_n\}$ be a sequence in K defined by* (2.6). *Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in* $(0,1)$ *such that* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ *and* $\liminf_{n\to\infty} \gamma_n(1-\gamma_n) > 0$. *Then $\{x_n\}$ Δ-converges to a fixed point of T.*

Proof Let $\omega_w(x_n) := \bigcup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We can complete the proof by showing that $\omega_w(x_n) \subseteq F(T)$ and $\omega_w(x_n)$ consists of exactly one point. Let $u \in \omega_w(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 2.2, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_n v_n = v \in K$. Hence $v \in F(T)$ by Lemma 3.1 and Lemma 3.2. Since $\lim_{n\to\infty} d(x_n, v)$ exists, so by Lemma 2.3, $v = u$, *i.e.*, $\omega_w(x_n) \subseteq F(T)$.

To show that $\{x_n\}$ Δ-converges to a fixed point of T, it is sufficient to show that $\omega_w(x_n)$ consists of exactly one point.

Let $\{w_n\}$ be a subsequence of $\{x_n\}$ with $A(\{w_n\}) = \{w\}$ and let $A(\{x_n\}) = \{x\}$. Since $w \in \omega_w(x_n) \subseteq F(T)$ and by Lemma 3.1, $\lim_{n\to\infty} d(x_n, w)$ exists. Again by Lemma 3.1, we have $x = w \in F(T)$. Thus $\omega_w(x_n) = \{x\}$. This shows that $\{x_n\}$ Δ-converges to a fixed point of T. This completes the proof. □

Theorem 3.4 *Let $k > 0$ and (X,d) be a complete* $\mathrm{CAT}(k)$ *space with* $\operatorname{diam}(X) = \frac{\pi/2-\varepsilon}{\sqrt{k}}$ *for some $\varepsilon \in (0,\pi/2)$. Let K be a nonempty closed convex subset of X, and let $T\colon K \to K$ be a uniformly continuous nearly asymptotically nonexpansive mapping with sequence $\{(a_n, \eta(T^n))\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty}(\eta(T^n) - 1) < \infty$. Let $\{x_n\}$ be a sequence in K defined by* (2.6). *Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in* $(0,1)$ *such that* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ *and* $\liminf_{n\to\infty} \gamma_n(1-\gamma_n) > 0$. *Suppose that T^m is semi-compact for some $m \in \mathbb{N}$. Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.*

Proof By Lemma 3.2, $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Since T is uniformly continuous, we have

$$d\bigl(x_n, T^m x_n\bigr) \le d(x_n, Tx_n) + d\bigl(Tx_n, T^2 x_n\bigr) + \cdots + d\bigl(T^{m-1}x_n, T^m x_n\bigr) \to 0,$$

as $n \to \infty$. That is, $\{x_n\}$ is an AFPS for T^m. By the semi-compactness of T^m, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $p \in K$ such that $\lim_{j\to\infty} x_{n_j} = p$. Again, by the uniform continuity of T, we have

$$d(Tp,p) \le d(Tp, Tx_{n_j}) + d(Tx_{n_j}, x_{n_j}) + d(x_{n_j}, p) \to 0 \quad \text{as } j \to \infty.$$

That is, $p \in F(T)$. By Lemma 3.1, $d(x_n, p)$ exists, thus p is the strong limit of the sequence $\{x_n\}$ itself. This shows that the sequence $\{x_n\}$ converges strongly to a fixed point of T. This completes the proof. □

Remark 3.1 Since T is completely continuous, the image of T^m, for some $m \in \mathbb{N}$, is semi-compact, $\{x_n\}$ is a bounded sequence and $d(x_n, T^m x_n) \to 0$ as $n \to \infty$. Thus T^m, for some $m \in \mathbb{N}$, is semi-compact, that is, the continuous image of a semi-compact space is semi-compact.

Example 3.1 ([47]) Let $X = K = [0,1]$, with the usual metric, and

$$T\colon K \to K, \quad T(x) = \begin{cases} \frac{x}{2} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

Then T is not continuous. However, T is semi-compact. In fact, if $\{x_n\}$ is a bounded sequence in K such that $|x_n - Tx_n| \to 0$ as $n \to \infty$, then by Balzano-Weierstrass theorem, it follows that $\{x_n\}$ has a convergent subsequence.

The following example shows that there is a semi-compact mapping that is not compact.

Example 3.2 ([47]) Let $X = \ell_2$ and $K = \{e_1, e_2, \ldots, e_n, \ldots\}$ be the usual orthonormal basis for ℓ_2. Define

$$T\colon K \to K, \quad T(e_i) = e_{i+1}, \quad i \in \mathbb{N}.$$

Then T is continuous (in fact, an isometry) but not compact. However, T is semi-compact. Indeed, if $\{e_i\}_{i\in\mathbb{N}}$ is a bounded sequence in K such that $e_i - Te_i$ converges, $\{e_i\}_{i\in\mathbb{N}}$ must be finite.

From Theorem 3.4 we can derive the following result as a corollary.

Corollary 3.3 *Let (X,d) be a complete* CAT(0) *space, K be a nonempty bounded, closed convex subset of X and $T\colon K \to K$ be a uniformly continuous nearly asymptotically nonexpansive mapping with sequence $\{(a_n, \eta(T^n))\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty}(\eta(T^n) - 1) < \infty$. Let $\{x_n\}$ be a sequence in K defined by* (2.6). *Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in* $(0,1)$ *such that* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ *and* $\liminf_{n\to\infty} \gamma_n(1-\gamma_n) > 0$. *Suppose that T^m is semi-compact for some $m \in \mathbb{N}$. Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.*

Example 3.3 ([40]) Let $E = \mathbb{R}$, $C = [0,1]$ and T be a mapping defined by

$$T: C \to C, \quad T(x) = \begin{cases} \frac{1}{2} & \text{if } x \in [0, \frac{1}{2}], \\ 0 & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

Here $F(T) = \{\frac{1}{2}\}$. Clearly, T is a discontinuous and non-Lipschitzian mapping. However, it is a nearly nonexpansive mapping and hence a nearly asymptotically nonexpansive mapping with sequence $\{a_n, \eta(T^n)\} = \{\frac{1}{2^n}, 1\}$. Indeed, for a sequence $\{a_n\}$ with $a_1 = \frac{1}{2}$ and $a_n \to 0$, we have

$$d(Tx, Ty) \leq d(x,y) + a_1 \quad \text{for all } x, y \in C$$

and

$$d\bigl(T^n x, T^n y\bigr) \leq d(x,y) + a_n \quad \text{for all } x, y \in C \text{ and } n \geq 2,$$

since

$$T^n x = \frac{1}{2} \quad \text{for all } x \in [0,1] \text{ and } n \geq 2.$$

Example 3.4 Let $X = K = [0,1]$ with the usual metric d, $\{x_n\} = \{\frac{1}{n}\}$, $\{u_{n_k}\} = \{\frac{1}{kn}\}$, for all $n, k \in \mathbb{N}$ are sequences in K. Then $A(\{x_n\}) = \{0\}$ and $A(\{u_{n_k}\}) = \{0\}$. This shows that $\{x_n\}$ Δ-converges to 0, that is, $\Delta\text{-}\lim_{n\to\infty} x_n = 0$. The sequence $\{x_n\}$ also converges strongly to 0, that is, $|x_n - 0| \to 0$ as $n \to \infty$. Also it is weakly convergent to 0, that is, $x_n \rightharpoonup 0$ as $n \to \infty$, by Proposition 2.1. Thus, we conclude that

$$\text{strong convergence} \quad \Rightarrow \quad \Delta\text{-convergence} \quad \Rightarrow \quad \text{weak convergence},$$

but the converse is not true in general.

The following example shows that, if the sequence $\{x_n\}$ is weakly convergent, then it is not Δ-convergent.

Example 3.5 ([37]) Let $X = \mathbb{R}$, d be the usual metric on X, $K = [-1,1]$, $\{x_n\} = \{1, -1, 1, -1, \ldots\}$, $\{u_n\} = \{-1, -1, -1, \ldots\}$ and $\{v_n\} = \{1, 1, 1, \ldots\}$. Then $A(\{x_n\}) = A_K(\{x_n\}) = \{0\}$, $A(\{u_n\}) = \{-1\}$ and $A(\{v_n\}) = \{1\}$. This shows that $\{x_n\} \rightharpoonup 0$ but it does not have a Δ-limit.

4 Conclusions

1. We proved strong and Δ convergence theorems of a modified three-step iteration process which contains a modified S-iteration process in the framework of CAT(k) spaces.
2. Theorem 3.1 extends Theorem 3.3 of Dhompongsa and Panyanak [30] to the case of a more general class of nonexpansive mappings which are not necessarily Lipschitzian, a modified three-step iteration scheme and from a CAT(0) space to a CAT(k) space considered in this paper.

3. Theorem 3.1 also extends Theorem 3.5 of Niwongsa and Panyanak [48] to the case of a more general class of asymptotically nonexpansive mappings which are not necessarily Lipschitzian, a modified three-step iteration scheme and from a CAT(0) space to a CAT(k) space considered in this paper.
4. Our results extend the corresponding results of Xu and Noor [22] to the case of a more general class of asymptotically nonexpansive mappings, a modified three-step iteration scheme and from a Banach space to a CAT(k) space considered in this paper.
5. Our results also extend and generalize the corresponding results of [35, 38, 49–52] for a more general class of non-Lipschitzian mappings, a modified three-step iteration scheme and from a uniformly convex metric space, a CAT(0) space to a CAT(k) space considered in this paper.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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